Proceedings

of the

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada

Gijón (Asturias), Spain

June 14-18, 2021

Sociedad Española de Matemática Aplicada

Universidad de Oviedo

Editors:
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Coordinadores: Rafael Gallego, Mariano Mateos (2021), Proceedings of the XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones / XVI Congreso de Matemática Aplicada. Universidad de Oviedo.

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Servicio de Publicaciones de la Universidad de Oviedo
Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
http: www.uniovi.es/publicaciones
servipub@uniovi.es

ISBN: 978-84-18482-21-2

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XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada
Gijón, 14-18 junio 2021
(pp. 44–50)

XVA for American options with two stochastic factors: modelling, mathematical analysis and numerical methods

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Abstract

In this work, we derive new linear and nonlinear partial differential equations (PDEs) models for pricing American options and total value adjustment in the presence of counterparty risk. Moreover, stochastic spreads are considered, which increases the dimension of the problem.

1. Introduction

Counterparty risk can be understood as the risk to each party of a contract from a future situation in which one of the counterparties cannot live up its contractual obligations. Since the last financial, crisis when several institutions went bankrupt, a relevant effort in quantitative finance research concerns to the consideration of counterparty risk in financial contracts, specially in the pricing of derivatives As a consequence, different adjustments on the value of the derivative without counterparty risk (hereafter referred as risk–free derivative) are being included in the derivative pricing. For example, the credit value adjustment (CVA) refers to the variation on the price of a contract due to the possibility of default of one (or both) of the counterparties. Adjustments on debit (DVA) and funding (FVA) are also important issues included in the so called total value adjustment (XVA). The XVA incorporates the sum of all the adjustments related to counterparty risk.

In a previous work [2], European and American options have been priced considering corunterparty risk. In such model, constant intensities of default for both counterparties have been assumed. So that a model depending on just one underlying stochastic factor (the underlying asset) is posed to price XVA. However, the intensity of default is not always constant, then stochastic intensities of default has been assumed in [3] as a result a model depending on two stochastic factors (the asset price and the spread from the investor) was deduced to price European options. In the current work, as we have done in [3], we consider that only the investor is defaultable and presents a stochastic intensity of default. Moreover, similar hypotheses as in the European options model introduced in [3] are assumed. Then, we extend the models introduced in [2], [3] to price the American options considering counterparty risk and compute the associated total value adjustment when stochastic intensity of default is assumed. So, we deduce a two dimensional PDE model for the American risky derivative value with stochastic intensity of deafult. The plan of the chapter is the following. In Section 1 we pose the complementarity problems deduced from the hedging arguments. In Section 2 we present the mathematical analysis of the previous problems. Section 3 presents the numerical methods and Section 4 shows some illustrative numerical results.

2. Mathematical model

In this section, we deduce the models for American options considering counterparty risk. With this aim, we consider self–financing portfolio and non–arbitrage scenarios. Moreover, we assume an investor as a risky counterparty and consider that the issuer's intensity of default is null. Thus, the underlying asset price S, and the short term CDS spread of the investor h are modelled by the following system of stochastic differential equations:

$$dS_t = (r(t) - q(t))\, S_t\, dt + \sigma^S(t)\, S_t\, dW_t^S\,,$$
$$dh_t = (\mu^h(t) - M^h(t)\sigma^h(t))\, dt + \sigma^h(t)\, dW_t^h\,,$$

where $(r(t) - q(t))$ and $(\mu^h(t) - M^h(t)\sigma^h(t))$ are the (respective) drifts of the processes. Moreover, $r(t)$ denotes the risk-free interest rate, $q(t)$ is the asset dividend yield rate, $M^h(t)$ is the market price of investor's credit risk, $\sigma^S(t,S)$ and $\sigma^h(t,h)$ are the volatility functions, and W_t^S and W_t^h are two correlated Wiener processes (i.e., $\rho\, dt = dW_t^S\, dW_t^h$) so that ρ is the instantaneous correlation between S_t and h_t.

Thus, we consider a derivative trade between a hedger and an investor, where only the investor has probability of default. The risky derivative value from the point of view of the investor, at time t, is denoted by $\widehat{V}(t, S_t, h_t, J_t^I)$, and depends on the spot value of the asset (S_t), on the spread of the investor (h_t) and on the investor's default state at time t (J_t^I). Remind that $J_t^I = 1$ in case of default before or at time t, otherwise $J_t^I = 0$. The risk–free

American option value, corresponding to the same contract between two free–bankruptcy counterparties, is denoted by $\widehat{V}(t,S_t)$ and does not include any counterparty risk adjustment, whereas the risky derivative price $\widehat{V}_t$ includes total value adjustment.

The risky derivative price in case of the investor makes default is given by:

$$\widehat{V}(t,S_t,h_t,1) = RM^+(t,S_t,h_t) + M^-(t,S_t,h_t), \tag{2.1}$$

where $M(t,S_t,h_t)$ denotes the mark–to–market price, $M^+ = \max(M,0)$ and $M^- = \min(0,M)$. In terms of the mark-to-market condition (2.1), we introduce $\Delta\widehat{V}$ as the variation of $\widehat{V}$ at default, which is given by:

$$\Delta\widehat{V}_t = RM_t^+ + M_t^- - \widehat{V}_t\,, \tag{2.2}$$

where $M_t = M(t,S_t,h_t)$. As it is usually assumed in the literature [4], and as we did in [2] and [3], we consider two possibilities for the mark–to–market value: either the risk–free value, either the derivative value including counterparty risk.

The hedger will trade with different financial instruments to hedge the market risk, the spread risk and the investor's default risk. Thus, in order to derive the value of American options with counterparty risk, we consider the same self–financing portfolio built for European options in [3], Π_t, which is designed to hedge all underlying risk factors:

$$\Pi_t = \alpha(t)H(t) + \beta(t) + \gamma(t)\mathrm{CDS}(t,T) + \varepsilon(t)\mathrm{CDS}(t,t+dt) + \Omega(t)B(t,t+dt)\,. \tag{2.3}$$

Furthermore, in order to avoid arbitrage opportunities we introduce the following hedging inequality:

$$d\widehat{V}_t \le d\Pi_t\,. \tag{2.4}$$

Next, by applying Itô's Lemma for jump diffusion processes, we obtain the variation $d\widehat{V}_t$ of the derivative value $\widehat{V}_t$. Thus, replacing the change of the portfolio and the change of the derivative in (2.4), the hedging equation is transformed into:

$$\begin{aligned}
&\frac{\partial\widehat{V}}{\partial t} + \frac{1}{2}(\sigma^S)^2S^2\frac{\partial^2\widehat{V}}{\partial S^2} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2\widehat{V}}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2\widehat{V}}{\partial S\partial h}\\
&\quad\le \frac{\partial\widehat{V}/\partial S}{\partial H/\partial S}\left(cH - (r-q)S\frac{\partial H}{\partial S}\right) + \frac{\partial\widehat{V}/\partial S}{\partial H/\partial S}(-fH)\\
&\quad+ \frac{\partial\widehat{V}/\partial h}{\partial\mathrm{CDS}(t,T)/\partial h}\left(-\frac{h}{1-R}\Delta\mathrm{CDS}(t,T) - \left(\mu^h - M\sigma^h\right)\frac{\partial\mathrm{CDS}(t,T)}{\partial h}\right)\\
&\quad+ \left(\frac{\partial\widehat{V}/\partial h}{\partial\mathrm{CDS}(t,T)/\partial h}\frac{\Delta\mathrm{CDS}(t,T)}{1-R} - \frac{\Delta\widehat{V}}{1-R}\right)h + f\widehat{V}\,,
\end{aligned} \tag{2.5}$$

in $[0,T)\times(0,\infty)\times(0,\infty)$. Then, the American option value when considering counterparty risk is modelled by the following complementarity problem:

$$\begin{cases}
\mathcal{L}(\widehat{V}) = \dfrac{\partial\widehat{V}}{\partial t} + \tilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta\widehat{V}}{1-R}h - f\widehat{V} \le 0\\
\widehat{V}(t,S,h) \ge G(S)\\
\mathcal{L}(\widehat{V})(\widehat{V}-G) = 0\\
\widehat{V}(T,S,h) = G(S)\,,
\end{cases} \tag{2.6}$$

where $G(S)$ represents the option payoff and the differential operator $\widetilde{\mathcal{L}}_{Sh}$ is

$$\mathcal{L}_{Sh}V \equiv \frac{1}{2}(\sigma^S)^2S^2\frac{\partial^2V}{\partial S^2} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2V}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2V}{\partial h\partial S} + (r-q)S\frac{\partial V}{\partial S} - \frac{\kappa}{1-R}h\frac{\partial V}{\partial h}\,.$$

According to the mark-to–market choices, two alternative linear complementarity problems are deduced:

- If $M = \widehat{V}$, we deduce the nonlinear complementarity problem:

$$\begin{cases}
\mathcal{L}_1(\widehat{V}) = \dfrac{\partial\widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - f\widehat{V} - h\widehat{V}^+ \le 0, \quad \text{in } [0,T)\times(0,\infty)\times(0,\infty)\\
\widehat{V}(t,S,h) \ge G(S)\\
\mathcal{L}_1(\widehat{V})(\widehat{V}-G) = 0\\
\widehat{V}(T,S,h) = G(S)\,.
\end{cases} \tag{2.7}$$

- If $M = V$, the following linear complementarity problem is derived:

$$\begin{cases} \mathcal{L}_2(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - \left(\dfrac{h}{1-R} + f\right)\widehat{V} \\ \qquad -((1-R)V^+ - V)\dfrac{h}{1-R} \leq 0, \quad \text{in } [0,T) \times (0,\infty) \times (0,\infty) \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}_2(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \tag{2.8}$$

Moreover, to compute the XVA value, we consider that $\widehat{V} = V + U$ where U denotes the XVA, then the adjustments can be obtained as the difference of the risky derivative value, $\widehat{V}$, and the risk–free derivative value, V, which is the solution of the classical Black-Scholes American problem:

$$\begin{cases} \mathcal{L}_3(V) = \dfrac{\partial V}{\partial t} + \mathcal{L}_S V - fV \leq 0\,, \quad \text{in } [0,T) \times (0,\infty) \\ V(t,S) \geq G(S) \\ \mathcal{L}_3(V)(V - G) = 0 \\ V(T,S) = G(S)\,, \end{cases} \tag{2.9}$$

where the operator $\mathcal{L}_S$ is given by

$$\mathcal{L}_S V \equiv \frac{(\sigma^S)^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S}\,.$$

In order to numerically solve problems (2.7) and (2.8) by a finite element method, we proceed to localize the problems on a bounded domain. For this purpose, let us consider $\Omega = (0, S_\infty) \times (0, h_\infty)$ for large enough values of S_∞ and h_∞, so that their choice does not affect the solution in the domain of financial interest. We need to impose appropiate boundary conditions on the risky derivative value problem in the bounded domain. For this purpose, we consider the same boundary conditions than for V and $\widehat{V}$ as in the case of European options in [3]. Then, at $S = 0$ and $S = S_\infty$, the derivative value is given by:

$$\begin{cases} \widehat{V}(t,0,h) = V(t,0) = V_0(t)\,, \\ \widehat{V}(t,S_\infty,h) = V(t,S_\infty) = V_\infty(t)\,, \end{cases}$$

where the values of $V_0(t)$ and $V_\infty(t)$ are respectively given by:

$$V_0(t) = \begin{cases} 0\,, & \text{for a call option,} \\ K\exp(-f(T-t))\,, & \text{for a put option,} \end{cases} \tag{2.10}$$

$$V_\infty(t) = \begin{cases} S_\infty - K\,, & \text{for a call option,} \\ 0\,, & \text{for a put option.} \end{cases} \tag{2.11}$$

In the next section, the existence and uniqueness of solution of problem (2.7) are studied. For this purpose, we introduce the problem which models the XVA in order to obtain a problem with homogeneous boundary conditions. Then, we split up the risky derivative value, $\widehat{V}$, as the sum of the XVA, U, plus the total value adjustment, V, i.e. $\widehat{V} = V + U$. Introducing this breakdown in (2.7), the following nonlinear complementarity problem is deduced:

$$\begin{cases} \mathcal{L}_t(U) = \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh}U - fU - h(U+V)^+ \leq -\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\,, \qquad t \in [0,T)\,, \quad (S,h) \in \Omega \\ U(t,S,h) \geq G(S) - V(t,S) \\ \left[\mathcal{L}_t(U) - \left(-\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\right)\right]\left[U - (G(S) - V(t,S))\right] = 0 \\ U(T,S,h) = 0 \\ U(t,0,h) = 0 \\ U(t,S_\infty,h) = 0 \\ U(t,S,0) = 0 \\ (A\nabla U \cdot \vec{n})(\tau, S, h_\infty) = 0\,. \end{cases} \tag{2.12}$$

For the linear problem (2.8), the same boundary conditions are considered.

3. Mathematical analysis

In this section we prove the existence and uniqueness of solution for the XVA problem (2.12) for a given function V. Then, taking into account the existence and uniqueness of solution V for the classical Black-Scholes problem, we obtain the existence and uniqueness of solution for problem (2.7). Introducing the time to maturity variable, $\tau = T - t$, as well as the new variables and unknown:

$$x = \ln \frac{S}{K}, \qquad u(\tau, x, h) = U(t, S, h), \qquad v(\tau, x) = V(t, S)\,.$$

we pose the nonlinear complementarity problem (2.12) as follows:

$$\begin{cases} \mathcal{L}_\tau(u) = \dfrac{\partial u}{\partial \tau} + \mathcal{A}u - \Phi(\tau, u) \geq \ell\,, \quad (x,h) \in \widehat{\Omega}, \quad \tau \in (0,T] \\ u \geq \psi \\ [\mathcal{L}_\tau(u) - \ell]\ [u - \psi] = 0 \\ u(0, S, h) = 0 \\ u(\tau, x_0, h) = 0 \\ u(\tau, x_\infty, h) = 0 \\ u(\tau, x, 0) = 0 \\ (\widehat{A}\nabla u \cdot \vec{n})(\tau, x, h_\infty) = 0\,, \end{cases} \tag{3.1}$$

Theorem 3.1 *The following statements are satisfied*:

1. *The continuous operator $\mathcal{A}$ satisfies Gårding's inequality, i.e.*:
$$(\mathcal{A}z, z) \geq \omega_1 \|z\|^2_{H^1_\Gamma(\widehat{\Omega})} - \omega_2 \|z\|^2_{L^2(\widehat{\Omega})}\,, \quad \forall z \in H^1_\Gamma(\widehat{\Omega})\,, \tag{3.2}$$
with $\omega_1 > 0$ and $\omega_2 \in \mathbb{R}$.
2. $\ell \in L^2(0,T; L^2(\widehat{\Omega})) \subset L^2(0,T; W^*)$.
3. *Let* $D(\phi) = \left\{ z \in H^1_\Gamma(\widehat{\Omega}) \;/\; \phi(z) < \infty \right\}$ *and* $u_0 = u(0,x,h)$. *Then,* $u_0 \in \overline{D(\phi)}$.
4. $\Phi(\tau, \varphi)$ *is Lipschitz continuous on variable φ, i.e.*
$$\|\Phi(\tau, \varphi_1) - \Phi(\tau, \varphi_2)\|_{L^2(\widehat{\Omega})} \leq L_G \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}\,.$$

Therefore, the nonlinear variational inequality (3.1) *has a unique solution* $u \in L^2(0,T; H^1_\Gamma(\widehat{\Omega})) \cap C([0,T]; L^2(\widehat{\Omega}))$; *in particular* $u \in W^{1,2}(0,T; L^2(\widehat{\Omega}))$ *and satisfies*

$$\|u\|_{W^{1,2}(0,T;L^2(\widehat{\Omega}))} \leq C_1 \left(1 + \|u_0\|_{L^2(\widehat{\Omega})} + \|\ell\|_{L^2(0,T;H^1_\Gamma(\widehat{\Omega}))}\right)\,. \tag{3.3}$$

4. Numerical simulation

The numerical approximation is mainly based on finite element methods combined with the method of characteristics. Moreover, a fixed point scheme is implemented for the nonlinear complementarity problem.

4.1. The method of characteristics

More precisely, taking into account the advective term, the risky derivative problem is approximated by

$$\begin{cases} \mathcal{L}_1^n(\widehat{V}^{n+1}) = \dfrac{\widehat{V}^{n+1} - \widehat{V}^n \circ \chi^n}{\Delta\tau^n} - \mathrm{div}(A\nabla\widehat{V}^{n+1}) + f\widehat{V}^{n+1} + h(\widehat{V}^{n+1})^+ \geq 0\,, \\ \widehat{V}^0(S,h) = 0\,, \\ \widehat{V}^{n+1}(S,h) \geq G(S)\,, \\ \mathcal{L}_1^n(\widehat{V}^{n+1})(\widehat{V}^{n+1} - G) = 0\,, \end{cases} \tag{4.1}$$

for $n = 0, 1, 2 \ldots, N_T - 1$, where $\widehat{V}^n(\cdot) \approx \widehat{V}(\tau^n, \cdot)$ and $\chi^n = \chi(\tau^n) = \chi((S,h), \tau^{n+1}; \tau^n)$ represents the characteristic curve passing through point (S,h) at time τ^{n+1}. Then function χ is the solution of the final value ODE problem:

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = \left((\sigma^S)^2 - (r-q)\right)\chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \dfrac{d\chi_2}{d\tau} = \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}\chi_2, \\ \chi_2(\tau^{n+1}) = h\,, \end{cases} \tag{4.2}$$

The components of χ^n can thus be deduced and are given by:

$$\chi_1^n = S \exp\left(-((\sigma^S)^2 - r + q)(\tau^{n+1} - \tau^n)\right),$$
$$\chi_2^n = -\frac{(1-R)\sigma^S \sigma^h \rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S \sigma^h \rho}{2\kappa}\right) \exp\left(\frac{-\kappa}{1-R}(\tau^{n+1} - \tau^n)\right).$$

4.2. Fixed point scheme

In order to solve the nonlinearity of problem (4.1), a fixed point scheme is proposed at each iteration of the characteristics method. Thus, the global scheme is shown in Algorithm 1.

Algorithm 1

Let $N_T > 1$, $n = 0$, $\varepsilon > 0$ and $\widehat{V}^0$ given
For $n = 1, 2, \ldots, N_T - 1$:

1. Let $\widehat{V}^{n+1,0} = \widehat{V}^n$, $k = 0$, $e = \varepsilon + 1$
2. For $k = 0, 1, \ldots$
   - Search $\widehat{V}^{n+1,k+1}$ solution of:

$$\begin{aligned}
&(1 + \Delta\tau^n f)\, \widehat{V}^{n+1,k+1} - \Delta\tau^n \operatorname{div}(A \nabla \widehat{V}^{n+1,k+1}) \\
&\qquad\qquad \geq \widehat{V}^n \circ \chi^n - \Delta\tau^n\, h\, (\widehat{V}^{n+1,k})^+ \\
&\widehat{V}^{n+1,k+1}(S,h) \geq G(S) \\
&\mathcal{L}_1^n(\widehat{V}^{n+1,k+1})(\widehat{V}^{n+1,k+1} - G) = 0
\end{aligned} \tag{4.3}$$

   - Compute the relative error $e = \dfrac{\|\widehat{V}^{n+1,k+1} - \widehat{V}^{n+1,k}\|}{\|\widehat{V}^{n+1,k+1}\|}$

   until $e < \varepsilon$.

4.3. Finite element method

For the spatial discretization of (4.3) a triangular mesh of Ω and the associated finite element space of piecewise linear Lagrange polynomials are considered. For fixed natural numbers $N_S > 0$ and $N_h > 0$, we consider a uniform mesh of the computational domain Ω, the nodes of which are (S_i, h_j), with $S_i = i\Delta S$ $(i = 0, \ldots, N_S + 1)$ and $h_j = j\Delta h$ $(j = 0, \ldots, N_h + 1)$, where $\Delta S = S_\infty/(N_S + 1)$ and $\Delta h = h_\infty/(N_h + 1)$ denote the constant mesh steps in each coordinate. Associated to this uniform mesh, a piecewise linear Lagrange finite element discretization is considered. More precisely, we introduce the finite element spaces

$$W_h = \{\varphi_h \in C(\Omega) \,/\, \widehat{\varphi}|_{T_j} \in \mathcal{P}_1,\ \forall T_j \in \mathcal{T}\},$$
$$\mathcal{K}_h = \{\varphi_h \in W_h \,/\, \varphi_h = \widehat{V} \text{ on } \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \text{and } \varphi_h \geq G(S)\},$$

in order to find $\widehat{V}_h^{n+1,k+1} \in \mathcal{K}_h$ satisfying the boundary conditions and such that:

$$\begin{aligned}
&\int_\Omega (1 + \Delta\tau^n f) \widehat{V}_h^{n+1,k+1} \left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\, dh \\
&\quad + \Delta\tau^n \int_\Omega A \nabla \widehat{V}_h^{n+1,k+1}\, \nabla\left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\, dh \\
&\quad - \Delta\tau^n \int_{\Gamma_2^{*,+}} (A \nabla V_h^{n+1,k+1}, n)(\varphi_h - \widehat{V}_h^{n+1,k+1}) \partial\gamma \\
&\quad \geq \int_\Omega \left(\widehat{V}_h^n \circ \chi^n\right)\left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\, dh - \Delta\tau^n \int_\Omega h \left(\widehat{V}_h^{n+1,k}\right)^+ \left(\varphi_h - \widehat{V}_h^{n+1,k+1}\right) dS\, dh,
\end{aligned}$$

for all $\varphi_h \in \mathcal{K}_h$. Quadrature formula based on the midpoints of the edges of the triangles has been used to obtain the coefficients of the matrix and the right hand side vector which define the linear system associated to the discretized problem.

After the time discretization with the method of characteristics and the spatial discretization with finite elements, the fully discretized problem can be written in the form:

$$\begin{cases} A_h \widehat{V}_h^{n+1,k+1} \geq b_h^{n+1,k+1}\,, \\ \widehat{V}_h^{n+1,k+1} \geq \Psi_h\,, \\ (A_h \widehat{V}_h^{n+1,k+1} - b_h^{n+1,k+1})(\widehat{V}_h^{n+1,k+1} - \Psi_h) = 0\,, \end{cases} \tag{4.4}$$

where Ψ_h denotes the discretized exercise value, $G(S)$, which also coincides with the value at maturity.

In order to solve problem (4.4), the augmented Lagrangian active set (ALAS) algorithm is employed.

5. Numerical results

Finally, in order to show the relevance of incorporating counterparty risk pricing derivatives we show some numerical results to understand the behaviour of the total value adjustment for American options. We focus on an American put option sold by the investor. The maturity time is $T = 0.5$ years and is discretized with $N_T = 700$ time steps. Firstly, we plot the risky and risk-free derivative value and the XVA. Moreover, we present the exercise region for both derivative value in order to show how affects the counterparty risk in the early exercise.

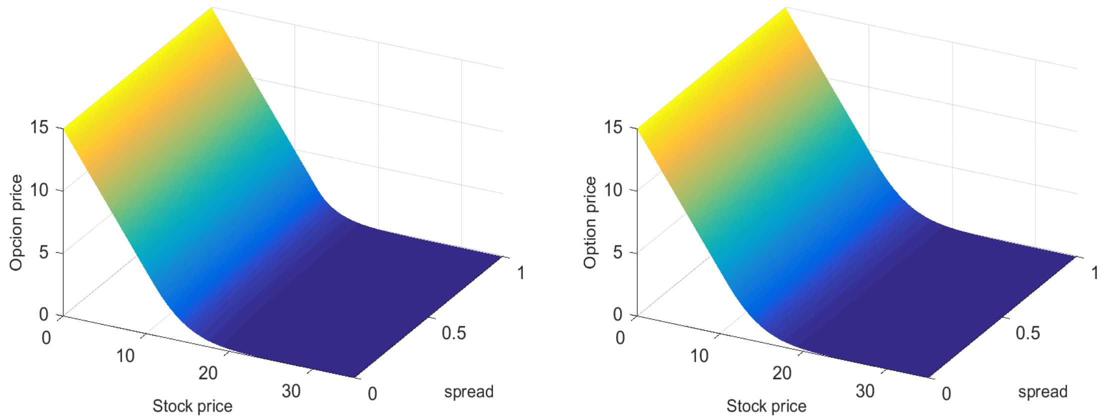


Fig. 1 American put option value risky valur (left), risk–free value (right)

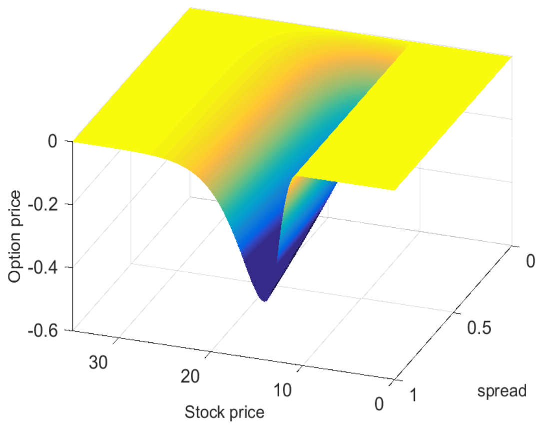
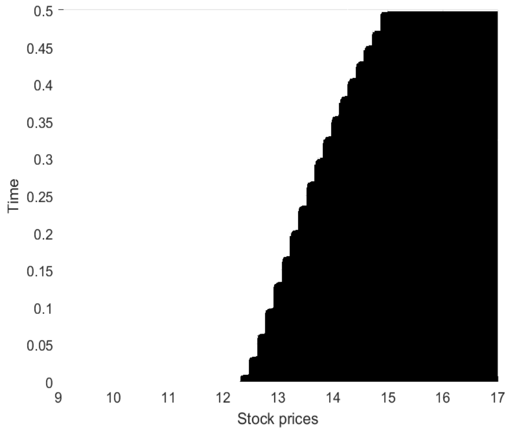


Fig. 2 Total value adjustment



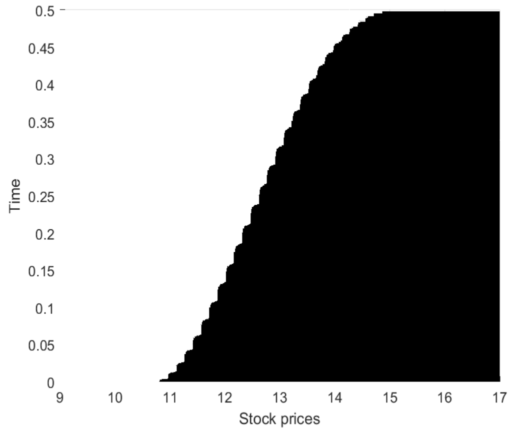


Fig. 3 Exercise regions (white) risky value (left) risk–free value (right)

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