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ORIGINAL PAPER

Mean curvature flow of closed curves evolving in two dimensional manifolds

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Abstract

We investigate the motion of a family of closed curves evolving according to the geometric evolution law on a given two dimensional manifold which is embedded or immersed in the three-dimensional Euclidean space. We derive a system of nonlinear parabolic equations describing the motion of curves belonging to a given two-dimensional manifold. Using the abstract theory of analytic semiflows, we prove the local existence, uniqueness of Hölder smooth solutions to the governing system of nonlinear parabolic equations for the position vector parametrization of evolving curves. We apply the method of flowing finite volumes in combination with the methods of lines for numerical approximation of the governing equations. Qualitative analytical results are illustrated by various numerical experiments.

Keywords Curvature-driven flow · Binormal flow · Analytic semi-flows · Hölder smooth solutions · Flowing finite-volume method

Mathematics Subject Classification Primary: 35K57 · 35K65 · 65N40 · 65M08 · Secondary: 53C80

1 Introduction

In this article, we investigate the motion of a family $\{\Gamma_t, t \geq 0\}$ of closed curves evolving in the three-dimensional Euclidean space $\mathbb{R}^3$ according to a subclass of the following geometric evolution law:

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$$\partial_t \mathbf{X} = v_N \mathbf{N} + v_B \mathbf{B} + v_T \mathbf{T}, \tag{1}$$

where the unit tangent $\mathbf{T}$, normal $\mathbf{N}$ and binormal $\mathbf{B}$ vectors form moving Frenet frame. In this paper, we restrict our interest to the investigation of the dynamics of three-dimensional closed curves on embedded and immersed manifolds in $\mathbb{R}^3$.

The three-dimensional motion of closed curves is motivated by various physical applications arising in materials sciences, fluid dynamics, or molecular biology. In fluid dynamics, the motion of a curve in space is often applied to the analysis of vortex structures first studied by Helmholtz [19]. For a detailed summary of recent advances in the analysis of three-dimensional motion of curves, we refer the reader to Kolář, Beneš and Ševčovič [7] and references therein.

Among many important physical and engineering applications of flows of space curves evolving on prescribed surfaces, we mention the dislocation dynamics. A geodesic description of space curves is a convenient mathematical framework to simulate a dislocation cross-slip phenomenon in the crystalline structure of solids (cf. Kolář et al. [24]). The dislocation curves evolve on a prescribed two-dimensional manifold in $\mathbb{R}^3$. Moreover, the dynamics of dislocation climbs [33, 34] investigated by Niu et al. motivates us to study the problem of diffusion and transport along a moving space curve (cf. Beneš et al. [8]). In nanomaterials manufacturing, a procedure called electrospinning is frequently used, i.e., by jetting polymer solutions in high electric fields into ultrafine nanofibers. The extruded fluid forms a curve evolving on the so-called Taylor conic surface. The jet oriface (outlet) is the vertex of the Taylor cone (cf. Reneker [36], Yarin et al. [16], He et al. [18] and particular references therein). Structures arising from the electrospinning procedure move in space according to (1) and under the effect of electric forces [38] to form nanofibers. Recently, Kolář and Ševčovič [25] analyzed, from the perspective of local existence and uniqueness, a motion of systems of space curves in the normal and binormal directions, mutually coupled by a Biot–Savart-type interaction. In [23], an area-preserving flow was subsequently investigated using a geodesic formulation of the curves. The structure of the Helfrich flow of curves with non-local constraints was recently investigated by Kenmochi, Miyatake, and Sakakibara in [21].

In [11], Deckelnick and Nürnberg proposed a novel framework for the evolution of parametric curves driven by the anisotropic curve shortening flow in $\mathbb{R}^3$. Furthermore, in [9], Binz investigates optimal error estimates for semidiscrete and fully discrete schemes that approximate isotropic curve shortening flow in three dimensions. Their formulation relies on a careful choice of the tangential component of the velocity in parameterization, which transforms the evolution problem into a strictly parabolic differential equation. This equation is written in divergence form, enabling the construction of a natural variational discretization. For a fully discrete finite element scheme based on piecewise linear elements, optimal error bounds are established. Numerical experiments support the theoretical findings and illustrate the effectiveness of the method.

The paper is organized as follows. In the second section, a parametric description of evolving curves is introduced. In the third section, we restrict our interest to the motion on a closed surface without boundary. We derive a force term that attaches the curve to the surface and formulate the system of governing equations for such a restricted 3D motion. In the fourth section, we discuss the conditions for the existence and uniqueness of classical Hölder solutions. We also recall the role of the tangential velocity when using the parametric description. In the fifth section, we propose the numerical approximation scheme based on the flowing finite-volume approach. Finally, several computational examples are presented in the sixth section.

2 Lagrangian description of evolving curves

Our methodology for solving (1) is based on the so-called direct Lagrangian approach investigated by Dziuk [12], Deckelnick [10], Gage and Hamilton [13], Mikula and Ševčovič [29–32], and references therein. We explore the direct Lagrangian approach for an analytical and numerical solution of the geometric motion law (1). The evolving family of curves Γ_t is parametrized by smooth mapping $\mathbf{X} : I \times [0, \infty) \to \mathbb{R}^3$, such that $\Gamma_t = \{\mathbf{X}(u, t), u \in I\}, t \geq 0$. In what follows, we denote by $I = \mathbb{R}/\mathbb{Z} \simeq S^1$ the periodic interval $I = [0, 1]$ isomorphic to the unit circle S^1. We assume that the scalar velocities v_N, v_T, v_B are smooth functions of the position vector $\mathbf{X} \in \mathbb{R}^3$, the curvature κ, the torsion τ, and possibly depend nonlocally on the quantities associated with the curve Γ_t itself, for example, its length. That is,

$$v_K = v_K(\mathbf{X}, \kappa, \tau, \mathbf{T}, \mathbf{N}, \mathbf{B}, \Gamma_t), \quad K \in \{T, N, B\}.$$

The unit tangent vector $\mathbf{T}$ to Γ_t is defined as $\mathbf{T} = \partial_s \mathbf{X}$, where s is the unit arc-length parameterization defined by $ds = |\partial_u \mathbf{X}| du$. Here, $|\mathbf{x}|$ and $\mathbf{x}^\mathsf{T}\mathbf{y} \equiv \mathbf{x} \cdot \mathbf{y}$ denote the Euclidean norm and the inner product of the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. The curvature κ of a curve Γ_t is defined as $\kappa = |\partial_s \mathbf{X} \times \partial_s^2 \mathbf{X}| = |\partial_s^2 \mathbf{X}|$. If $\kappa > 0$, we can define the Frenet frame along the curve Γ_t with unit normal $\mathbf{N} = \kappa^{-1} \partial_s^2 \mathbf{X}$ and binormal vectors $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, respectively. Recall the Frenet-Serret formulae:

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

where τ is the torsion of Γ_t.

We study a coupled system of evolutionary equations that describes the evolution of closed 3D curves evolving in normal and binormal directions. More specifically, we focus on the motion of a family of curves evolving in 3D and satisfying the geometric law

$$\partial_t \mathbf{X} = a \partial_s^2 \mathbf{X} + \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}) + \alpha \partial_s \mathbf{X}, \tag{2}$$

where $a = a(\mathbf{X}, \partial_s \mathbf{X}) > 0$, and $\mathbf{F} = \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X})$ are bounded and smooth functions of their arguments. Here, $\mathbf{F}$ is an external force term that restricts the movement of Γ_t to a given manifold. The forcing term is discussed further in Theorem 1. Since $\partial_s^2 \mathbf{X} = \kappa \mathbf{N}$ and $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ the relationship between the geometric equations (1) and (2) is as follows:

$$v_N = a\kappa + \mathbf{F}^\mathsf{T}\mathbf{N}, \quad v_B = \mathbf{F}^\mathsf{T}\mathbf{B}, \quad v_T = \mathbf{F}^\mathsf{T}\mathbf{T} + \alpha. \tag{3}$$

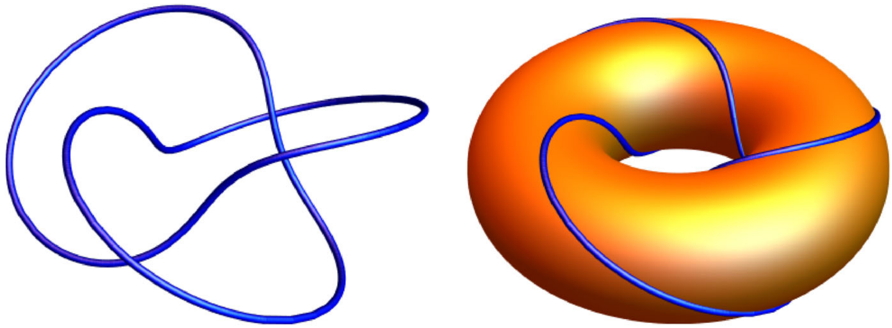


Fig. 1 A knotted curve (**a**) belonging to the embedded torus surface (**b**)

The system of equations (2) is subject to the initial condition

$$\mathbf{X}(u,0) = \mathbf{X}_0(u),\, u \in I = \mathbb{R}/\mathbb{Z} \simeq S^1, \tag{4}$$

representing parameterization of the initial curve Γ_0.

3 Evolution of closed curves on a closed surface without boundary

In this section, we analyze the evolution of closed curves on a two-dimensional surface without boundary. First, we discuss the evolution of curves in embedded manifolds.

3.1 Evolution of curves on embedded manifolds

Assume $\mathcal{M} \in \mathbb{R}^3$ is an embedded manifold given by $\mathcal{M} = \{\mathbf{X} = (X_1, X_2, X_3)^\mathsf{T} \in \mathbb{R}^3,\ f(\mathbf{X}) = 0\}$ where $f(\mathbf{X}) : \mathbb{R}^3 \to \mathbb{R}$ is a C^4 smooth regular map. This means that $\nabla f(\mathbf{X}) \neq 0$ for $\mathbf{X} \in \mathcal{M}$.

Let $\phi(u,t) = f(\mathbf{X}(u,t))$. Then

$$\begin{aligned}
\partial_t \phi &= \nabla f(\mathbf{X})^\mathsf{T} \partial_t \mathbf{X} \\
&= \nabla f(\mathbf{X})^\mathsf{T} \left(a\partial_s^2 \mathbf{X} + \mathbf{F} + \alpha \partial_s \mathbf{X} \right) = a \nabla f(\mathbf{X})^\mathsf{T} \partial_s^2 \mathbf{X} + \nabla f(\mathbf{X})^\mathsf{T} \mathbf{F} + \alpha \nabla f(\mathbf{X})^\mathsf{T} \partial_s \mathbf{X} \\
&= a\partial_s \left(\nabla f(\mathbf{X})^\mathsf{T} \partial_s \mathbf{X} \right) - a\partial_s \mathbf{X}^\mathsf{T} \nabla^2 f(\mathbf{X}) \partial_s \mathbf{X} + \nabla f(\mathbf{X})^\mathsf{T} \mathbf{F} + \alpha \partial_s \phi \\
&= a\partial_s^2 \phi - a\partial_s \mathbf{X}^\mathsf{T} \nabla^2 f(\mathbf{X}) \partial_s \mathbf{X} + \nabla f(\mathbf{X})^\mathsf{T} \mathbf{F} + \alpha \partial_s \phi
\end{aligned}$$

because $\partial_s \phi = \nabla f(\mathbf{X})^\mathsf{T} \partial_s \mathbf{X}$.

Theorem 1 *Suppose that the external force* $\mathbf{F}$ *is given by*

$$\mathbf{F}(\mathbf{X}, \mathbf{T}) = a \frac{\mathbf{T}^\mathsf{T} \nabla^2 f(\mathbf{X}) \mathbf{T} + h(f(\mathbf{X}))}{|\nabla f(\mathbf{X})|^2} \nabla f(\mathbf{X}), \tag{5}$$

$\mathbf{T} = \partial_s \mathbf{X}$. *Here $h : \mathbb{R} \to \mathbb{R}$ is a smooth function, $h(0) = 0$. If the initial curve $\Gamma_0 \subset \mathcal{M}$ then the family of curves Γ_t, for $t > 0$, evolving with respect to the geometric equation (2) belongs to the manifold $\mathcal{M} = \{\mathbf{X} \in \mathbb{R}^3,\ f(\mathbf{X}) = 0\}$. Furthermore, the tangential velocity $v_T = \alpha$.*

Proof If the initial curve $\Gamma_0 \subset \mathcal{M}$ then $\phi(\cdot, 0) = f(\mathbf{X}(\cdot, 0)) = 0$. The function ϕ is a solution to the parabolic equation:

$$\partial_t \phi = a \partial_s^2 \phi + a h(\phi) + \alpha \partial_s \phi, \tag{6}$$

with a zero initial condition $\phi(\cdot, 0) = 0$. Since $h(0) = 0$ we have $\phi(\cdot, t) = 0$ for all $t > 0$. That is $\Gamma_t \subset \mathcal{M}$. The projection $\mathbf{F}^\top \mathbf{T}$ of $\mathbf{F}$ to the tangent direction is vanishing. In fact, as $0 = \partial_s \phi = \nabla f(\mathbf{X})^\top \mathbf{T}$, we have $\mathbf{F}(\mathbf{X}, \mathbf{T})^\top \mathbf{T} = a \frac{\mathbf{T}^\top \nabla^2 f(\mathbf{X}) \mathbf{T} + h(f(\mathbf{X}))}{|\nabla f(\mathbf{X})|^2} \nabla f(\mathbf{X})^\top \mathbf{T} = 0$.

□

Theorem 2 *Suppose that the external force $\mathbf{F}$ is defined by equation (5), and the initial curve $\Gamma_0 \subset \mathcal{M}$. Then the geometric flow of the curves, which satisfies the law (2), is the length-shortening flow on the surface $\mathcal{M}$, i.e., $\frac{d}{dt} L(\Gamma_t) \le 0$, where $L(\Gamma_t)$ denotes the total length of the curve Γ_t.*

Proof Suppose that the initial curve $\Gamma_0 \subset \mathcal{M}$. With regard to Theorem 1 we have $\Gamma_t \subset \mathcal{M}$ for any $t \ge 0$. That is, $\phi(s, t) \equiv f(\mathbf{X}(s, t)) = 0$ for all $s \in [0, L(\Gamma_t)]$ and $t \ge 0$. Hence $0 = \partial_s \phi = \nabla f(\mathbf{X})^\top \mathbf{T}$, and

$$0 = \partial_s^2 \phi = \mathbf{T}^\top \nabla^2 f(\mathbf{X}) \mathbf{T} + \nabla f(\mathbf{X})^\top \partial_s \mathbf{T} = \mathbf{T}^\top \nabla^2 f(\mathbf{X}) \mathbf{T} + \kappa \nabla f(\mathbf{X})^\top \mathbf{N}.$$

The projection of the velocity $\partial_t \mathbf{X}$ in the normal direction $\mathbf{N}$ is given by

$$\begin{aligned} v_N &= (a\, \partial_s^2 \mathbf{X} + \mathbf{F})^\top \mathbf{N} = a\,\kappa + a \frac{\mathbf{T}^\top \nabla^2 f(\mathbf{X}) \mathbf{T}}{|\nabla f(\mathbf{X})|^2} \nabla f(\mathbf{X})^\top \mathbf{N} = a\,\kappa - a\,\kappa \frac{(\nabla f(\mathbf{X})^\top \mathbf{N})^2}{|\nabla f(\mathbf{X})|^2} \\ &= a\,\kappa - a\,\kappa \frac{(\nabla f(\mathbf{X})^\top \mathbf{N})^2}{|\nabla f(\mathbf{X})|^2} = a\,\kappa \frac{(\nabla f(\mathbf{X})^\top \mathbf{B})^2}{|\nabla f(\mathbf{X})|^2} \end{aligned} \tag{7}$$

because $|\nabla f(\mathbf{X})|^2 = (\nabla f(\mathbf{X})^\top \mathbf{N})^2 + (\nabla f(\mathbf{X})^\top \mathbf{T})^2 + (\nabla f(\mathbf{X})^\top \mathbf{B})^2 = (\nabla f(\mathbf{X})^\top \mathbf{N})^2 + (\nabla f(\mathbf{X})^\top \mathbf{B})^2$.

$$\frac{d}{dt} L(\Gamma_t) = -\int_{\Gamma_t} \kappa v_N ds = -\int_{\Gamma_t} a\,\kappa^2 \frac{(\nabla f(\mathbf{X})^\top \mathbf{B})^2}{|\nabla f(\mathbf{X})|^2} ds \le 0. \tag{8}$$

□

Any monotonically increasing modification of the normal velocity v_N given by (7) again represents a length-shortening flow.

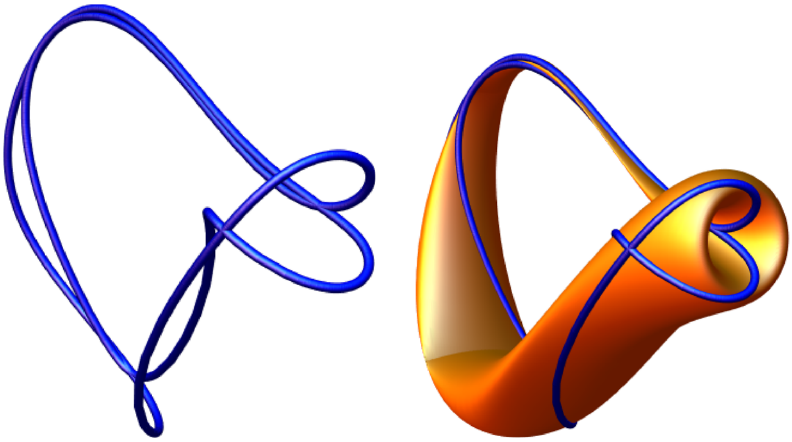


Fig. 2 a) An initial knotted curve belonging to the immersed Klein bottle surface b)

Remark 1 The geodesic curvature κ^g of a curve $\Gamma \subset \mathcal{M}$ can be defined as a projection of the derivative $\partial_s \mathbf{T}$ to the unit vector $\mathbf{N}^g$ perpendicular to $\mathbf{T}$ and belonging to the tangent space $T_X(\mathcal{M})$ at the point $\mathbf{X} \in \mathcal{M}$. Both vectors belong to the tangent space $T_X(\mathcal{M})$ that is perpendicular to the outer normal vector $\nabla f(\mathbf{X})$ to the surface $\mathcal{M}$. The unit normal vector to the surface $\mathcal{M}$ is given by the vector $\nabla f(\mathbf{X})/|\nabla f(\mathbf{X})|$. Therefore, $\mathbf{N}^g = (\nabla f(\mathbf{X}) \times \mathbf{T})/|\nabla f(\mathbf{X})|$. Hence,

$$\begin{aligned} \kappa^g &= \partial_s \mathbf{T}^\intercal \mathbf{N}^g = \kappa \mathbf{N}^\intercal \mathbf{N}^g = \kappa \mathbf{N}^\intercal (\nabla f(\mathbf{X}) \times \mathbf{T})/|\nabla f(\mathbf{X})| \\ &= \kappa \nabla f(\mathbf{X})^\intercal (\mathbf{T} \times \mathbf{N})/|\nabla f(\mathbf{X})| = \kappa (\nabla f(\mathbf{X})^\intercal \mathbf{B})/|\nabla f(\mathbf{X})|. \end{aligned}$$

This means that the flow driven by the geodesic curvature with normal velocity $v_{N^g} = a\kappa^g$ is the same as the flow of a surface with v_N given by (7).

According to Mikula and Ševčovič [32, Eq. (12)] we have $\frac{d}{dt} L(\Gamma_t) = -\int_{\Gamma_t} \kappa^g v_{N^g} ds$. If the normal velocity v_{N^g} is proportional to the geodesic curvature, $v_{N^g} = a\kappa^g$ then we obtain

$$\frac{d}{dt} L(\Gamma_t) = -\int_{\Gamma_t} \kappa^g v_{N^g} ds = -\int_{\Gamma_t} a (\kappa^g)^2 ds = -\int_{\Gamma_t} a\,\kappa^2 \frac{(\nabla f(\mathbf{X})^\intercal \mathbf{B})^2}{|\nabla f(\mathbf{X})|^2} ds \le 0, \tag{9}$$

which is the relation (8).

3.2 Evolution of curves on immersed manifolds

In this section, we discuss the evolution of 3D curves on immersed manifolds. We consider an immersed manifold $\mathcal{M} = \{\mathbf{X} = \mathcal{X}(\mathbf{Y}), \quad \mathbf{Y} \in I \times I\}$ where $\mathbf{Y} = (Y_1, Y_2)^\intercal$ and $\mathbf{X} = (X_1, X_2, X_3)^\intercal$ are parameterized by immersion $\mathcal{X} : I \times I \to \mathbb{R}^3$, where $I = \mathbb{R}/\mathbb{Z} \simeq S^1$ is the periodic interval.

Proposition 1 *Suppose that the function* $\mathbf{Y}(\cdot, t) \subset I \times I$ *is a 1-periodic solution to the parabolic equation:*

$$\partial_t \mathbf{Y} = a\partial_s^2 \mathbf{Y} + \mathbf{G}(\mathbf{Y}, \partial_s \mathbf{Y}) + \alpha \partial_s \mathbf{Y}, \quad \mathbf{Y}(\cdot, 0) = \mathbf{Y}_0(\cdot), \tag{10}$$

where $ds = |\nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_u \mathbf{Y}| du$, $a = a(\mathcal{X}(\mathbf{Y}), \nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_s \mathbf{Y})$, *and* $\mathbf{G} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *is a* C^4 *smooth function.*

Then the closed curve $\Gamma_t = \{\mathbf{X}(s, t), s \in [0, L(\Gamma_t)]\} \subset \mathcal{M}$, $\mathbf{X}(\cdot, t) = \mathcal{X}(\mathbf{Y}(\cdot, t))$ *evolves according to the second order parabolic geometric equation:*

$$\partial_t \mathbf{X} = a\partial_s^2 \mathbf{X} + \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}) + \alpha \partial_s \mathbf{X}, \quad \mathbf{X}(\cdot, 0) = \mathbf{X}_0(\cdot), \tag{11}$$

where $\mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}) = \nabla \mathcal{X}(\mathbf{Y})^\intercal \mathbf{G}(\mathbf{Y}, \partial_s \mathbf{Y}) - a\partial_s \mathbf{Y}^\intercal \nabla^2 \mathcal{X}(\mathbf{Y}) \partial_s \mathbf{Y}$. *Here* $\mathbf{Y} \in I \times I$ *is such that* $\mathbf{X} = \mathcal{X}(\mathbf{Y}) \in \mathcal{M}$ *and* $\partial_s \mathbf{Y} = (\nabla \mathcal{X}(\mathbf{Y}) \nabla \mathcal{X}(\mathbf{Y})^\intercal)^{-1} \nabla \mathcal{X}(\mathbf{Y}) \partial_s \mathbf{X}$ *where the* 2×3 *matrix*

$(\nabla \mathcal{X}(\mathbf{Y}) \nabla \mathcal{X}(\mathbf{Y})^\intercal)^{-1} \nabla \mathcal{X}(\mathbf{Y})$ *is the left Moore-Penrose pseudoinversion of* 3×2 *matrix* $\nabla \mathcal{X}(\mathbf{Y})^\intercal$.

The curve $\Gamma_t, t \geq 0$, *evolves according to the geometric equation (1) with the normal, binormal and tangential velocities given by (3).*

Proof Sin‘ce $\mathbf{X} = \mathcal{X}(\mathbf{Y})$ we have $\partial_t \mathbf{X} = \nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_t \mathbf{Y}$, $\partial_s \mathbf{X} = \nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_s \mathbf{Y}$, $\partial_s^2 \mathbf{X} = \nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_s^2 \mathbf{Y} + \partial_s \mathbf{Y}^\intercal \nabla^2 \mathcal{X}(\mathbf{Y}) \partial_s \mathbf{Y}$, where $\nabla \mathcal{X} = \nabla \mathcal{X}(\mathbf{Y})$ is the 2×3 matrix:

$$\nabla \mathcal{X} = \begin{pmatrix} \frac{\partial \mathcal{X}_1}{\partial Y_1} & \frac{\partial \mathcal{X}_2}{\partial Y_1} & \frac{\partial \mathcal{X}_3}{\partial Y_1} \\ \frac{\partial \mathcal{X}_1}{\partial Y_2} & \frac{\partial \mathcal{X}_2}{\partial Y_2} & \frac{\partial \mathcal{X}_3}{\partial Y_2} \end{pmatrix}.$$

The vector $\partial_s \mathbf{Y}^\intercal \nabla^2 \mathcal{X}(\mathbf{Y}) \partial_s \mathbf{Y} \in \mathbb{R}^3$ is constructed as follows:

$$\partial_s \mathbf{Y}^\intercal \nabla^2 \mathcal{X}(\mathbf{Y}) \partial_s \mathbf{Y} = (\partial_s \mathbf{Y}^\intercal \nabla^2 \mathcal{X}_k(\mathbf{Y}) \partial_s \mathbf{Y})_{k=1,2,3} \in \mathbb{R}^3, \quad \nabla^2 \mathcal{X}_k = \left(\frac{\partial^2 \mathcal{X}_k}{\partial Y_i \partial Y_j} \right)_{i,j=1,2}.$$

The 2×3 matrix $\nabla \mathcal{X}(\mathbf{Y})$ has full rank because the mapping $\mathcal{X}$ is assumed to be an immersion. Furthermore, as $\mathbf{X} = \mathcal{X}(\mathbf{Y})$ we have $ds = |\partial_u \mathbf{X}| du = |\nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_u \mathbf{Y}| du$. Therefore, the derivative of $\mathbf{Y}$ with respect to the arc-length parameterization s of the curve $\mathbf{X}$ can be written as

$$\frac{\partial \mathbf{Y}}{\partial s} = \frac{1}{|\nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_u \mathbf{Y}|} \frac{\partial \mathbf{Y}}{\partial u}.$$

Then

$$\begin{aligned} \partial_t \mathbf{X} = \nabla \mathcal{X}(\mathbf{Y})^\intercal \partial_t \mathbf{Y} &= \nabla \mathcal{X}(\mathbf{Y})^\intercal \left(a\partial_s^2 \mathbf{Y} + \mathbf{G}(\mathbf{Y}, \partial_s \mathbf{Y}) + \alpha \partial_s \mathbf{Y} \right) \\ &= a\partial_s^2 \mathbf{X} + \alpha \partial_s \mathbf{X} - a\partial_s \mathbf{Y}^\intercal \nabla^2 \mathcal{X}(\mathbf{Y}) \partial_s \mathbf{Y} + \nabla \mathcal{X}(\mathbf{Y})^\intercal \mathbf{G}(\mathbf{Y}, \partial_s \mathbf{Y}) \\ &= a\partial_s^2 \mathbf{X} + \alpha \partial_s \mathbf{X} + \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}), \end{aligned}$$

as claimed. □

As an example of a non-orientable immersed manifold in $\mathbb{R}^3$, we can consider the Klein bottle surface parameterized by $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$
where

$$\begin{aligned}
\mathcal{X}_1(u,v) &= -(2/15)\cos(2\pi u)(3\cos(2\pi v) - 30\sin(2\pi u) + 90\cos(2\pi u)^4\sin(2\pi u) \\
&\quad -60\cos(2\pi u)^6\sin(2\pi u) + 5\cos(2\pi u)\cos(2\pi v)\sin(2\pi u)), \\
\mathcal{X}_2(u,v) &= -(1/15)\sin(2\pi u)(3\cos(2\pi v) - 3\cos(2\pi u)^2\cos(2\pi v) \\
&\quad -48\cos(2\pi u)^4\cos(2\pi v) \\
&\quad +48\cos(2\pi u)^6\cos(2\pi v) + 60\sin(2\pi u) + 5\cos(2\pi u)\cos(2\pi v)\sin(2\pi u) \\
&\quad -5\cos(2\pi u)^3\cos(2\pi v)\sin(2\pi u) - 80\cos(2\pi u)^5\cos(2\pi v)\sin(2\pi u) \\
&\quad +80\cos(2\pi u)^7\cos(2\pi v)\sin(2\pi u)), \\
\mathcal{X}_3(u,v) &= (2/15)(3 + 5\cos(2\pi u)\sin(2\pi u))\sin(2\pi v).
\end{aligned}$$

The surface of the Klein bottle is shown in Fig. 2b. The initial curve $\mathbf{X}_0$ is parameterized by

$$\mathbf{X}_0(u) = \mathcal{X}(ku, lu), \quad u \in I,$$

where $k = 1, l = 4$. It is shown in Fig. 2a. The Klein bottle is an immersed manifold in $\mathbb{R}^3$. It is well known that it can be embedded in $\mathbb{R}^4$ but it cannot be embedded in $\mathbb{R}^3$.

Remark 2 The 2×2 matrix $\nabla\mathcal{X}(\mathbf{Y})\nabla\mathcal{X}(\mathbf{Y})^\intercal$ is positive definite because the mapping $\mathcal{X}$ is immersion. For the torus surface, we have

$$det(\nabla\mathcal{X}(\mathbf{Y})\nabla\mathcal{X}(\mathbf{Y})^\intercal) = 16\pi^4 r^2 (R + r\cos(2\pi v))^2 \geq 16\pi^4 r^2 (R - r)^2 > 0.$$

On the other hand, for the Klein bottle surface shown in Fig. 2, b) we observe large spectral variations in the 2×2 positive definite matrix $\nabla\mathcal{X}(\mathbf{Y})\nabla\mathcal{X}(\mathbf{Y})^\intercal$. That is,

$$det(\nabla\mathcal{X}(\mathbf{Y})\nabla\mathcal{X}(\mathbf{Y})^\intercal) \in (0.0145, 32020).$$

4 Existence and uniqueness of classical Hölder smooth solutions

In this section, we present theoretical results on the existence and uniqueness of the classical Hölder smooth solution to the system of equations (2) for the motion of the time-dependent family of curves $\Gamma_t = \{\mathbf{X}(u,t),\ u \in I\},\ t \geq 0$, evolving in $\mathbb{R}^3$. Furthermore, we prove the existence and uniqueness of solutions $\mathbf{Y} = \mathbf{Y}(u,t)$ of the non-linear equation (10) (see Proposition 1). We employ the analytical framework developed by Beneš, Kolář, Ševčovič [7] in the context of curve evolution in $\mathbb{R}^3$ and Mikula and Ševčovič [29–32] for the evolution of planar curves. The proof of the existence and uniqueness of solutions in Hölder spaces is based on the abstract theory of analytic semiflows in Banach spaces, as established by DaPrato and Grisvard [17], Angenent [1, 2] and Lunardi [26]. In the proof technique, the position vector equation (2) with uniform tangential velocity v_T is analyzed.

The nonlinear parabolic equation (2), and similarly (10), can be rewritten as the abstract parabolic equation: $\partial_t \mathbf{X} = \mathscr{F}(\mathbf{X}),\ \mathbf{X}(0) = \mathbf{X}_0$, on a scale of suitable Banach spaces. Suppose $0 < \varepsilon < 1$, and $k \in \mathbf{N}$. The little Hölder space $h^{k+\varepsilon}(S^1)$ is the Banach space defined as the closure of C^∞ smooth functions defined in the periodic domain S^1. The norm is the sum of the C^k norm and the ε-Hölder semi-norm of the k-th derivative. Next, we define the following scale of Banach spaces consisting of $(2k+\varepsilon)$-Hölder continuous functions in the periodic domain $I \simeq S^1$:

$$\mathcal{E}_k^X = h^{2k+\varepsilon}(S^1) \times h^{2k+\varepsilon}(S^1) \times h^{2k+\varepsilon}(S^1), \quad \mathcal{E}_k^Y = h^{2k+\varepsilon}(S^1) \times h^{2k+\varepsilon}(S^1), \quad k = 0, \frac{1}{2}, 1. \tag{12}$$

For the application of the theory of nonlinear analytic semiflovs due to DaPrato and Grisvard [17], Angenent [1, 2], and Lunardi [26], it is sufficient to prove that, for any $\tilde{\mathbf{X}}$ linearization $\mathscr{A} = \mathscr{F}'(\tilde{\mathbf{X}})$ generates an analytic semigroup in the space in $\mathcal{E}_0^Z$, and it belongs to the maximal regularity class between Banach spaces $\mathcal{E}_1^Z$ and $\mathcal{E}_0^Z$ for $Z \in \{X, Y\}$. Now we can state the following result, stating the local existence and uniqueness of solutions to the system of nonlinear geometric equations (2).

Theorem 3 *Assume $\mathcal{M} \in \mathbb{R}^3$ is an embedded manifold given by $\mathcal{M} = \{\mathbf{X} \in \mathbb{R}^3,\ f(\mathbf{X}) = 0\}$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^4 smooth regular map, i.e., $\nabla f(\mathbf{X}) \neq 0$ for $\mathbf{X} \in \mathcal{M}$, and $a = a(\mathbf{X}, \mathbf{T}) > 0$ is a C^4 smooth function, $a : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. Assume that $h : \mathbb{R} \to \mathbb{R}$ is a C^4 smooth function, $h(0) = 0$. Suppose that the parameterization $\mathbf{X}_0$ of the initial curve Γ_0 belongs to the Hölder space $\mathcal{E}_1^X$, and is uniformly parameterized $|\partial_u \mathbf{X}_0(u)| = L(\Gamma_0) > 0$ for all $u \in I$. Assume that the total tangential velocity v_T preserves the relative local length. Then there exists $T > 0$ and the unique solution $\mathbf{X}$ to the initial value problem:*

$$\partial_t \mathbf{X} = a\partial_s^2 \mathbf{X} + \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}) + \alpha \partial_s \mathbf{X}, \quad \mathbf{X}(\cdot, 0) = \mathbf{X}_0(\cdot),\ u \in I,\ t \in [0, T),$$

where the external force $\mathbf{F}$ is given by (5). Moreover, $\mathbf{X} \in C([0, T], \mathcal{E}_1^X) \cap C^1([0, T], \mathcal{E}_0^X)$.

Proof We rewrite the non-linear parabolic equation (2) in the form: $\partial_t \mathbf{X} = \mathscr{F}(\mathbf{X})$, where $\mathscr{F}(\mathbf{X}) = a\partial_s^2 \mathbf{X} + \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}) + \alpha \partial_s \mathbf{X}$. Under the assumptions made on functions a, f, and h, the mapping

$$\mathcal{E}_{\frac{1}{2}}^X \ni \mathbf{X} \mapsto \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}) + \alpha \partial_s \mathbf{X} \in \mathcal{E}_0^X$$

is C^1 mapping from the Banach space $\mathcal{E}_{\frac{1}{2}}^X$ to $\mathcal{E}_0^X$.

Assume $\tilde{\mathbf{X}}$ belongs to the Hölder space $\mathcal{E}_1^X$, and is uniformly parameterized, $|\partial_u \tilde{\mathbf{X}}| = L(\tilde{\Gamma}) > 0$ for all $u \in I$. Then the linearization $\mathscr{A} = \mathscr{F}'(\tilde{\mathbf{X}})$ can be decomposed as follows: $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$, where the principal part $\mathscr{A}_0$ containing the second order derivative has the form $\mathscr{A}_0 \mathbf{X} = \frac{\tilde{a}}{L(\tilde{\Gamma})^2} \partial_u^2 \mathbf{X}$, where $\tilde{a} = a(\tilde{\mathbf{X}}, \tilde{\mathbf{T}})$, $\tilde{\mathbf{T}} = \partial_s \tilde{\mathbf{X}}$. It is known that $\mathscr{A}_0$ generates an analytic semigroup in space $\mathcal{E}_0^X$, and it belongs to the maximal regularity class between Banach spaces $\mathcal{E}_1^X$ and $\mathcal{E}_0^X$ (cf. [1, 2]). Furthermore, the operator $\mathscr{A}_1 = \mathscr{A} - \mathscr{A}_0$ contains the first order derivative of $\mathbf{X}$. It is a bounded linear operator $\mathscr{A}_1 : \mathscr{E}_{\frac{1}{2}} \to \mathscr{E}_0$. Therefore, the operator $\mathscr{A}_1$ considered as a mapping from $\mathscr{E}_1 \to \mathscr{E}_0$ has the relative zero norm with respect to $\mathscr{A}_0$. Therefore, linearization $\mathscr{A}$ belongs to the maximal regularity class $\mathcal{M}(\mathscr{E}_1, \mathscr{E}_0)$ because this class is closed with respect to perturbation with the relative zero norm (cf. [2, Lemma 2.5], DaPrato and Grisvard [17], Lunardi [26]). The proof now follows [2, Theorem 2.7] due to Angenent. □

Theorem 4 *Assume $\mathcal{M}$ is a manifold immersed in $\mathbb{R}^3$ given by $\mathcal{M} = \{\mathbf{X} = \mathcal{X}(\mathbf{Y}),\ \mathbf{Y} \in I \times I\}$ where $\mathcal{X} : I \times I \to \mathbb{R}^3$, is a C^4 smooth immersion, $rank(\nabla\mathcal{X}(\mathbf{Y})) = 2$ for all $\mathbf{Y} \in I \times I$, and $a = a(\mathbf{X}, \mathbf{T}) > 0$ is a C^4 smooth function, $a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. Suppose that the initial condition $\mathbf{Y}_0$ belongs to the Hölder space $\mathcal{E}_1^Y$, and $|\partial_u \mathbf{Y}_0(u)| > 0$ for all $u \in I$. Then there exists $T > 0$ and the unique solution $\mathbf{Y} \in C([0, T], \mathcal{E}_1^Y) \cap C^1([0, T], \mathcal{E}_0^Y)$ to the initial value problem:*

$$\partial_t \mathbf{Y} = a\partial_s^2 \mathbf{Y} + \mathbf{G}(\mathbf{Y}, \partial_s \mathbf{Y}) + \alpha\partial_s \mathbf{Y}, \quad \mathbf{Y}(\cdot, 0) = \mathbf{Y}_0(\cdot),\ u \in I,\ t \in [0, T),$$

where $ds = |\nabla\mathcal{X}(\mathbf{Y})^\intercal \partial_u \mathbf{Y}|du$, $a = a(\mathcal{X}(\mathbf{Y}), \nabla\mathcal{X}(\mathbf{Y})^\intercal \partial_s \mathbf{Y})$ and $\mathbf{G} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is a C^4 smooth function.

Proof Based on the assumptions made on the immersion mapping $\mathcal{X} : I \times I \to \mathbb{R}^3$, we have the following:

$$\mathcal{E}_{\frac{1}{2}}^Y \ni \mathbf{Y} \mapsto \mathbf{G}(\mathbf{Y}, \partial_s \mathbf{Y}) + \alpha\partial_s \mathbf{Y} \in \mathcal{E}_0^Y$$

is a C^1 mapping from the Banach space $\mathcal{E}_{\frac{1}{2}}^Y$ to $\mathcal{E}_0^Y$. The rest of the proof is essentially the same as that of Theorem 3. □

In general, we cannot take $T = +\infty$. Indeed, a closed curve on a torus or plane surface may shrink to a point in finite time.

5 Flowing finite volumes numerical discretization scheme

In this section, we present a numerical discretization scheme for solving the system of equations (2) with tangential velocity α. The discretization utilizes the method of lines with spatial discretization achieved through the finite-volume method. We focus on the evolution of the curves Γ_t, $t \geq 0$, satisfying the governing equation:

$$\partial_t \mathbf{X} = a\partial_s^2 \mathbf{X} + \mathbf{F}(\mathbf{X}, \partial_s \mathbf{X}) + \alpha\mathbf{T}. \tag{13}$$

We consider M discrete nodes $\mathbf{X}_k = \mathbf{X}(u_k)$, $k = 0, 1, 2, \ldots M$, $u_k = kh \in [0, 1]$, where $h = 1/M$, $\mathbf{X}_0 = \mathbf{X}_M$ along the curve Γ_t. The dual nodes are defined as $\mathbf{X}_{k\pm\frac{1}{2}} = \mathbf{X}(u_k \pm h/2)$ (see Fig. 3) and $(\mathbf{X}_k + \mathbf{X}_{k+1})/2$ is the midpoint of the line segment connecting nodes $\mathbf{X}_k$ and $\mathbf{X}_{k+1}$. This midpoint differs from $\mathbf{X}_{k\pm\frac{1}{2}} \in \Gamma_t$. The

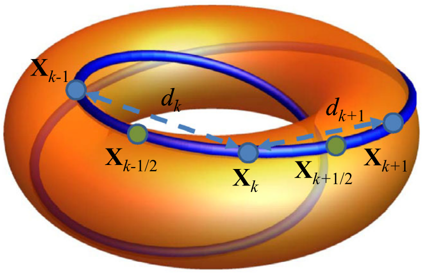


Fig. 3 Discretization of a segment of a 3D curve on a surface by the method of flowing finite volumes

k-th segment $\mathcal{S}_k$ of Γ_t between the nodes $\mathbf{X}_{k+\frac{1}{2}}$ and $\mathbf{X}_{k-\frac{1}{2}}$ represents the finite volume. Integration of equation (13) over such a volume yields

$$\int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \partial_t \mathbf{X}|\partial_u \mathbf{X}|du = \int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} a\frac{\partial}{\partial u}\left(\frac{\partial_u \mathbf{X}}{|\partial_u \mathbf{X}|}\right)du + \int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \mathbf{F}|\partial_u \mathbf{X}|du + \int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \alpha\partial_u \mathbf{X}du. \tag{14}$$

To discretize the governing system of equations, we assume that $\partial_t\mathbf{X}$, $\partial_u\mathbf{X}$, $\mathbf{F}$, α, κ, a, b, $\mathbf{T}$, and $\mathbf{N}$ remain constant on the finite volume $\mathcal{S}_k$ bounded by the nodes $\mathbf{X}_{k-\frac{1}{2}}$ and $\mathbf{X}_{k+\frac{1}{2}}$. These variables assume the values $\partial_t\mathbf{X}_k$, $\partial_u\mathbf{X}_k$, $\mathbf{F}_k$, α_k, κ_k, $\mathbf{T}_k$, and $\mathbf{N}_k$, respectively. In the approximation of the nonlocal vector function $\mathbf{F}_k$, the curve Γ_t used to define $\mathbf{F}$ is replaced by a polygonal curve having vertices at $(\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_M)$. The numerical approximation of the tangential velocity α_k is summarized at the end of this section.

Let us denote $d_k = |\mathbf{X}_k - \mathbf{X}_{k-1}|$ for $k = 1, 2, \ldots, M, M+1$, where $\mathbf{X}_M = \mathbf{X}_0$ and $\mathbf{X}_{M+1} = \mathbf{X}_1$ for the closed curve Γ and we approximate the integral expressions in (14) by means of the flowing finite volume method as follows:

$$\int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \partial_t \mathbf{X}|\partial_u \mathbf{X}|du \approx \frac{d\mathbf{X}_k}{dt}\frac{d_{k+1}+d_k}{2},$$

$$\int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} a\partial_u\frac{\partial_u \mathbf{X}}{|\partial_u \mathbf{X}|}du \approx a_k\left(\frac{\mathbf{X}_{k+1}-\mathbf{X}_k}{d_{k+1}} - \frac{\mathbf{X}_k-\mathbf{X}_{k-1}}{d_k}\right),$$

$$\int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \mathbf{F}|\partial_u \mathbf{X}|du \approx \mathbf{F}_k\frac{d_{k+1}+d_k}{2}, \qquad \int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \alpha\partial_u \mathbf{X}du \approx \alpha_k\frac{\mathbf{X}_{k+1}-\mathbf{X}_{k-1}}{2}.$$

Here $a_k = a(\mathbf{X}_k, \mathbf{T}_k)$ is the diffusion coefficient evaluated at $(\mathbf{X}_k, \mathbf{T}_k)$. The estimation of the nonnegative curvature κ along with the tangent vector $\mathbf{T}$ and the normal vector $\mathbf{N}$, where $\kappa\mathbf{N} = \partial_s\mathbf{T}$, can be expressed as follows:

$$\mathbf{N}_k \approx \frac{1}{\delta+\kappa_k}\frac{2}{d_k+d_{k+1}}\left(\frac{\mathbf{X}_{k+1}-\mathbf{X}_k}{d_{k+1}} - \frac{\mathbf{X}_k-\mathbf{X}_{k-1}}{d_k}\right), \qquad \mathbf{T}_k \approx \frac{\mathbf{X}_{k+1}-\mathbf{X}_{k-1}}{d_{k+1}+d_k},$$

$$\kappa_k \approx \left| \frac{2}{d_k + d_{k+1}} \left(\frac{\mathbf{X}_{k+1} - \mathbf{X}_k}{d_{k+1}} - \frac{\mathbf{X}_k - \mathbf{X}_{k-1}}{d_k} \right) \right|,$$

where $0 < \delta \ll 1$ is a small regularization parameter.

The semidiscrete scheme for resolving (13) is expressed as follows.

$$\frac{d\mathbf{X}_k}{dt} = a_k \frac{2}{d_{k+1} + d_k} \left(\frac{\mathbf{X}_{k+1} - \mathbf{X}_k}{d_{k+1}} - \frac{\mathbf{X}_k - \mathbf{X}_{k-1}}{d_k} \right) + \mathbf{F}_k + \alpha_k \frac{\mathbf{X}_{k+1} - \mathbf{X}_{k-1}}{d_{k+1} + d_k},$$
$$\mathbf{X}_k(0) = \mathbf{X}_{ini}(u_k), \quad \text{for } k = 1, \dots, M. \tag{15}$$

Recall that the tangential component α of the velocity vector for evolving closed curves $\{\Gamma_t, t \geq 0\}$ maintains their shape unchanged (cf. Epstein [14]). However, for a numerical solution of (13), the careful selection of the tangential velocity functional α is key to preserve the stability of the computational procedure (cf. Mikula and Ševčovič [29–31]). The significance of tangential velocity is also substantial in theoretical analyzes of curve evolution, as shown by the works of Hou et al. [20], Kimura [22].

Barrett et al.[3, 4] and Elliott and Fritz [27], explored gradient and elastic flows for closed and open curves in $\mathbb{R}^d$, where $d \geq 2$, and formulated a numerical approximation method to effectively redistribute the tangential component. Furthermore, the relevance of tangential velocity is recognized in material science investigations by Beneš, Kolář, and Ševčovič [5] and in the context of interactive evolving curves [6]. In a different field, Garcke, Kohsaka, and Ševčovič [15] applied uniform tangential redistribution to theoretically confirm the nonlinear stability of curvature-induced flows with triple junctions in planes. Remešíková et al. [28] analyzed the tangential redistribution effects for the flows of closed manifolds in $\mathbb{R}^n$.

Calculating the time change of the ratio of the relative local length $|\partial_u \mathbf{X}(u,t)|$ and the total curve length $L(\Gamma_t) = \int_0^1 |\partial_u \mathbf{X}(u,t)| du$, it is possible to derive an equation for the unknown α (cf. [37]):

$$\frac{\partial}{\partial t} \frac{|\partial_u \mathbf{X}(u,t)|}{L(\Gamma_t)} = \frac{|\partial_u \mathbf{X}(u,t)|}{L(\Gamma_t)} \left(\partial_s \alpha - \kappa v_N + \langle \kappa v_N \rangle \right),$$
$$\text{where } \langle \kappa v_N \rangle = \frac{1}{L(\Gamma_t)} \int_{\Gamma_t} \kappa v_N ds. \tag{16}$$

The meaning of $\langle \cdot \rangle$ is the average value of a scalar quantity along the curve Γ_t. Therefore, the ratio $|\partial_u \mathbf{X}(u,t)|/L(\Gamma_t)$ is constant with respect to the time t, i.e.

$$\frac{|\partial_u \mathbf{X}(u,t)|}{L(\Gamma_t)} = \frac{|\partial_u \mathbf{X}(u,0)|}{L(\Gamma_0)}, \quad \text{for any } t \geq 0, \tag{17}$$

provided that the tangential velocity satisfies $\partial_s \alpha = \kappa v_N - \langle \kappa v_N \rangle$. Another suitable choice of the tangential velocity α is the so-called asymptotically uniform tangential velocity proposed and analyzed by Mikula and Ševčovič in [30, 31].

Taking into account a positive damping parameter ω and tangential velocity α as the solution of the following equation

$$\partial_s \alpha = \kappa v_N - \langle \kappa v_N \rangle + \left(\frac{L(\Gamma_t)}{|\partial_u \mathbf{X}(u,t)|} - 1 \right) \omega, \tag{18}$$

then, using (16) we obtain $\lim_{t\to\infty} \frac{|\partial_u \mathbf{X}(u,t)|}{L(\Gamma_t)} = 1$ uniformly with respect to $u \in [0,1]$ provided that $\omega > 0$ is a positive constant. This means that the redistribution becomes asymptotically uniform. The numerical approximation of the tangential velocity (18) follows from [37]. It requires discrete values of curvature κ_k, normal velocity $v_{N,k}$, and segment lengths d_k, and a straightforward trapezoidal integration is used (cf. [37]). The values $\alpha_0 = \alpha_M$ are chosen so that $\sum_{i=1}^{M} \alpha_i (d_{i+1} + d_i)/2 = 0$. Then the values α_k for $k = 0, 1, \ldots, M$ are uniquely given and the direct integration of (18) leads to the following formulae

$$\alpha_i = \alpha_1 + \sum_{k=2}^{i} \left[\kappa_i v_{N,i} d_i - \langle \kappa v_N \rangle d_i + \left(\frac{L(\Gamma_t)}{M} - d_i \right) \omega \right], \quad i = 2, \ldots, M,$$
$$\alpha_1 = -\frac{1}{\sum_{i=1}^{M} \frac{(d_{i+1}+d_i)}{2}} \left\{ \sum_{i=2}^{M} \frac{(d_{i+1}+d_i)}{2} \left(\sum_{k=2}^{i} \left[\kappa_i v_{N,i} d_i - \langle \kappa v_N \rangle d_i + \left(\frac{L(\Gamma_t)}{M} - d_i \right) \omega \right] \right) \right\}. \tag{19}$$

Remark 3 We present a finite difference numerical approximation of the parabolic equation (10) that governs the flow on an immersed manifold. The second derivative $\partial_s^2 \mathbf{Y}_k$ appearing in (10) is approximated as follows:

$$\frac{|\mathbf{Y}_{k+1} - \mathbf{Y}_k| + |\mathbf{Y}_k - \mathbf{Y}_{k-1}|}{2} \partial_s^2 \mathbf{Y}_k \approx \int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \frac{1}{|\nabla \mathcal{X}(\mathbf{Y})^\intercal \mathbf{t}(\mathbf{Y})|} \frac{\partial}{\partial u} \left(\frac{1}{|\nabla \mathcal{X}(\mathbf{Y})^\intercal \mathbf{t}(\mathbf{Y})|} \frac{\partial_u \mathbf{Y}}{|\partial_u \mathbf{Y}|} \right) du$$
$$\approx \frac{1}{|\nabla \mathcal{X}(\mathbf{Y}_k)^\intercal \mathbf{t}(\mathbf{Y}_k)|} \left(\frac{1}{|\nabla \mathcal{X}(\mathbf{Y}_{k+1})^\intercal \mathbf{t}(\mathbf{Y}_{k+1})|} \frac{\mathbf{Y}_{k+1} - \mathbf{Y}_k}{|\mathbf{Y}_{k+1} - \mathbf{Y}_k|} - \frac{1}{|\nabla \mathcal{X}(\mathbf{Y}_k)^\intercal \mathbf{t}(\mathbf{Y}_k)|} \frac{\mathbf{Y}_k - \mathbf{Y}_{k-1}}{|\mathbf{Y}_k - \mathbf{Y}_{k-1}|} \right)$$

where $\mathbf{t}(\mathbf{Y}_k) \approx (\mathbf{Y}_k - \mathbf{Y}_{k-1})/|\mathbf{Y}_k - \mathbf{Y}_{k-1}|$. Similarly, the first derivatives $\partial_s \mathbf{Y}$ and $\partial_t \mathbf{Y}$ in (10) are approximated by means of finite differences as follows:

$$\partial_s \mathbf{Y}_k = \frac{1}{|\nabla \mathcal{X}(\mathbf{Y}_k)^\intercal \mathbf{t}(\mathbf{Y}_k)|} \partial_r \mathbf{Y}_k \approx \frac{1}{|\nabla \mathcal{X}(\mathbf{Y}_k)^\intercal \mathbf{t}(\mathbf{Y}_k)|} \frac{\mathbf{Y}_k - \mathbf{Y}_{k-1}}{|\mathbf{Y}_k - \mathbf{Y}_{k-1}|},$$

$$\int_{u_{k-\frac{1}{2}}}^{u_{k+\frac{1}{2}}} \partial_t \mathbf{Y} |\partial_u \mathbf{Y}| du \approx \frac{d\mathbf{Y}_k}{dt} \frac{d_{k+1} + d_k}{2}, \quad d_k = |\mathbf{X}_k - \mathbf{X}_{k-1}|.$$

We approximate the partial derivatives $\frac{\partial \mathcal{X}_i}{\partial Y_j}$, $i = 1, 2, 3$, $j = 1, 2$, by means of finite diferences. The 2×2 matrix $\nabla \mathcal{X}(\mathbf{Y}) \nabla \mathcal{X}(\mathbf{Y})^\intercal$ can be explicitly inverted. Notice that this matrix is regular because $\mathcal{X}$ is assumed to be an immersion.

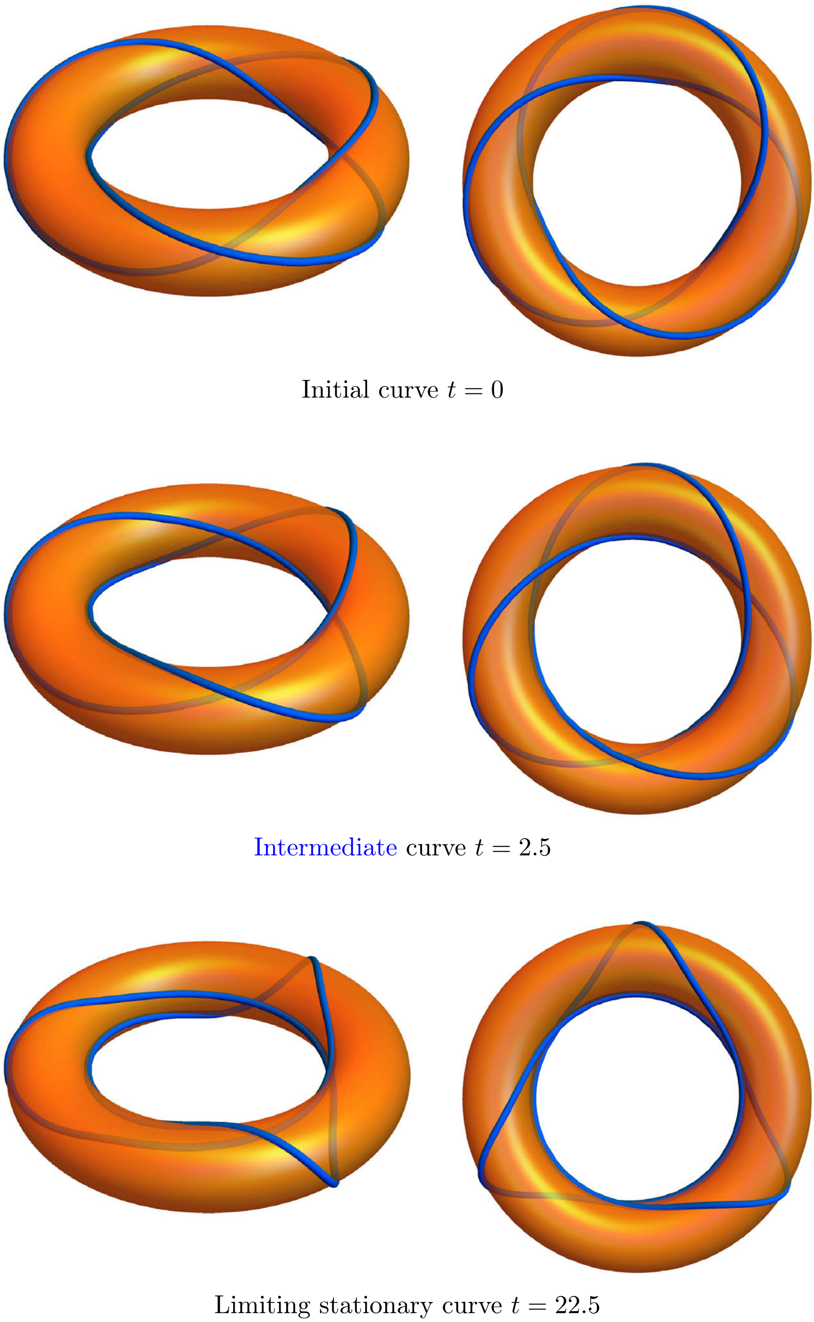
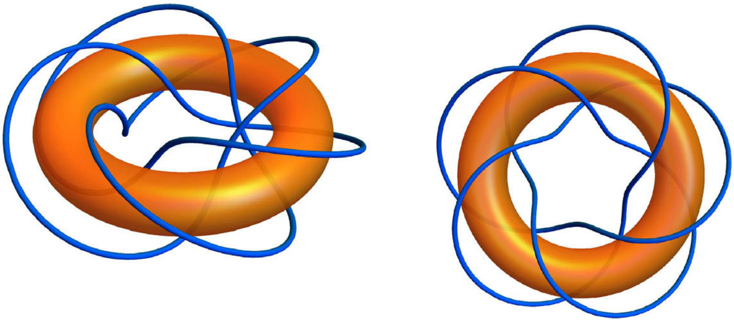
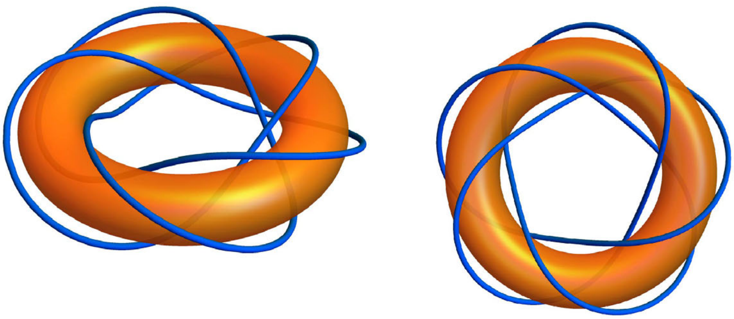


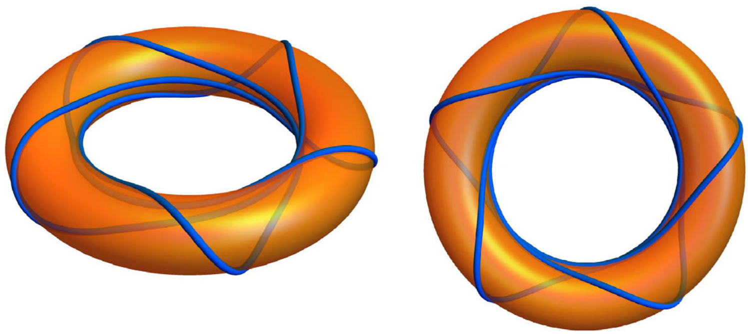
Fig. 4 Time evolution of a knotted Fourier curve belonging to the orientable torus surface with parameters $r = 1$, $R = 4$, and $k = 2$, $l = 3$ (see 25). Initial curve $t = 0$ Intermediate curve $t = 2.5$ Limiting stationary curve $t = 22.5$



Initial curve $t = 0$

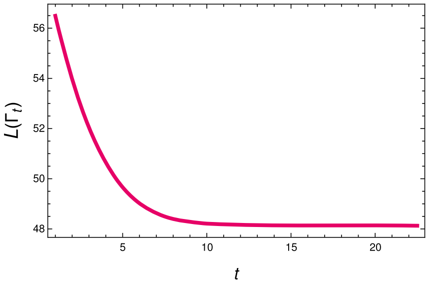


Intermediate curve $t = 1$



Limiting stationary curve $t = 19.75$

Fig. 5 Time evolution of initially inflated knotted curve belonging to the orientable torus surface with parameters $r = 1$, $R = 4$. Topologically more complex case corresponding to the choice $k = 3$, $l = 5$ in initial condition (25)



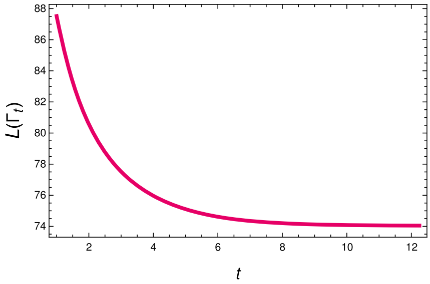


Fig. 6 Lengths $L(\Gamma_t)$ of shrinking curves on the torus (20) - (**a**) and attaching to the torus (20) - (**b**)

6 Examples

In our examples, we consider the evolution of an initial Fourier curve parameterized by a finite trigonometric series in the parameter $u \in I$. Here we remind the reader that I is identified with the unit circle and $I = \mathbb{R}/\mathbb{Z} \simeq S^1$. In all of our numerical experiments, we use $M = 200$ discretization nodes, uniform tangential redistribution, and the regularization parameter δ in the discrete approximation of curvature was set to $\delta = 10^{-5}$. The resulting system of ODEs (15) is numerically solved using the fourth-order explicit Runge–Kutta-Merson method, incorporating automatic time step control with a tolerance level of 10^{-3} (refer to [35]). The initial time step was selected as $4h^2$, where $h = 1/M$ denotes the spatial mesh size.

6.1 Evolution of knotted curves on torus

As an example of a torus, we can consider immersion $\mathcal{X} : I \times I \to \mathbb{R}^3$ defined as:

$$\mathcal{X}(u, v) = ((r\cos(2\pi v) + R)\sin(2\pi u),\ (r\cos(2\pi v) + R)\cos(2\pi u),\ r\sin(2\pi v))^{\mathsf{T}}, \quad (20)$$

where $0 < r < R$ and $u, v \in I$. The torus surface can also be defined as an embedded manifold $\mathcal{M} = \{\mathbf{X} = (X_1, X_2, X_3)^{\mathsf{T}} \in \mathbb{R}^3,\ f(\mathbf{X}) = 0\}$ where

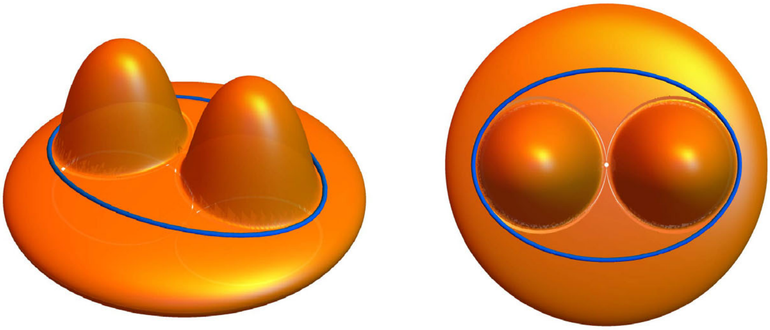
$$f(\mathbf{X}) = ((X_1^2 + X_2^2)^{\frac{1}{2}} - R)^2 + X_3^2 - r^2.$$

Its gradient $\nabla f(\mathbf{X})$ and the Hesse matrix $\nabla^2 f(\mathbf{X})$ are given by

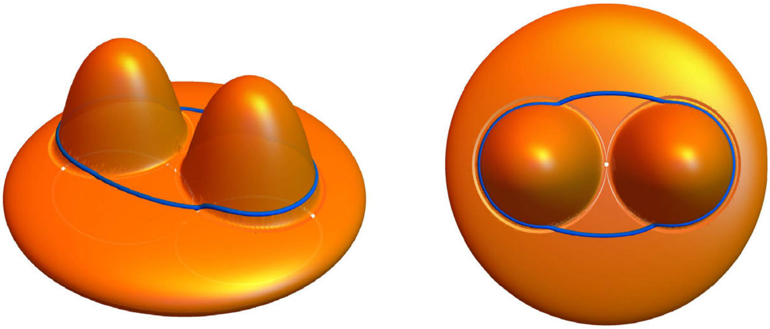
$$\nabla f(\mathbf{X}) = 2\mathbf{X} - 2R\left(X_1/(X_1^2 + X_2^2)^{\frac{1}{2}},\ X_2/(X_1^2 + X_2^2)^{\frac{1}{2}},\ 0\right)^{\mathsf{T}}, \quad (21)$$

$$|\nabla f(\mathbf{X})| = 2\left(X_1^2 + X_2^2 + X_3^2 - 2R(X_1^2 + X_2^2)^{\frac{1}{2}} + R^2\right)^{\frac{1}{2}}, \quad (22)$$

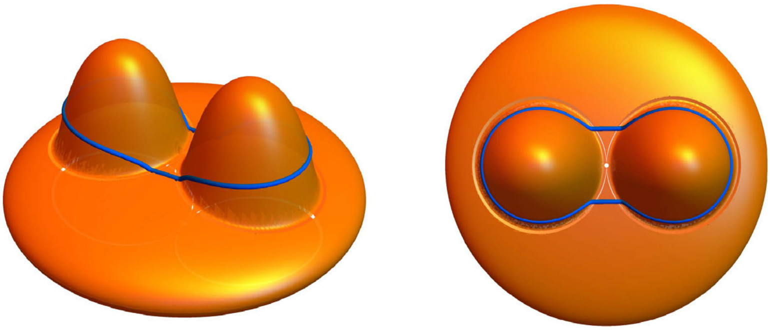
$$\nabla^2 f(\mathbf{X}) = 2\mathbf{I} - 2\frac{R}{(X_1^2 + X_2^2)^{\frac{3}{2}}}(X_2, -X_1, 0)(X_2, -X_1, 0)^{\mathsf{T}}, \quad (23)$$



Initial curve $t = 0$



Intermediate curve $t = 4.5$



Limiting stationary curve $t = 13.5$

Fig. 7 Evolution of a simple curve belonging to the orientable surface of genus 0 with humps

$$\mathbf{T}^\intercal \nabla^2 f(\mathbf{X})\mathbf{T} = 2 - 2\frac{R}{(X_1^2+X_2^2)^{\frac{3}{2}}}(T_1 X_2 - T_2 X_1)^2, \tag{24}$$

where the unit tangent vector $\mathbf{T} = (T_1, T_2, T_3)^\intercal$. The torus surface is shown in Fig. 1. The initial curve $\mathbf{X}_0$ derived from the mapping (20) is parameterized by

$$\mathbf{X}_0(u) = \mathcal{X}(ku, lu), \quad u \in I, \tag{25}$$

where $k = 2, l = 3, r = 1$ and $R = 4$. Its time evolution is shown in Fig. 4. The curve shrinks and converges to the stationary state with constant length as suggested in Fig. 6 a).

6.2 Attraction of curves by a torus surface

In this part, we present an example of the evolution of initial closed curves belonging to a small neighborhood of the given surface $\mathcal{M}$.

The following computational example demonstrates how an initially knotted curve evolves according to the geometric evolution equation (2) driven by the force term (5). The reference surface is the torus given by immersion (20) with $r = 1$, $R = 4$. The initial curve $\mathbf{X}_0$ is parameterized by mapping (20) as

$$\mathbf{X}_0(u) = \mathcal{X}(ku, lu), \quad u \in I,$$

where $k = 3, l = 5, r = 2$ and $R = 4$. The time evolution of such an inflated initial curve is shown in Fig. 5. The curve continues to shrink until it attaches to the torus surface and eventually finds its stationary state with constant length, as suggested in Fig. 6b.

6.3 Evolution of simple curves on surface with genus 0 with humps

The last example is shown in Fig. 7. $\mathcal{M} = \{\mathbf{X} = (X_1, X_2, X_3)^\intercal \in \mathbb{R}^3, \; f(\mathbf{X}) = 0\}$ where

$$f(\mathbf{X}) = X_1^2 + X_2^2 + c^2(X_3 - \phi(X_1, X_2))^2 - r^2,$$
$$\phi(X_1, X_2) = h(X_1 - 1, X_2) + h(X_1 + 1, X_2).$$

Here h is a smooth bump function, $h(X_1, X_2) = v\, 2^{-1/(1-X_1^2-X_2^2)}$ for $X_1^2 + X_2^2 < 1$, and $h(X_1, X_2) = 0$, otherwise. In the example shown in Fig. 7 we set $r = 2.5$, $c = 4$, and $v = 3$. The initial curve is an ellipse projected onto the surface, i.e.

$$\Gamma_0 = \{\mathbf{X} = (X_1, X_2, X_3), \quad X_1^2/2 + X_2^2 = 2, \quad X_3 = c^{-1}(r^2 - X_1^2 - X_2^2)^{\frac{1}{2}} + \phi(X_1, X_2)\}.$$

7 Conclusion

In this paper, we investigated the curvature-driven flow of a family of closed curves evolving on an embedded or immersed manifold in the three-dimensional Euclidean space. We analyzed the qualitative behavior of this flow. Using the abstract theory of analytic semi-flows in Banach spaces, we prove the local existence and uniqueness of Hölder smooth solutions to the governing system of evolution equations for the curve parametrization. We demonstrate the behavior of sulutions in several computational examples constructed by means of the flowing finite-volume method and asymptotically uniform tangential redistribution of discretization points.

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Data Availability All data in this paper are available from the author on a reasonable request.

Declarations

Conflict of interest The authors declare that there are no conflict of interest with respect to the publication of this paper.

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