EVOLUTION OF PLANE CURVES DRIVEN BY A NONLINEAR FUNCTION OF CURVATURE AND ANISOTROPY

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Abstract. We study the intrinsic heat equation governing the motion of plane curves. The normal velocity v of the motion is assumed to be a nonlinear function of the curvature and tangential angle of a plane curve Γ. By contrast to the usual approach, the intrinsic heat equation is modified to include an appropriate nontrivial tangential velocity functional α. Short time existence of a regular family of evolving curves is shown in the case when $v = \gamma(\nu)|k|^{m-1}k$, $0 < m \le 2$ and the governing system of equations includes a nontrivial tangential velocity functional.

We study the evolution of a closed smooth plane curve $\Gamma : S^1 \to \mathbb{R}^2$ with the normal velocity speed v depending on the curvature k and the tangential angle ν, i.e. $v = \beta(k,\nu)$. As a motivation one can consider e.g. the multiphase thermomechanics where the plane curve evolution satisfying $v = \beta(k,\nu)$ is an appropriate model for describing the motion of phase interfaces (see [AG]). Another application arises from the image processing where the affine invariant scale with $v = k^{1/3}$ has special conceptual and practical importance (see [ST], [AST]).

In our approach a family of evolving curves $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$, is represented by the position vector $x : Q_T = S^1 \times (0,T) \to R^2$ satisfying the intrinsic heat equation

$$\partial_t x = \theta_1^{-1}\partial_s\left(\theta_2^{-1}\partial_s x\right), \quad x(.,0) = x^0(.) \tag{1}$$

where s is the arc-length parameter and θ_1, θ_2 are geometric quantities, i.e. functions whose definition is independent of particular parameterization of Γ^t. By using Frenet's formulae, the intrinsic heat equation can be rewritten as

$$\partial_t x = \beta\vec{N} + \alpha\vec{T}, \quad \text{where } \theta_1\theta_2 = k/\beta(k,\nu) \text{ and } \alpha = \theta_1^{-1}\partial_s\left(\theta_2^{-1}\right) . \tag{2}$$

Given a function β, the only constraint imposed on θ_1, θ_2 is the condition $\theta_1\theta_2 = k/\beta$. This gives raise to various choices of θ_1, θ_2 and subsequently to various tangential velocities α. It is well known (cf. [AST]) that the tangential velocity functional α does not change the shape of evolving curves. On the other hand, the presence of a suitable tangential velocity is very important in order to suggest a powerful numerical scheme for solving the geometric equation $v = \beta(k,\nu)$ (cf. [MS1], [MS2]). The choice of a trivial $\alpha = 0$ may lead to computational instabilities caused by merging of numerical grid points representing a discrete curve or by formation of the so-called swallow tails.

If we denote $g = |\partial_u x|$ then the intrinsic heat equation can be rewritten in terms k, ν and g as follows

$$\begin{aligned}
&\partial_t k = g^{-1}\partial_u\left(g^{-1}\partial_u\beta(k,\nu)\right) + \alpha g^{-1}\partial_u k + k^2\beta(k,\nu) \\
&\partial_t \nu = \beta_k'(k,\nu)g^{-1}\partial_u\left(g^{-1}\partial_u\nu\right) + k(\alpha + \beta_\nu'(k,\nu)) \qquad (u,t) \in Q_T \\
&\partial_t g = -gk\beta(k,\nu) + \partial_u\alpha
\end{aligned} \tag{3}$$

(cf. [MS2]). A solution of (3) is subject to the initial conditions $k(.,0) = k^0$, $\nu(.,0) = \nu^0$, $g(.,0) = g^0$ corresponding to the initial curve $\Gamma^0 = \mathrm{Image}(x^0)$. Notice that $\partial_u\nu^0 = g^0k^0$.

In this paper we propose a special choice of the tangential velocity functional α such that that the ratio of the local length element $g = |\partial_u x|$ to the the total length $|\Gamma^t|$ is constant with respect to time, i.e. $\partial_t(g/|\Gamma^t|) = 0$. Combining the third equation in (3) with the equation for the total length $\frac{d}{dt}|\Gamma^t| + \int_{\Gamma^t} k\beta(k,\nu)ds = 0$ it turns out that $\partial_t(g/|\Gamma^t|) = 0$ iff α is a solution of the nonlocal equation

$$\partial_s\alpha = k\beta(k,\nu) - \frac{1}{|\Gamma|}\int_\Gamma k\beta(k,\nu)\ .ds \tag{4}$$

Notice that there is a unique α satisfying (4) up to an additive constant which can be determined from the normalization condition $\theta_2(0) = 1$ (see (2)). This choice of α leads to a powerful numerical scheme having the property of uniform in time redistribution of grid points and preventing the computed numerical solution from forming the above mentioned numerical instabilities (see [MS2]).

Let us denote by E_0, E_1 the following Banach spaces

$$E_0 = c^\sigma(S^1) \times c^\sigma(S^1) \times c^{1+\sigma}(S^1), \quad E_1 = c^{2+\sigma}(S^1) \times c^{2+\sigma}(S^1) \times c^{1+\sigma}(S^1) \quad 0 < \sigma < 1 \tag{5}$$

where $c^{k+\sigma}, k = 0, 1, 2$, is the little Hölder space, i.e., the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{k+\sigma}(S^1)$ (see [A1]). If $\beta = \beta(k, \nu)$ is a C^2 smooth function such that

$$0 < \lambda_- \le \beta'_k(k, \nu) \le \lambda_+ < \infty, \quad \text{for any } k, \nu \tag{6}$$

where $\lambda_\pm > 0$ are constants then by using the abstract theory due to Angenent (cf. [A1], [A2]) we can prove the local existence and uniqueness of solutions of (3).

Theorem 1. *([MS2, Th. 4.1]) Assume $\Gamma^0 = \text{Image}(x^0)$ is such that $(k^0, \nu^0, g^0) \in E_1$ and $g^0 = |\partial_u x^0| > 0$. If $\beta = \beta(k, \nu)$ is a C^2 smooth function satisfying (6) and α is the normalized solution of (4) then there exists a unique classical solution $\Phi = (k, \nu, g) \in C([0, T], E_1) \cap C^1([0, T], E_0)$ of the governing system of Eqs. (3) defined on some small time interval $[0, T]$. Moreover, if Φ is a maximal solution defined on $[0, T_{max})$ and $T_{max} < \infty$ then $\max_{\Gamma^t} |k(., t)| \to \infty$ as $t \to T_{max}$.*

This result can not be however applied to the singular case when $\beta(k, \nu) = \gamma(\nu)|k|^{m-1}k, \; m > 0$. Here $\gamma : \mathbb{R} \to \mathbb{R}^+$ is a given C^∞ smooth anisotropy function satisfying $0 < C_1^{-1} \le \gamma(\nu) \le C_1$ $|\gamma'_\nu(\nu)| \le C_1$ for any $\nu \in \mathbb{R}$. To make use of the result established in Theorem 1 we must go through a regularization argument. A similar technique was applied in [AST] for the case of the so-called affine invariant scale $v = k^{1/3}$. We slightly modify their approach for more general anisotropic power like function $\beta(k, \nu)$ including both fast $(0 < m < 1)$ and slow $(1 < m)$ diffusion case and for the case when the system of governing equations (3) involves a nontrivial tangential velocity term α given by (4). The main idea is to regularize β by some β^ε satisfying (6) and then to provide necessary a-priori estimates which are independent of the regularization parameter ε. Similarly as in [AST] the key step is to find L^∞ estimate for the gradient of β^ε. This can be done by following the Nash-Moser iterative technique for estimating $X_p(t) = \int_{\Gamma^t} |\partial_s \beta^\varepsilon|^p ds$ where $p = 2^k$ and k tends to ∞. Now suppose that either $0 < m \le 1$ or $1 < m \le 2$ and the initial curve Γ^0 satisfies the structural condition

$$\int_{\Gamma^0} \frac{k^0}{\beta(k^0, \nu^0)} \, ds < \infty \; . \tag{7}$$

Then one can show that there is a constant $M > 0$ such that $\max_{\Gamma^t} |\partial_s \beta^\varepsilon| \le M t^{-\frac{3}{4}}$ for any $0 < \varepsilon \ll 1$ and $0 < t \le T$ (cf [MS2, Lemma 5.4]). Having this bound on the gradient of β^ε it can be shown by letting $\varepsilon \to 0^+$ that the geometric equation (2) has a regular solution.

Theorem 2. *([MS2, Th. 6.3]) Suppose that $\beta(k, \nu) = \gamma(\nu)|k|^{m-1}k$ where $0 < m \le 2$. Let $\Gamma^0 = \text{Image}(x^0)$ be a smooth regular plane curve as in Theorem 1. If $1 < m \le 2$ we also suppose that Γ^0 satisfies the condition (7). Then there exists $T > 0$ and a family of regular plane curves $\Gamma^t = \text{Image}(x(., t)), t \in [0, T]$, such that*

a) $x, \partial_u x \in (C(\overline{Q_T}))^2, \;\; \partial_u^2 x, \; \partial_t x, \; \partial_u \partial_t x \in (L^\infty(Q_T))^2$;

b) the flow $\Gamma^t = \text{Image}(x(., t)), t \in [0, T]$ of regular plane curves satisfies the geometric equation $\partial_t x = \beta(k, \nu)\vec{N} + \alpha \vec{T}$ where α is the tangential velocity preserving the relative local length, i.e. $\partial_t \left(|\partial_u x(u, t)| / |\Gamma^t|\right) = 0$ for any $(u, t) \in Q_T$.

It is worth to note that the condition (7) is fulfilled in the case when $0 < m \leq 1$ or Γ^0 is strictly convex or in the case when $1 < m$ and Γ^0 is a nonconvex smooth curve whose inflection points have at most $2 + \frac{1}{m-1}$ order contact with their tangents.

In the next Figures 1-2 we have computed affine evolution of the same initial curve. The initial curve has been discretized uniformly. First, we have used the tangential velocity preserving the relative local length. As it can be seen from Fig. 1., the uniform initial distribution is then preserved during evolution. The numerical blow up time is $T_{max} = 0.694$, solution stabilizes on an ellipse with the isoperimetric ratio tending to 1.33 which is in a good agreement with analytical results due to Sapiro and Tannenbaum ([ST]). On the other hand, without tangential redistribution (i.e. $\alpha = 0$) one can see a rapid merging of several grid points corresponding to the vanishing of the local length element $|\partial_u x|$. In Fig. 2 we see the evolution until $t = 0.38$ just before collapse.

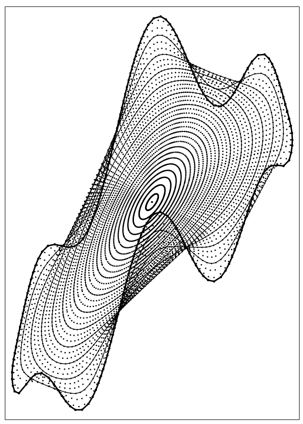

Fig. 1. $\beta(k) = k^{1/3}$, discrete evolution using tangential redistribution of grid points preserving the relative local length.

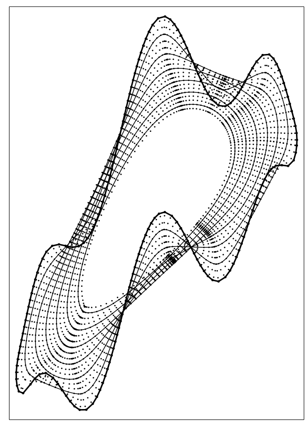

Fig. 2. $\beta(k) = k^{1/3}$, without redistribution, computation collapses due to vanishing of the local length $|\partial_u x|$.

References.

[AL] U.Abresch, J.Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geom. **23** (1986), 175-196.

[A1] S.B.Angenent, *Parabolic equations for curves on surfaces I: Curves with p-integrable curvature.*, Annals of Mathematics **132, No.3** (1990), 451-483.

[A2] S.B. Angenent, *Nonlinear analytic semiflows.*, Proc. R. Soc. Edinb., Sect. A **115** (1990), 91-107.

[AG] S.B.Angenent, M.E.Gurtin, *General contact angle conditions with and without kinetics*, J. Quarterly Appl. Math. **54, 3** (1996), 557-569.

[AST] S.B.Angenent, G.Sapiro, A.Tannenbaum, *On affine heat equation for non-convex curves*, Journal of the Amer. Math. Soc. **11** (1998), 601-634.

[D] G.Dziuk, *Convergence of a semi discrete scheme for the curve shortening flow*, M^3AS **4, No. 4** (1994), 589-606.

[MK] K.Mikula, J.Kačur, *Evolution of convex plane curves describing anisotropic motions of phase interfaces*, SIAM J. Sci. Comput. **17, No. 6** (1996).

[M] K.Mikula, *Solution of nonlinear curvature driven evolution of plane convex curves*, Applied Numerical Mathematics **21** (1997), 1-14.

[MS1] K.Mikula, D.Ševčovič, *Solution of nonlinearly curvature driven evolution of plane curves*, Applied Numerical Mathematics **31, No. 2** (1999).

[MS2] K.Mikula, D.Ševčovič, *Evolution of plane curves driven by a nonlinear function of curvature and anisotropy*, Preprint 99-02, Slovak Technical University (1999).
(available at: http://www.iam.fmph.uniba.sk/institute/sevcovic/papers/cl17.pdf or cl17.ps.gz)

[ST] G.Sapiro, A.Tannenbaum, *On affine plane curve evolution*, J.Funct.Anal. **119, No. 1** (1994), 79-120.