# Analysis of the free boundary for the pricing of an American call option 

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#### Abstract

The purpose of this paper is to analyse the free boundary problem for the Black-Scholes equation for pricing the American call option on stocks paying a continuous dividend. Using the Fourier integral transformation method, we derive and analyse a nonlinear singular integral equation determining the shape of the free boundary. Numerical experiments based on this integral equation are also presented.


## 1 Introduction

In the past years, the Black-Scholes equation for pricing derivatives has attracted a lot of attention from both theoretical as well as practical point of view. One of the interesting problems in this field is the analysis of the early exercise boundary and the optimal stopping time for American options on stocks paying a continuous dividend. It can be easily reduced to a problem of solving a certain free boundary problem for the Black-Scholes equation (cf. Black \& Scholes [2]). However, the exact analytical expression for the free boundary profile is not known yet. Many authors have investigated various approximate models leading to approximate expressions for valuing American call and put options $[5,6,9,10,11,12,15,16]$. For the purpose of studying the free boundary profile near expiry, many different integral equations have been derived $[1,3,8,14,17]$. We refer to the book by Salopek [17] for a comprehensive survey of both theoretical and computational aspects of valuing American options.

In this paper, we present an alternative integral equation which will provide an accurate numerical method for calculating the early exercise boundary near expiry. The derivation of the nonlinear integral equation is based on the Fourier transform. A solution to this integral equation is the free boundary profile. The novelty of this approach consists in three steps:
(1) The fixed domain transformation.
(2) Derivation of a nonlocal nonlinear PDE for the so-called synthetic portfolio.
(3) Construction of a solution by means of Fourier sine and cosine integral transforms.

Let us recall that the equation governing the time evolution of the price $V(S, t)$ of the

American call option is the following parabolic PDE:

$$
\begin{align*}
& \frac{\partial V}{\partial t}+(r-D) S \frac{\partial V}{\partial S}+\frac{\sigma^{2}}{2} S^{2} \frac{\partial^{2} V}{\partial S^{2}}-r V=0, \quad 0<t<T, 0<S<S_{f}(t) \\
& V(0, t)=0, V\left(S_{f}(t), t\right)=S_{f}(t)-E, \frac{\partial V}{\partial S}\left(S_{f}(t), t\right)=1  \tag{1.1}\\
& V(S, T)=\max (S-E, 0)
\end{align*}
$$

defined on a time-dependent domain $S \in\left(0, S_{f}(t)\right)$, where $t \in(0, T)$. Here $S>0$ is the stock price, $E>0$ is the exercise price, $r>0$ is the risk-free rate, $D>0$ is the continuous stock dividend rate and $\sigma>0$ is the volatility of the underlying stock process.

In this paper, we restrict our attention to the case when $r>D>0$. It is well known that, for $r>D>0$, the free boundary $\varrho(\tau)=S_{f}(T-\tau)$ starts at $\varrho(0)=r E / D$, whereas $\varrho(0)=E$ for the case $r \leq D$ (cf. Dewynne et al. [4]). Thus, the free boundary profile develops an initial jump in the case $r>D>0$. Notice that the case $0<r \leq D$ can be also treated by other methods based on integral equations. Kwok [13] derived another integral equation which covers both cases $0<r \leq D$, as well as $r>D>0$ (see equation (3.13) and Remark 3.1). However, in the latter case equation (3.13) becomes singular as $t \rightarrow T^{-}$, leading to numerical instabilities near expiry.

It is the purpose of this paper to investigate the behaviour of the free boundary $S_{f}(t)$. We present a method of reducing the free boundary problem for (1.1) to a nonlinear integral equation with a singular kernel. Notice that our method of reducing the free boundary problem to a nonlinear integral equation can be also successfully adopted for valuing the American put option paying no dividends (cf. Stamicar et al. [18]).

The paper is organized as follows. In § 2 we present a fixed domain transformation of the Black-Scholes equation. The method to reduce the free boundary problem to a nonlinear integral equation is discussed in § 3. We derive a PDE for the synthetic portfolio $\Pi=V-S \partial V / \partial S$, and transform the resulting PDE into a parameterized system of a nonautonoumous ODE for sine and cosine Fourier transforms of $\Pi$. Using the inverse Fourier transforms, we derive a nonlinear integral equation for the free boundary $S_{f}(t)$. In § 4 we present numerical results of the free boundary problem obtained by means of solving the nonlinear integral equation. We compare our results with results obtained by several classical methods for computing values of an American call option.

## 2 Fixed domain transformation of the free boundary problem

In this section we will perform a fixed domain transformation of the free boundary problem (1.1)-(1.3) into a parabolic equation defined on a fixed spatial domain. As will be shown below, imposing of the free boundary condition will result in a nonlinear timedependent term involved in the resulting equation.

To transform equation (1.1) defined on a time dependent spatial domain $\left(0, S_{f}(t)\right)$, we introduce the following change of variables:

$$
\begin{equation*}
\tau=T-t, \quad x=\ln \left(\frac{\varrho(\tau)}{S}\right) \quad \text { where } \varrho(\tau)=S_{f}(T-\tau) \tag{2.1}
\end{equation*}
$$

Clearly, $\tau \in(0, T)$ and $x \in(0, \infty)$ whenever $S \in\left(0, S_{f}(t)\right)$. Let us furthermore define the
auxiliary function $\Pi=\Pi(x, \tau)$ as follows:

$$
\begin{equation*}
\Pi(x, \tau)=V(S, t)-S \frac{\partial V}{\partial S}(S, t) \tag{2.2}
\end{equation*}
$$

Notice that the quantity $\Pi$ has an important financial meaning as it is a synthetic portfolio consisting of one long option and $\Delta=\frac{\partial V}{\partial S}$ underlying short stocks. It follows from (2.1) that

$$
\begin{align*}
& \frac{\partial \Pi}{\partial x}=S^{2} \frac{\partial^{2} V}{\partial S^{2}}, \quad \frac{\partial^{2} \Pi}{\partial x^{2}}+2 \frac{\partial \Pi}{\partial x}=-S^{3} \frac{\partial^{3} V}{\partial S^{3}} \\
& \frac{\partial \Pi}{\partial \tau}+\frac{\dot{\varrho}}{\varrho} \frac{\partial \Pi}{\partial x}=S \frac{\partial^{2} V}{\partial S \partial t}-\frac{\partial V}{\partial t} \tag{2.3}
\end{align*}
$$

where $\dot{\varrho}=d \varrho / d \tau$. Now assuming that $V=V(S, t)$ is a sufficiently smooth solution of (1.1), we may differentiate (1.1) with respect to $S$. Plugging expressions (2.3) into (1.1), we finally obtain that the function $\Pi=\Pi(x, \tau)$ is a solution of the parabolic equation

$$
\begin{equation*}
\frac{\partial \Pi}{\partial \tau}+a(\tau) \frac{\partial \Pi}{\partial x}-\frac{\sigma^{2}}{2} \frac{\partial^{2} \Pi}{\partial x^{2}}+r \Pi=0 \tag{2.4}
\end{equation*}
$$

with a time-dependent coefficient

$$
a(\tau)=\frac{\dot{\varrho}(\tau)}{\varrho(\tau)}+\left(r-D-\frac{\sigma^{2}}{2}\right)
$$

It follows from the boundary condition $V\left(S_{f}(t), t\right)=S_{f}(t)-E$ and $V_{S}\left(S_{f}(t), t\right)=1$ that

$$
\begin{equation*}
\Pi(0, \tau)=-E, \quad \Pi(+\infty, \tau)=0 \tag{2.5}
\end{equation*}
$$

The initial condition $\Pi(x, 0)$ can be deduced from the pay-off diagram for $V(S, T)$. We obtain

$$
\Pi(x, 0)= \begin{cases}-E & \text { for } x<\ln \left(\frac{\varrho(0)}{E}\right)  \tag{2.6}\\ 0 & \text { otherwise }\end{cases}
$$

Notice that equation (2.4) is a parabolic PDE with a time-dependent coefficient $a(\tau)$. In what follows, we will show the function $a(\tau)$ depends upon a solution $\Pi$ itself. This dependence is non-local in the spatial variable $x$. Moreover, the initial position of the interface $\varrho(0)$ enters the initial condition $\Pi(x, 0)$. Therefore we first have to determine the relationship between the solution $\Pi(x, \tau)$ and the free boundary function $\varrho(\tau)$ first. To this end, we make use of the boundary condition imposed on $V$ at the interface $S=S_{f}(t)$. Since $S_{f}(t)-E=V\left(S_{f}(t), t\right)$ we have

$$
\frac{d}{d t} S_{f}(t)=\frac{\partial V}{\partial S}\left(S_{f}(t), t\right) \frac{d}{d t} S_{f}(t)+\frac{\partial V}{\partial t}\left(S_{f}(t), t\right)
$$

As $\frac{\partial V}{\partial S}\left(S_{f}(t), t\right)=1$ we obtain $\frac{\partial V}{\partial t}(S, t)=0$ at $S=S_{f}(t)$. Assuming the function $\Pi_{x}$ has a trace at $x=0$, and taking into account (2.3), we may conclude that, for any $t=T-\tau \in[0, T)$,

$$
S^{2} \frac{\partial^{2} V}{\partial S^{2}}(S, t) \rightarrow \frac{\partial \Pi}{\partial x}(0, \tau), \quad S \frac{\partial V}{\partial S}(S, t) \rightarrow \varrho(\tau) \text { as } S \rightarrow S_{f}(t)^{-}
$$

If $\frac{\partial V}{\partial t}(S, t) \rightarrow \frac{\partial V}{\partial t}\left(S_{f}(t), t\right)=0$ as $S \rightarrow S_{f}(t)^{-}$, then it follows from the Black-Scholes
equation (1.1) that

$$
\begin{gathered}
(r-D) \varrho(\tau)+\frac{\sigma^{2}}{2} \frac{\partial \Pi}{\partial x}(0, \tau)-r(\varrho(\tau)-E) \\
=\lim _{S \rightarrow S_{f}(t)^{-}}\left(\frac{\partial V}{\partial t}(S, t)+(r-D) S \frac{\partial V}{\partial S}(S, t)+\frac{\sigma^{2}}{2} S^{2} \frac{\partial^{2} V}{\partial S^{2}}(S, t)-r V(S, t)\right)=0
\end{gathered}
$$

and hence

$$
\begin{equation*}
\varrho(\tau)=\frac{r E}{D}+\frac{\sigma^{2}}{2 D} \frac{\partial \Pi}{\partial x}(0, \tau) \text { for } 0<\tau \leq T \tag{2.7}
\end{equation*}
$$

It remains to determine the initial position of the interface $\varrho(0)$. According to Dewynne et al. [4], the initial position $\varrho(0)$ of the free boundary is $r E / D$ for the case $0<D<r$. Alternatively, we can derive this condition from (2.4)-(2.6) by assuming the smoothness of $\Pi$ in the $x$ variable up to the boundary $x=0$ uniformly for $\tau \rightarrow 0^{+}$. More precisely, we assume that

$$
\lim _{\tau \rightarrow 0^{+}} \frac{\partial \Pi}{\partial x}(0, \tau)=\lim _{\tau \rightarrow 0^{+}, x \rightarrow 0^{+}} \frac{\partial \Pi}{\partial x}(x, \tau)=\lim _{x \rightarrow 0^{+}} \frac{\partial \Pi}{\partial x}(x, 0)=0
$$

because $\Pi(x, 0)=-E$ for $x$ close to $0^{+}$. By (2.7) we obtain

$$
\begin{equation*}
\varrho(0)=\frac{r E}{D} . \tag{2.8}
\end{equation*}
$$

In summary, we have shown that, under suitable regularity assumptions imposed on a solution $\Pi$ to $(2.4),(2.5),(2.6)$, the free boundary problem (1.1) can be transformed into the initial boundary value problem for parabolic PDE

$$
\begin{align*}
& \frac{\partial \Pi}{\partial \tau}+a(\tau) \frac{\partial \Pi}{\partial x}-\frac{\sigma^{2}}{2} \frac{\partial^{2} \Pi}{\partial x^{2}}+r \Pi=0, \\
& \Pi(0, \tau)=-E, \quad \Pi(+\infty, \tau)=0,  \tag{2.9}\\
& \Pi(x, 0)= \begin{cases}-E & \text { for } x<\ln \left(\frac{r}{D}\right) \\
0 & \text { otherwise }\end{cases}
\end{align*}
$$

where $a(\tau)=\frac{\dot{\varrho}(\tau)}{\varrho(\tau)}+r-D-\frac{\sigma^{2}}{2}$ and

$$
\begin{equation*}
\varrho(\tau)=\frac{r E}{D}+\frac{\sigma^{2}}{2 D} \frac{\partial \Pi}{\partial x}(0, \tau), \quad \varrho(0)=\frac{r E}{D} . \tag{2.10}
\end{equation*}
$$

We repeat that the problem (2.9) is a nonlinear parabolic equation with a nonlocal constraint given by (2.10).

Remark 2.1 In our derivation of the free boundary function $\varrho(\tau)$ and its initial condition $\varrho(0)$ we did not assume that the solution $V(S, t)$ is smooth up to the free boundary $S=S_{f}(t)$. Such an assumption would lead to an obvious contradiction $\Gamma:=\partial^{2} V / \partial S^{2}=0$ at $S=S_{f}(t)$. On the other hand, the jump in $\Gamma$ at the free boundary is the only driving force for the evolution of the free boundary function $\varrho$ (see (2.7)). Construction of a PDE for the synthetic portfolio function $\Pi$ is a crucial step in our approach because the derivative $\partial_{x} \Pi$ admits a trace at the boundary $x=0$ and the unknown free boundary function $\varrho$ can be determined via (2.10) (see also Remark 3.2).

## 3 Reduction to a nonlinear integral equation

The main purpose of this section is to show how the fully nonlinear nonlocal problem (2.9)-(2.10) can reduced to a single nonlinear integral equation for $\varrho(\tau)$ giving the explicit formula for the solution $\Pi(x, \tau)$ to (2.9). The idea is to apply the Fourier sine and cosine integral transforms (cf. Stein \& Weiss [19]).

Recall that for any Lebesgue integrable function $f \in L^{1}\left(R^{+}\right)$the sine and cosine transformations are defined as follows:

$$
F_{S}(f)(\omega)=\int_{0}^{\infty} f(x) \sin \omega x d x, \quad F_{C}(f)(\omega)=\int_{0}^{\infty} f(x) \cos \omega x d x
$$

Their inverse transforms are given by

$$
F_{S}^{-1}(g)(x)=\frac{2}{\pi} \int_{0}^{\infty} g(\omega) \sin \omega x d \omega, \quad F_{C}^{-1}(g)(x)=\frac{2}{\pi} \int_{0}^{\infty} g(\omega) \cos \omega x d \omega
$$

Clearly, for any function $f$ such that $f, f^{\prime}, f^{\prime \prime} \in L^{1}\left(R^{+}\right)$and $f(+\infty)=f^{\prime}(+\infty)=0$ we have

$$
\begin{align*}
F_{S}\left(f^{\prime}\right) & =-\omega F_{C}(f), & F_{C}\left(f^{\prime}\right) & =-f\left(0^{+}\right)+\omega F_{S}(f) \\
F_{S}\left(f^{\prime \prime}\right) & =\omega f\left(0^{+}\right)-\omega^{2} F_{S}(f), & F_{C}\left(f^{\prime \prime}\right) & =-f^{\prime}\left(0^{+}\right)-\omega^{2} F_{C}(f) \tag{3.1}
\end{align*}
$$

Now we suppose that the function $\varrho(\tau)$ and subsequently $a(\tau)$ are already know. Let $\Pi=\Pi(x, \tau)$ be a solution of (2.9) corresponding to a given function $a(\tau)$. Let us denote

$$
\begin{equation*}
p(\omega, \tau)=F_{S}(\Pi(., \tau))(\omega), \quad q(\omega, \tau)=F_{C}(\Pi(., \tau))(\omega) \tag{3.2}
\end{equation*}
$$

where $\omega \in R^{+}, \tau \in(0, T)$. By applying the sine and cosine integral transforms formulae (3.1) to equation (2.9) and taking into account (2.10), we finally obtain a linear nonautonomous $\omega$-parameterized system of ODEs

$$
\begin{align*}
\frac{d}{d \tau} p(\omega, \tau)-a(\tau) \omega q(\omega, \tau)+\alpha(\omega) p(\omega, \tau) & =-E \omega \frac{\sigma^{2}}{2} \\
\frac{d}{d \tau} q(\omega, \tau)+a(\tau) \omega p(\omega, \tau)+\alpha(\omega) q(\omega, \tau) & =-E a(\tau)-D \varrho(\tau)+r E \tag{3.3}
\end{align*}
$$

for the sine and cosine transforms of $\Pi$ (see (3.2)) where

$$
\alpha(\omega)=\frac{1}{2}\left(\sigma^{2} \omega^{2}+2 r\right)
$$

The system of equations (3.3) is subject to initial conditions at $\tau=0, p(\omega, 0)=$ $F_{S}\left(\Pi(., 0)(\omega), \quad q(\omega, 0)=F_{C}(\Pi(., 0)(\omega)\right.$. In the case of the call option, we deduce from the initial condition for $\Pi$ (see (2.9)) that

$$
\begin{equation*}
p(\omega, 0)=\frac{E}{\omega}\left(\cos \left(\omega \ln \frac{r}{D}\right)-1\right), \quad q(\omega, 0)=-\frac{E}{\omega} \sin \left(\omega \ln \frac{r}{D}\right) \tag{3.4}
\end{equation*}
$$

Taking into account (3.4) and by using the variation of constants formula for solving linear non-autonomous ODEs, we obtain an explicit formula for $p(\omega, \tau)=-E \omega^{-1}+\tilde{p}(\omega, \tau)$ where

$$
\begin{gather*}
\tilde{p}(\omega, \tau)=\frac{E}{\omega} e^{-\alpha(\omega) \tau} \cos (\omega(A(\tau, 0)+\ln (r / D))) \\
+\int_{0}^{\tau} \mathrm{e}^{-\alpha(\omega)(\tau-s)}\left[\frac{r E}{\omega} \cos (\omega A(\tau, s))+(r E-D \varrho(s)) \sin (\omega A(\tau, s))\right] d s \tag{3.5}
\end{gather*}
$$

Here we have denoted by $A$ the function defined as

$$
\begin{equation*}
A(\tau, s)=\int_{s}^{\tau} a(\xi) d \xi=\ln \frac{\varrho(\tau)}{\varrho(s)}+\left(r-D-\frac{\sigma^{2}}{2}\right)(\tau-s) \tag{3.6}
\end{equation*}
$$

As $F_{S}^{-1}\left(\omega^{-1}\right)=1$ we have $\Pi(x, \tau)=F_{S}^{-1}(p(\omega, \tau))=-E+\frac{2}{\pi} \int_{0}^{\infty} \tilde{p}(\omega, \tau) \sin (\omega x) d \omega$. From (2.10) we conclude that the free boundary function $\varrho$ satisfies the following equation:

$$
\begin{equation*}
\varrho(\tau)=\frac{r E}{D}+\frac{\sigma^{2}}{D \pi} \int_{0}^{\infty} \omega \tilde{p}(\omega, \tau) d \omega . \tag{3.7}
\end{equation*}
$$

To compute the right-hand side of (3.7) we recall that

$$
\begin{aligned}
& \int_{0}^{\infty} e^{-\alpha(\omega) \tau} \cos (\omega A) d \omega=\sqrt{\frac{\pi}{2 \sigma^{2} \tau}} \exp \left(-r \tau-\frac{A^{2}}{2 \sigma^{2} \tau}\right) \\
& \int_{0}^{\infty} e^{-\alpha(\omega) \tau} \omega \sin (\omega A) d \omega=\frac{A}{\sigma^{2} \tau} \sqrt{\frac{\pi}{2 \sigma^{2} \tau}} \exp \left(-r \tau-\frac{A^{2}}{2 \sigma^{2} \tau}\right),
\end{aligned}
$$

for any $A \in R, \tau>0$ where $\alpha(\omega)=\left(\sigma^{2} \omega^{2}+2 r\right) / 2$. Using the above integrals and taking into account (3.5) and (3.7) we end up with the following nonlinear singular integral equation for the free boundary function $\varrho(\tau)$ :

$$
\begin{align*}
\varrho(\tau) & =\frac{r E}{D}\left(1+\frac{\sigma}{r \sqrt{2 \pi \tau}} \exp \left(-r \tau-\frac{(A(\tau, 0)+\ln (r / D))^{2}}{2 \sigma^{2} \tau}\right)\right.  \tag{3.8}\\
& \left.+\frac{1}{\sqrt{2 \pi}} \int_{0}^{\tau}\left[\sigma+\frac{1}{\sigma}\left(1-\frac{D \varrho(s)}{r E}\right) \frac{A(\tau, s)}{\tau-s}\right] \frac{\exp \left(-r(\tau-s)-\frac{A(\tau, s)^{2}}{2 \sigma^{2}(\tau-s)}\right)}{\sqrt{\tau-s}} d s\right)
\end{align*}
$$

where the function $A$ depends upon $\varrho$ via equation (3.6). To simplify this integral equation, we introduce a new auxiliary function $H:[0, \sqrt{T}] \rightarrow R$ as follows:

$$
\begin{equation*}
\varrho(\tau)=\frac{r E}{D}(1+\sigma \sqrt{2} H(\sqrt{\tau})) . \tag{3.9}
\end{equation*}
$$

Using the change of variables $s=\xi^{2} \cos ^{2} \theta$, one can rewrite the integral equation (3.7) in terms of the function $H$ as follows:

$$
\begin{equation*}
H(\xi)=f_{H}(\xi)+\frac{1}{\sqrt{\pi}} \int_{0}^{\frac{\pi}{2}}\left[\xi \cos \theta-2 \operatorname{cotg} \theta H(\xi \cos \theta) g_{H}(\xi, \theta)\right] \mathrm{e}^{-r \xi^{2} \sin ^{2} \theta-g_{H}^{2}(\xi, \theta)} d \theta \tag{3.10}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{H}(\xi, \theta)=\frac{1}{\sigma \sqrt{2}} \frac{1}{\xi \sin \theta} \ln \left(\frac{1+\sigma \sqrt{2} H(\xi)}{1+\sigma \sqrt{2} H(\xi \cos \theta)}\right)+\frac{\Lambda}{\sqrt{2}} \xi \sin \theta \tag{3.11}
\end{equation*}
$$

for $\xi \in[0, \sqrt{T}], \theta \in(0, \pi / 2)$,

$$
\Lambda=\frac{r-D}{\sigma}-\frac{\sigma}{2}
$$

and

$$
\begin{equation*}
f_{H}(\xi)=\frac{1}{2 r \sqrt{\pi} \xi} \mathrm{e}^{-r \xi^{2}-\left(g_{H}\left(\xi, \frac{\pi}{2}\right)+\frac{1}{\xi} \frac{1}{\sigma \sqrt{2}} \ln (r / D)\right)^{2}} . \tag{3.12}
\end{equation*}
$$

Notice that equations (3.8) and (3.10) are integral equations with a singular kernel (cf. Gripenberg et al. [7]).

Remark 3.1 Kwok [13] derived another integral equation for the early exercise boundary for the American call option on a stock paying continuous dividend. According to Kwok [13, Section 4.2.3], $\varrho(\tau)$ satisfies the integral equation

$$
\begin{align*}
\varrho(\tau) & =E+\varrho(\tau) e^{-D \tau} N(d)-E \mathrm{e}^{-r \tau} N(d-\sigma \sqrt{\tau}) \\
& +\int_{0}^{\tau} D \varrho(\tau) e^{-D \xi} N\left(d_{\xi}\right)-r E \mathrm{e}^{-r \xi} N\left(d_{\xi}-\sigma \sqrt{\xi}\right) d \xi \tag{3.13}
\end{align*}
$$

where

$$
d=\frac{1}{\sigma \sqrt{\tau}} \ln \left(\frac{\varrho(\tau)}{E}\right)+\Lambda \sqrt{2 \tau}, \quad d_{\xi}=\frac{1}{\sigma \sqrt{\xi}} \ln \left(\frac{\varrho(\tau)}{\varrho(\tau-\xi)}\right)+\Lambda \sqrt{2 \xi}
$$

and $N(u)$ is the cumulative distribution function for the normal distribution. The above integral equation covers both cases: $r \leq D$ as well as $r>D$. However, in the case $r>D$ this equation becomes singular as $\tau \rightarrow 0^{+}$. It is worth noting that the above integral equation and equation (3.8) differ, and it is not obvious - at least to the author - how to transform (3.8) into (3.13) and vice versa.

In the rest of this section we derive a formula for pricing American call options based on the solution $\varrho$ to the integral equation (3.10). With regard to (2.2), we have

$$
\frac{\partial}{\partial S}\left(S^{-1} V(S, t)\right)=-S^{-2} \Pi\left(\ln \left(S^{-1} \varrho(T-t)\right), T-t\right)
$$

Taking into account the boundary condition $V\left(S_{f}(t), t\right)=S_{f}(t)-E$ and integrating the above equation from $S$ to $S_{f}(t)=\varrho(T-t)$, we obtain

$$
\begin{equation*}
V(S, T-\tau)=\frac{S}{\varrho(\tau)}\left(\varrho(\tau)-E+\int_{0}^{\ln \frac{\varrho(\tau)}{S}} \mathrm{e}^{x} \Pi(x, \tau) d x\right) \tag{3.14}
\end{equation*}
$$

Remark 3.2 Suppose that $H$ is a $C^{1}$ smooth solution to (3.10) whose derivative $H^{\prime}$ is bounded uniformly in the interval $[0, \sqrt{T}]$. Let us construct the free boundary function $S_{f}(t)=\varrho(T-t)$ where $\varrho$ is given by (3.9). Then there exists a classical solution $V$ to (1.1) satisfying the Dirichlet boundary conditions $V(0, t)=0, V\left(S_{f}(t), t\right)=S_{f}(t)-E$ for $0<t<T$. The function $V(., t)$ is smooth on the interval $\left(0, S_{f}(t)\right)$ for $0<t<T$ and this is why the synthetic portfolio function $\Pi$ can be constructed via (2.2), $\Pi$ satisfies (2.9) and $V$ is given by (3.14). Moreover, $\partial_{x} \Pi$ has a trace at $x=0$. Now, the fact that $\varrho$ is assumed to be solution to (3.8) together with (2.10) enables us to conclude that $\partial_{S} V(S, t)=1$ at the free boundary $S=S_{f}(t)$. Hence, $V$ is a solution to the free boundary problem (1.1).

To evaluate the integral part of the right-hand side of (3.14), let us introduce the following auxiliary functions:

$$
\begin{align*}
& I_{1}(A, L, \tau)=\frac{2}{\pi} \int_{0}^{L} \int_{0}^{\infty} \mathrm{e}^{x-\alpha(\omega) \tau} \sin (\omega A) \sin (\omega x) d \omega d x \\
& I_{2}(A, L, \tau)=\frac{2}{\pi} \int_{0}^{L} \int_{0}^{\infty} \mathrm{e}^{x-\alpha(\omega) \tau} \omega^{-1} \cos (\omega A) \sin (\omega x) d \omega d x \tag{3.15}
\end{align*}
$$

It is a straightforward calculation that $I_{1}=I_{1}(A, L, \tau)$ and $I_{2}=I_{2}(A, L, \tau)$ can be
computed as

$$
\begin{align*}
I_{1}= & \frac{e^{-\left(r-\sigma^{2} / 2\right) \tau}}{2}\left[e^{A} M\left(\frac{-A-\sigma^{2} \tau}{\sigma \sqrt{2 \tau}}, \frac{L}{\sigma \sqrt{2 \tau}}\right)-e^{-A} M\left(\frac{A-\sigma^{2} \tau}{\sigma \sqrt{2 \tau}}, \frac{L}{\sigma \sqrt{2 \tau}}\right)\right] \\
I_{2}= & \frac{e^{-r \tau} e^{L}}{2} M\left(\frac{A-L}{\sigma \sqrt{2 \tau}}, \frac{2 L}{\sigma \sqrt{2 \tau}}\right)  \tag{3.16}\\
& -\frac{e^{-\left(r-\sigma^{2} / 2\right) \tau}}{2}\left[e^{A} M\left(\frac{-A-\sigma^{2} \tau}{\sigma \sqrt{2 \tau}}, \frac{L}{\sigma \sqrt{2 \tau}}\right)+e^{-A} M\left(\frac{A-\sigma^{2} \tau}{\sigma \sqrt{2 \tau}}, \frac{L}{\sigma \sqrt{2 \tau}}\right)\right]
\end{align*}
$$

where

$$
M(x, y)=\operatorname{erf}(x+y)-\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}} \int_{x}^{x+y} e^{-\xi^{2}} d \xi
$$

Using the auxiliary functions defined in (3.15) and (3.16) and inserting the expressions (3.5) and (3.7) (recall that $\Pi(x, \tau)=-E+\frac{2}{\pi} \int_{0}^{\infty} \tilde{p}(\omega, \tau) \sin \omega x d \omega$ ) into (3.14), we end up with the formula for pricing the American call option:

$$
\begin{align*}
V(S, T-\tau) & =S-E+\frac{S}{\varrho(\tau)} E I_{2}(A(\tau, 0)+\ln (r / D), \ln (\varrho(\tau) / S), \tau) \\
& +\frac{S}{\varrho(\tau)} \int_{0}^{\tau}\left[r E I_{2}(A(\tau, s), \ln (\varrho(\tau) / S), \tau-s)\right.  \tag{3.17}\\
& \left.+(r E-D \varrho(s)) I_{1}(A(\tau, s), \ln (\varrho(\tau) / S), \tau-s)\right] d s
\end{align*}
$$

for any $S \in\left(0, S_{f}(t)\right)$ and $t \in[0, T]$, where $A(\tau, s)=\ln \frac{\varrho(\tau)}{\varrho(s)}+\left(r-D-\frac{\sigma^{2}}{2}\right)(\tau-s)$. We will refer to (3.17) as the semi-explicit formula for pricing the American call option. We use the term 'semi-explicit' because (3.17) contains the free boundary function $\varrho(\tau)=S_{f}(T-\tau)$ which has to be determined first by solving the nonlinear integral equation (3.10).

## 4 Numerical experiments

In this section we focus on numerical experiments. We will compute the free boundary profile

$$
\begin{equation*}
S_{f}(t)=\varrho(T-t)=\frac{r E}{D}(1+\sigma \sqrt{2} H(\sqrt{T-t})) \tag{4.1}
\end{equation*}
$$

(see (3.4)) by solving the nonlinear integral equation (3.10). We also present a comparison of the results obtained by our methods to those obtained by other known methods for solving the American call option problem.

The computation of a solution of the nonlinear integral equation is based on an iterative method. We will construct a sequence of approximate solutions to (3.10). Let $H^{0}$ be an initial approximation of a solution to (3.10). If we suppose $H^{0}(\xi)=h_{1} \xi$, then, plugging this ansatz into (3.10) yields the well-known first order approximation of a solution $H(\xi)$ in the form

$$
H^{0}(\xi)=0.451381 \xi
$$

i.e. $\varrho^{0}(\tau)=\frac{r E}{D}(1+0.638349 \sigma \sqrt{\tau})$ (cf. Dewynne et al. [4]). For $n=0,1, \ldots$ we will define the $n+1$ approximation $H^{n+1}$ as follows:

$$
H^{n+1}(\xi)=
$$

$$
\begin{equation*}
f_{H^{n}}(\xi)+\frac{1}{\sqrt{\pi}} \int_{0}^{\frac{\pi}{2}}\left[\xi \cos \theta-2 \operatorname{cotg} \theta H^{n}(\xi \cos \theta) g_{H^{n}}(\xi, \theta)\right] \mathrm{e}^{-r \xi^{2} \sin ^{2} \theta-g_{H^{n}}^{2}(\xi, \theta)} d \theta \tag{4.2}
\end{equation*}
$$

for $\xi \in[0, \sqrt{T}]$. With regard to (3.11) and (3.12), we have

$$
\begin{gathered}
g_{H^{n}}(\xi, \theta)=\frac{1}{\sigma \sqrt{2}} \frac{1}{\xi \sin \theta} \ln \left(\frac{1+\sigma \sqrt{2} H^{n}(\xi)}{1+\sigma \sqrt{2} H^{n}(\xi \cos \theta)}\right)+\frac{\Lambda}{\sqrt{2}} \xi \sin \theta \\
f_{H^{n}}(\xi)=\frac{1}{2 r \sqrt{\pi} \xi} \mathrm{e}^{-r \xi^{2}-\left(g_{H^{n}}\left(\xi, \frac{\pi}{2}\right)+\frac{1}{\xi} \frac{1}{\sigma \sqrt{2}} \ln \left(\frac{r}{D}\right)\right)^{2}}
\end{gathered}
$$

Notice that the function $g_{H}$ is bounded provided that $H$ is nonnegative and Lipschitz continuous on $[0, \sqrt{T}]$. Recall that we have assumed $r>D>0$. Then the function $\xi \mapsto f_{H}(\xi)$ is bounded for $\xi \in[0, \sqrt{T}]$ and it vanishes at $\xi=0$. Moreover, if $H$ is smooth then $f_{H}$ is a flat function at $\xi=0$, i.e. $f_{H}(\xi)=o\left(\xi^{n}\right)$ as $\xi \rightarrow 0^{+}$for all $n \in N$. From the numerical point of view such a flat function can be omitted from computations.

To compute the integral part of the right-hand side in (4.1) we have to decompose the integral $\int_{0}^{\frac{\pi}{2}}$ into two parts: $\int_{\varepsilon}^{\frac{\pi}{2}}$ and $\int_{0}^{\varepsilon}$. The first regular part can be computed by a simple trapezoidal integration rule. The singular integral $\int_{0}^{\varepsilon}$ must be treated separately. For small values of $\theta$ we approximate the function $\operatorname{cotg} \theta g_{H^{n}}(\xi, \theta)$ by its limit as $\theta \rightarrow 0^{+}$. It yields the approximation of the singular term in (4.1) in the form

$$
\operatorname{cotg} \theta g_{H}(\xi, \theta) \approx \frac{1}{2} \frac{H^{\prime}(\xi)}{1+\sigma \sqrt{2} H(\xi)}+\frac{\Lambda}{\sqrt{2}} \xi \quad \text { for } \quad 0<\theta \leq \varepsilon \ll 1
$$

where $H=H^{n}$. In the following numerical simulations we chose $\varepsilon \approx 10^{-5}$.
Example 1. In Figure 1 we show five iterates of the free boundary function $S_{f}(t)$, where the auxiliary function $H(\xi)$ is constructed by means of successive iterations of the nonlinear integral operator defined by the right-hand side of (3.10). This sequence converges to a fixed point of such a map, i.e. to a solution of (3.10). Parameter values were chosen as $E=10, r=0.1, \sigma=0.2, D=0.05, T=1$. Figure 2 depicts the final tenth iteration of the function $S_{f}(t)$. The mesh contained 100 grid points.


Figure 1. Five iterates of $S_{f}(t)$ obtained from equation (3.10).


Figure 2. The profile of the solution $S_{f}(t)$.

In general, one can observe very rapid convergence of iterates to a fixed point. In practice, no more than six iterates are sufficient to obtain the fixed point of (3.10). It is worth noting that in all our numerical simulations the convergence was monotone, i.e. the curve moves only up in the iteration process.

In Figure 3 we show the long time behaviour of the free boundary $S_{f}(t), t \in[0, T]$ for large values of the expiration time $T$. For the parameter values $T=50, r=0.1, D=$ $0.05, \sigma=0.35$ and $E=10$ the theoretical value of $S_{f}(+\infty)$ is 36.8179 (see Dewynne et al. [4]).


Figure 3. Long-time behavior of $S_{f}(t)$.

In Table 1 we show a comparison of results obtained by our method based on the semi-explicit formula (3.17) and those obtained by known methods based on trinomial trees (both with the depth of the tree equal to 100), finite difference approximation (with 200 spatial and time grids) and analytic approximation of Barone-Adesi \& Whaley (cf. [1], [8, Ch. 15, p. 384]), resp.

Table 1. Comparison of the method based on formula (3.16) with other methods for the parameter values $E=10, T=1, \sigma=0.2, r=0.1, D=0.05$. The position $S_{f}(0)=\varrho(T)$ of the free boundary at $t=0$ (i.e. at $\tau=T$ ) was computed as $S_{f}(0)=\varrho(T)=22.3754$

| Method $\backslash$ The asset value $S$ | 15 | 18 | 20 | 21 | 22.3754 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Our method | 5.15 | 8.09 | 10.03 | 11.01 | 12.37 |
| Trinomial tree | 5.15 | 8.09 | 10.03 | 11.01 | 12.37 |
| Finite differences | 5.49 | 8.48 | 10.48 | 11.48 | 12.48 |
| Analytic approximation | 5.23 | 8.10 | 10.04 | 11.02 | 12.38 |

It also turned out that the method based on solving the integral equation (3.10) is 5-10 times faster than other methods based on trees or finite differences. The reason is that the computation of $V(S, t)$ for a wide range of values of $S$ based on the semi-explicit pricing formula (3.17) is very fast provided that the free boundary function $\varrho$ has already been computed.


Figure 4. The early exercise boundary $S_{f}(t)$ for $r=0.1, D=0.05$.
In Figures 4-5 the early exercise boundary $S_{f}(t)$ is computed for various values of the parameter $D$. In these computations we chose $E=10, T=0.01, \sigma=0.45$. Of particular interest is the case where $D$ is close to $r$ (see Figure 5).

## 5 Discussion

In this paper, we have derived and analysed a nonlinear integral equation determining the shape of the early exercise boundary for an American call option on stocks paying continuous dividend. The integral equation can be effectively solved by means of successive iterations. The results of numerical simulations have been compared to other known methods for solving the corresponding free boundary problem. It also turns out from numerical simulations that our method is faster than methods based on trees or finite differences, and allows for the sensitivity analysis of the dependence of the early exercise boundary on various parameters.


Figure 5. The early exercise boundary $S_{f}(t)$ for $r=0.1, D=0.09$.

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