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Evolution of curves on a surface driven by the geodesic curvature and external force

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We study a flow of closed curves on a given graph surface driven by the geodesic curvature and external force. Using vertical projection of surface curves to the plane we show how the geodesic curvature-driven flow can be reduced to a solution of a fully nonlinear system of parabolic differential equations. We show that the flow of surface curves is gradient-like, i.e. there exists a Lyapunov functional nonincreasing along trajectories. Special attention is placed on the analysis of closed stationary surface curves. We present sufficient conditions for their dynamic stability. Several computational examples of evolution of surface curves driven by the geodesic curvature and external force on various surfaces are presented in this article. We also discuss a link between the geodesic flow and the edge detection problem arising from the image segmentation theory.

Keywords: Geodesic curvature; External force; Flow of surface curves; Linearized stability; Lyapunov functional; Closed geodesic curve

AMS Classifications: 35K65; 35B35; 35K55; 53C44

1. Introduction

In this article we study a flow of curves on a given two-dimensional surface $\mathcal{M}$ in $\mathbb{R}^3$ represented by a smooth graph. We consider the simplest possible case in which the normal velocity $\mathcal{V}$ of a curve $\mathcal{G}$ on $\mathcal{M}$ is a linear function of its geodesic curvature $\mathcal{K}_g$ and external force,

$$\mathcal{V} = \mathcal{K}_g + \mathcal{F} \tag{1}$$

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where $\mathcal{F}$ is the normal component of a given external force $\vec{G}$, i.e. $\mathcal{F} = \vec{G} \cdot \vec{\mathcal{N}}$ and $\vec{\mathcal{N}}$ is the unit inward normal vector to a curve $\mathcal{G}$ belonging to the tangent space $T_x(\mathcal{M})$.

The idea how to analyze the flow of curves on a surface $\mathcal{M}$ consists in vertical projection of surface curves onto the plane. It allows for reducing the problem to the analysis of evolution of planar curves $\Gamma_t\colon S^1 \to \mathbb{R}^2$, $t \geq 0$ instead of surface ones. Although the geometric equation $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ is simple, the description of the normal velocity v of the family of projected planar curves is rather involved. Nevertheless, it can be written in the form of the equation

$$v = \beta(x, k, \nu) \equiv a(x, \nu)k + c(x, \nu) \tag{2}$$

where the normal velocity v is an affine function of the curvature k, nonlinearly depending on the tangential angle ν and the position vector $x \in \Gamma_t$. The precise form of the function β can be found in the next section. Recall that geometric equations of the form (2) can be often found in a variety of applied problems such as material science, combustion, robotics, image processing and computer vision. For an overview of important applications of (2) we refer to recent books by Sethian [26], Sapiro [25] and Osher and Fedkiw [24].

Our methodology for solving (2) is based on the so-called direct approach investigated by Dziuk, Deckelnick, Gage and Hamilton, Grayson, Mikula and Ševčovič and other authors (see [5–7,10,11,18–23] and references therein). The main idea is to represent the flow of planar curves by the position vector x which is a solution to the geometric equation $\partial_t x = \beta\vec{N} + \alpha\vec{T}$ where $\vec{N}, \vec{T}$ are the unit inward normal and tangent vectors, respectively. It turns out that one can construct a closed system of parabolic–ordinary differential equations for relevant geometric quantities: the curvature, tangential angle, local length and position vector. Other well-known techniques, such as level-set method due to Osher and Sethian (cf [24,26]) or phase-field approximations (see e.g. Beneš [2]) treat the geometric equation (2) by means of a solution to a higher-dimensional parabolic problem. In comparison to these methods, in the direct approach one space dimensional evolutionary problems are only solved. Notice that the direct approach for solving (2) can be accompanied by a proper choice of tangential velocity α significantly improving and stabilizing numerical computations as it was documented by many authors (see [5,12,13,16,20–23]).

The main purpose of this article is to study the qualitative properties of solutions to the geometric equation (1). We focus our attention to the linearized stability of stationary geodesic curves. We give sufficient conditions for their linearized stability. These conditions are shown to be sharp in the case of a flow of radially symmetric curves on radially symmetric surface. We, furthermore, prove that the flow of surface curves is gradient-like, i.e. there exists a Lyapunov functional nonincreasing along trajectories. Several computational examples of evolution of surface curves driven by the geodesic curvature and external force on various surfaces are presented in this article.

The outline of the article is as follows. In the next section we show how to project the flow of surface curves into the plane. We construct a normal velocity for the family of projected planar curves. In section 2.1 we present the governing system of partial differential equations (PDEs) describing the evolution of plane curves satisfying (2). The system consists of coupled parabolic–ordinary differential equations for the curvature, tangential angle, local length and position vector. Qualitative aspects of solutions like existence and their limiting behavior are investigated in section 3. Various Lyapunov-like functionals are derived in this section. Special attention is placed on the analysis of closed stationary surface curves in section 3.2. Here we present necessary and sufficient conditions for their stability. Furthermore, we analyze radially symmetric solutions. We also show that the stability criteria are sharp. Results of numerical approximation of the flow of curves on various complex surfaces, numerical study of stability results given in the article as well as a possible application to edge detection problem in the image segmentation are presented in section 4.

2. Preliminaries

2.1. *Projection of a flow of surface curves to the plane*

The main idea how to solve the geometric problem (1) is to project the flow of surface curves into the plane. A surface $\mathcal{M} = \{(x,z) \in \mathbb{R}^3, z = \phi(x), x \in \Omega\}$ is assumed to be a graph of a smooth function $\phi: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ defined in some domain $\Omega \subset \mathbb{R}^2$. The symbol (x,z) stands for a vector $(x_1, x_2, z) \in \mathbb{R}^3$ where $x = (x_1, x_2) \in \mathbb{R}^2$. With this notation any smooth closed curve $\mathcal{G}$ on the surface $\mathcal{M}$ can then be represented by its vertical projection to the plane, i.e.

$$\mathcal{G} = \{(x,z) \in \mathbb{R}^3, x \in \Gamma, z = \phi(x)\}$$

where Γ is a closed planar curve in $\Omega \subset \mathbb{R}^2$. Throughout the article we assume that the driving force $\mathcal{F}$ is a projection of a given external vector field $\vec{G}$ to the inward unit normal vector $\vec{\mathcal{N}} \in T_x(\mathcal{M})$ to a surface curve $\mathcal{G} \subset \mathcal{M}$ relative to $\mathcal{M}$. Thus $\mathcal{F} = \vec{G} \cdot \vec{\mathcal{N}}$. The external vector field $\vec{G}$ is assumed to be perpendicular to the plane $\mathbb{R}^2$ and it may depend on the vertical coordinate $z = \phi(x)$ only, i.e.

$$\vec{G}(x) = -(0, 0, \gamma)$$

where $\gamma = \gamma(z) = \gamma(\phi(x))$ is a given scalar 'gravity' functional. In [23] we have shown that under the above assumptions made on the surface $\mathcal{M}$ and the vector field $\vec{G}$, one can express $\mathcal{K}_g$ of the family of surface curves $\mathcal{G}_t$ as well as the external force $\mathcal{F}$ in terms k, ν and the position vector x of the vertically projected plane curve Γ_t.

Following the so-called direct approach (cf [5–7,12,19–22]) the evolution of planar curves $\Gamma_t, t \geq 0,$ can be described by a solution $x = x(.,t) \in \mathbb{R}^2$ to the position vector equation

$$\partial_t x = \beta \vec{N} + \alpha \vec{T} \tag{3}$$

where β and α are normal and tangential velocities of Γ_t, respectively. Assuming the family of surface curves $\mathcal{G}_t$ satisfies (1) it has been shown in [23] that the geometric equation $v = \beta(x, k, \nu)$ for the normal velocity v of the vertically projected planar curve Γ_t can be written in the following form:

$$v = \beta(x, k, \nu) \equiv a(x, \nu)\, k - b(x, \nu)\, \nabla\phi(x) \cdot \vec{N} \tag{4}$$

where $a = a(x, \nu) > 0$ and $b = b(x, \nu)$ are smooth functions given by

$$a(x,\nu) = \frac{1}{1 + (\nabla\phi \cdot \vec{T})^2}, \quad b(x,\nu) = \frac{1}{1 + |\nabla\phi|^2}\left(\gamma(\phi) - \frac{\vec{T}^T \nabla^2\phi\, \vec{T}}{1 + (\nabla\phi \cdot \vec{T})^2}\right), \tag{5}$$

$\phi = \phi(x)$ and $k, \vec{N} = (-\sin\nu, \cos\nu), \vec{T} = (\cos\nu, \sin\nu)$ are the curvature, unit inward normal and tangent vectors to a curve Γ_t. Thus ν is a tangent angle. Notice that the function β is a 2π-periodic function in the variable ν and is C^{k-2} smooth provided that $\phi \in C^k$. Moreover, the function b is positive provided that $\gamma > \sup|\nabla^2\phi|$.

2.2. *Local existence, uniqueness and continuation of classical solutions*

In this section we present a closed system of PDEs governing the evolution of a flow of plane curves satisfying geometric equation (2). An embedded regular plane curve Γ will be parameterized by a smooth function x: $S^1 \to \mathbb{R}^2$. It means that $\Gamma = \mathrm{Image}(x) := \{x(u), u \in S^1\}$ and $g = |\partial_u x| > 0$. Taking into account the periodic boundary conditions at $u = 0, 1$, we can hereafter identify S^1 with the interval $[0, 1]$. The unit arc-length parameterization of a curve $\Gamma = \mathrm{Image}(x)$ is denoted by s, $ds = g\,du$. The tangent vector $\vec{T}$ and the signed curvature k of Γ satisfy $\vec{T} = \partial_s x = g^{-1}\partial_u x$, $k = \partial_s x \wedge \partial_s^2 x = g^{-3}\partial_u x \wedge \partial_u^2 x$. We choose orientation of the unit inward normal vector $\vec{N}$ in such a way that $\vec{T} \wedge \vec{N} = 1$ where $\vec{a} \wedge \vec{b}$ is the determinant of the 2×2 matrix with column vectors $\vec{a}$, $\vec{b}$. By ν we denote the tangent angle to Γ, i.e. $\nu = \arg(\vec{T})$. Then $\vec{T} = (\cos\nu, \sin\nu)$ and, by Frenét's formulas, $\partial_s\vec{T} = k\vec{N}$, $\partial_s\vec{N} = -k\vec{T}$ and $\partial_s\nu = k$.

Let a regular smooth initial curve $\Gamma_0 = \mathrm{Image}(x_0)$ be given. A family of planar curves $\Gamma_t = \mathrm{Image}(x(., t))$, $t \in [0, T)$, satisfying (2) can be represented by a solution $x = x(u, t)$ to the position vector equation (3). Notice that $\beta = \beta(x, k, \nu)$ depends on x, k, ν and this is why we have to provide equation for the variables k, ν as well as local length $g = |\partial_u x|$, also. The governing system of equations for a general position vector equation (3) has been derived and analyzed by the authors in [21–23] for a wide class of normal velocities β. They are straightforward modifications of well-known geometric equations derived for the case of a zero tangential velocity α (see e.g. [10]). In the case of a nontrivial tangential velocity functional α, the system of parabolic–ordinary governing equations has the following form:

$$\partial_t k = \partial_s^2\beta + \alpha\partial_s k + k^2\beta, \tag{6}$$

$$\partial_t\nu = \beta'_k\partial_s^2\nu + (\alpha + \beta'_\nu)\partial_s\nu + \nabla_x\beta \cdot \vec{T}, \tag{7}$$

$$\partial_t g = -gk\beta + \partial_u\alpha, \tag{8}$$

$$\partial_t x = \beta\vec{N} + \alpha\vec{T} \tag{9}$$

where $(u, t) \in [0, 1] \times (0, T)$, $ds = g\,du$, $\vec{T} = \partial_s x = (\cos\nu, \sin\nu)$, $\vec{N} = \vec{T}^\perp = (-\sin\nu, \cos\nu)$, $\beta = \beta(x, k, \nu)$. A solution (k, ν, g, x) to (6)–(9) is subject to initial conditions

$$k(., 0) = k_0, \quad \nu(., 0) = \nu_0, \quad g(., 0) = g_0, \quad x(., 0) = x_0(\cdot)$$

and periodic boundary conditions at $u = 0, 1$ except of ν for which we require the boundary condition $\nu(1,t) \equiv \nu(0,t) \bmod(2\pi)$. The initial conditions for k_0, ν_0, g_0 and x_0 must satisfy natural compatibility constraints: $g_0 = |\partial_u x_0| > 0$, $k_0 = g_0^{-3}\partial_u x_0 \wedge \partial_u^2 x_0$, $\partial_u \nu_0 = g_0 k_0$ following from the equation $k = \partial_s x \wedge \partial_s^2 x$ and Frenét's formulas applied to the initial curve $\Gamma_0 = \text{Image}(x_0)$. Notice that the system of governing equations consists of coupled parabolic–ordinary differential equations.

Since α enters the governing equations, a solution k, ν, g, x to (6)–(9) does depend on α. On the other hand, the family of planar curves $\Gamma_t = \text{Image}(x(.,t)), t \in [0, T)$, is independent of a particular choice of the tangential velocity α as it does not change the shape of a curve. The tangential velocity α can therefore be considered as a free parameter to be determined in a suitable way. For example, in the plain vanilla curve shortening equation $\nu = k$, we can write equation (3) in the form $\partial_t x = \partial_s^2 x = g^{-1}\partial_u(g^{-1}\partial_u x) + \alpha g^{-1}\partial_u x$, where $g = |\partial_u x|$. Epstein and Gage [9] showed how this degenerate parabolic equation (g need not be smooth enough) can be turned into the strictly parabolic equation $\partial_t x = \partial_s^2 x = g^{-2}\partial_u^2 x)$ by choosing the tangential term α in the form $\alpha = g^{-1}\partial_u(g^{-1})\partial_u x$. This trick is known as 'De Turck's trick' named after De Turck [8] who used this approach to prove short time existence for the Ricci flow. Numerical aspects of this 'trick' have been discussed by Deckelnick in [5]. In general, we allow the tangential velocity functional α appearing in (6)–(9) to be dependent on k, ν, g, x in various ways including nonlocal dependence, in particular (see section 4 for details).

Let us denote $\Phi = (k, \nu, g, x)$. Let $0 < \varrho < 1$ be fixed. By E_k we denote the following scale of Banach spaces (manifolds)

$$E_k = c^{2k+\varrho} \times c_*^{2k+\varrho} \times c^{1+\varrho} \times (c^{2k+\varrho})^2 \tag{10}$$

where $k = 0, 1$, and $c^{2k+\varrho} = c^{2k+\varrho}(S^1)$ is the 'little' Hölder space, i.e. the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{2k+\varrho}(S^1)$ (see [1]). By $c_*^{2k+\varrho}(S^1)$ we have denoted the Banach manifold $c_*^{2k+\varrho}(S^1) = \{\nu\colon \mathbb{R} \to \mathbb{R}, \vec{T} = (\cos\nu, \sin\nu) \in (c^{2k+\varrho}(S^1))^2\}$. Concerning the tangential velocity α we will assume

$$\alpha \in C^1(\mathcal{O}_{\frac{1}{2}}, c^{2+\varrho}(S^1)) \tag{11}$$

for any bounded open subset $\mathcal{O}_{\frac{1}{2}} \subset E_{\frac{1}{2}}$ such that $g > 0$ for any $(k, \nu, g, x) \in \mathcal{O}_{\frac{1}{2}}$.

In the rest of this section we recall a general result on local existence and uniqueness a classical solution of the governing system of equations (6)–(9). The normal velocity β defined as in (4) belongs to a wide class of normal velocities for which local existence of classical solutions has been shown in [22,23]. This result is based on the abstract theory of nonlinear analytic semigroups developed by Angenent in [1] and it utilizes the so-called maximal regularity theory for abstract parabolic equations.

THEOREM 2.1 ([22, Theorem 3.1]) *Assume $\Phi_0 = (k_0, \nu_0, g_0, x_0) \in E_1$ where k_0 is the curvature, ν_0 is the tangential vector, $g_0 = |\partial_u x_0| > 0$ is the local length element of an initial regular closed curve $\Gamma_0 = \text{Image}(x_0)$ and the Banach space E_k is defined as in* (10). *Assume $\beta = \beta(x, k, \nu)$ is a C^4 smooth and 2π-periodic function in the ν variable such that $\min_{\Gamma_0} \beta'_k(x_0, k_0, \nu_0) > 0$ and α satisfies* (11). *Then there exists a unique solution $\Phi = (k, \nu, g, x) \in C([0, T], E_1) \cap C^1([0, T], E_0)$ of the governing system of equations* (6)–(9)

defined on some small time interval $[0, T]$, $T > 0$. *Moreover, if* Φ *is a maximal solution defined on* $[0, T_{\max})$ *then we have either* $T_{\max} = +\infty$ *or* $\liminf_{t\to T_{\max}^-} \min_{\Gamma_t} \beta'_k(x, k, \nu) = 0$ *or* $T_{\max} < +\infty$ *and* $\max_{\Gamma_t} |k| \to \infty$ *as* $t \to T_{\max}$.

3. Qualitative behavior of solutions

3.1. *First integrals and conserved quantities*

The aim of this section is to show that the flow of surface curves driven by geometric equation (1) is gradient-like, i.e. there exists a Lyapunov functional nonincreasing along the trajectories. In the case there is no external force $\mathcal{F}$ in (1), the length functional $\mathcal{L}_t = \text{Length}\ (\mathcal{G}_t)$ is a Lyapunov functional because its time derivative $(\mathrm{d}/\mathrm{d}t)\mathcal{L}_t$ satisfies the well-known geometric identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_t = -\int_{\mathcal{G}_t} \mathcal{K}_g \mathcal{V}\, \mathrm{d}S \tag{12}$$

and the right-hand side of (12) is nonpositive in the case $\mathcal{V} = \mathcal{K}_g$. The main purpose of the next proposition is to generalize (12) for the case of a nontrivial external force $\mathcal{F}$.

PROPOSITION 3.1 *Let* $\mathcal{H}\colon \mathbb{R} \to \mathbb{R}$ *be a solution to the ODE*: $\mathcal{H}'(z) = \gamma(z)\mathcal{H}(z)$, $z \in \mathbb{R}$. *If the family* $\mathcal{G}_t, t \in [0, T)$, *of surface curves evolves according to the normal velocity* $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ *where* $\mathcal{F} = \vec{G}\cdot\vec{\mathcal{N}}$ *and* $\vec{G} = -(0, 0, \gamma(z))$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{G}_t} \mathcal{H}(z)\, \mathrm{d}S = -\int_{\mathcal{G}_t} \mathcal{V}^2 \mathcal{H}(z)\, \mathrm{d}S.$$

Proof For the sake of simplicity we take $\alpha = 0$ in the proof of this statement. Other choices of α, however, do not change the result as the curve $\Gamma_t = \text{Image}(x(., t))$ is independent of a particular choice of tangential redistribution and so does any other geometric quantity evaluated over the curve Γ_t. To simplify notation, we write $\mathcal{H}$ instead of $\mathcal{H}(\phi(x))$ and use identity $\partial_s\phi = \nabla\phi\cdot\vec{T}$. Clearly,

$$\int_{\mathcal{G}} \mathcal{H}\, \mathrm{d}S = \int_{\Gamma} \mathcal{H}\big(1 + (\partial_s\phi)^2\big)^{1/2} \mathrm{d}s = \int_{S^1} \mathcal{H}\big(1 + (\partial_s\phi)^2\big)^{1/2} g\, \mathrm{d}u.$$

Moreover, as $\partial_t\vec{T} = \partial_t(\cos\nu, \sin\nu) = \partial_t\nu\vec{N} = \partial_s\beta\vec{N}$ (see (7)) we have

$$\partial_t\big(1 + (\partial_s\phi)^2\big)^{1/2} = \frac{\partial_s\phi}{\big(1 + (\partial_s\phi)^2\big)^{1/2}}\Big(\partial_s(\beta\nabla\phi\cdot\vec{N}) + k\beta\partial_s\phi\Big)$$

and

$$\partial_s\frac{\partial_s\phi}{(1 + (\partial_s\phi)^2)^{1/2}} = \frac{\vec{T}^T\nabla^2\phi\,\vec{T} + k\nabla\phi\cdot\vec{N}}{\big(1 + (\partial_s\phi)^2\big)^{3/2}}. \tag{13}$$

By the assumption made on $\mathcal{H}$, due to (8), and using integration by parts we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{G}_t}\mathcal{H}\,\mathrm{d}S &= \int_{\Gamma_t}\partial_t\Big(\big(1+(\partial_s\phi)^2\big)^{1/2}\mathcal{H}\Big) - \big(1+(\partial_s\phi)^2\big)^{1/2}\mathcal{H}k\beta\,\mathrm{d}s\\
&= \int_{\Gamma_t}(1+(\partial_s\phi)^2)^{1/2}\beta\Big(\mathcal{H}'\nabla\phi\cdot\vec{N} + \frac{\mathcal{H}k(\partial_s\phi)^2}{1+(\partial_s\phi)^2} - \mathcal{H}k\Big)\,\mathrm{d}s\\
&\quad + \int_{\Gamma_t}\frac{\mathcal{H}\partial_s\phi}{\big(1+(\nabla\phi\cdot\vec{T})^2\big)^{1/2}}\partial_s(\beta\nabla\phi\cdot\vec{N})\,\mathrm{d}s\\
&= \int_{\Gamma_t}\big(1+(\partial_s\phi)^2\big)^{1/2}\beta\mathcal{H}\Big(\gamma\nabla\phi\cdot\vec{N} - \frac{k}{1+(\partial_s\phi)^2}\Big)\,\mathrm{d}s\\
&\quad - \int_{\Gamma_t}\beta\mathcal{H}\nabla\phi\cdot\vec{N}\Bigg(\partial_s\frac{\partial_s\phi}{\big(1+(\partial_s\phi)^2\big)^{1/2}} + \frac{\gamma(\partial_s\phi)^2}{\big(1+(\partial_s\phi)^2\big)^{1/2}}\Bigg)\,\mathrm{d}s\\
&= \int_{\Gamma_t}\frac{\beta\mathcal{H}}{\big(1+(\partial_s\phi)^2\big)^{1/2}}\Bigg(\Big(\gamma - \frac{\vec{T}^T\nabla^2\phi\,\vec{T}}{1+(\partial_s\phi)^2}\Big)\nabla\phi\cdot\vec{N} - k\frac{1+|\nabla\phi|^2}{1+(\partial_s\phi)^2}\Bigg)\\
&= -\int_{\Gamma_t}\frac{1+|\nabla\phi|^2}{\big(1+(\partial_s\phi)^2\big)^{1/2}}\beta^2\mathcal{H}\,\mathrm{d}s = -\int_{\mathcal{G}_t}\mathcal{V}^2\mathcal{H}\,\mathrm{d}S,
\end{aligned}
$$

as claimed. Note that we have used the identities $1+(\partial_s\phi)^2+(\nabla\phi\cdot\vec{N})^2 = 1+|\nabla\phi|^2$ and $\mathcal{V}^2 = \beta^2(1+|\nabla\phi|^2)/(1+(\partial_s\phi)^2)$ throughout the derivation of the above identities. ■

Clearly, if $\mathcal{V}=\mathcal{K}_g$ then $\gamma=0$ and $H\equiv 1$ is a solution to $H'=\gamma H$. As $\mathcal{L}_t=\int_{\mathcal{G}_t}\mathrm{d}S$, we can conclude from Proposition 3.1 that $\mathrm{d}/\mathrm{d}t\,\mathcal{L}_t = -\int_{\mathcal{G}_t}\mathcal{V}^2\,\mathrm{d}S$ which is exactly equation (12). Furthermore, it follows from Proposition 3.1 that the functional $\int_{\mathcal{G}}\mathcal{H}(z)\,\mathrm{d}S$ is a Lyapunov-like functional nonincreasing along trajectories of solutions to (1). The next result is, therefore, a consequence of Proposition 3.1.

Corollary 3.1 *There exists no nontrivial time periodic family of surface curves* $\{\mathcal{G}_t, t\geq 0\}$, *with the normal velocity* $\mathcal{V}$ *satisfying the geometric equation* $\mathcal{V}=\mathcal{K}_g+\mathcal{F}$ *where* $\mathcal{F}=\vec{G}\cdot\vec{\mathcal{N}}$ *and* $\vec{G}=-(0,0,\gamma(z))$.

3.2. *Closed stationary curves and their stability*

In this section we analyze the stationary surface curves with respect to the normal velocity $\mathcal{V}=\mathcal{K}_g+\mathcal{F}$, i.e. the surface curves satisfying $\mathcal{K}_g+\mathcal{F}=0$. Since there is one-to-one correspondence between the flow of curves on a given surface and the flow of vertically projected planar curves, we are only concerned with stationary planar curves satisfying $\beta(x,k,\nu)=0$ where β is given by (4). We will also analyze the stability of such curves with respect to small perturbations in the normal velocity.

Definition 3.1 A closed smooth planar curve $\bar{\Gamma}=\mathrm{Image}(\bar{x})$ is called a stationary curve with respect to the normal velocity β iff $\beta(\bar{x},\bar{k},\bar{\nu})=0$ on $\bar{\Gamma}$ where $\bar{x}$, $\bar{k}$ and $\bar{\nu}$ are the position vector, curvature and tangential angle of the curve $\bar{\Gamma}$.

3.3. *Principle of linearized stability*

Since the presence of arbitrary tangential velocity functional in the system of governing equations has no impact on the shape of evolving curves $\Gamma_t = \mathrm{Image}(x(.,t))$, we take $\alpha = 0$ in the analysis of stability of stationary curves. The governing system of equations (6)–(9) reduces to:

$$\begin{aligned} \partial_t k &= g^{-1}\partial_u(g^{-1}\partial_u\beta) + k^2\beta, & \partial_t \nu &= g^{-1}\partial_u\beta, \\ \partial_t g &= -gk\beta, & \partial_t x &= \beta\vec{N}, \end{aligned} \tag{14}$$

$u \in S^1, t \in (0,T)$. Let $\bar{\Gamma} = \mathrm{Image}(\bar{x})$ be a stationary curve having the curvature $\bar{k}$, tangential angle $\bar{\nu}$, the local length $\bar{g}$, position vector $\bar{x}$ and the unit normal vector $\bar{\vec{N}}$. In order to analyze stability of $\bar{\Gamma}$ we have to investigate the behavior of infinitesimal variations of k, ν, g and x. Variations from a steady state $(\bar{k}, \bar{\nu}, \bar{g}, \bar{x})$ will be denoted by $(\delta^k, \delta^\nu, \delta^g, \delta^x)$. Since $\bar{\beta} = \beta(\bar{x}, \bar{k}, \bar{\nu}) = 0$ on $\bar{\Gamma}$ we have $\partial_u\bar{\beta} = \partial_u^2\bar{\beta} = 0$ on $\bar{\Gamma}$. Hence infinitesimally small variations δ^k, δ^ν, δ^g and δ^x satisfy the linearized system

$$\begin{aligned} \partial_t\delta^k &= \bar{g}^{-1}\partial_u(\bar{g}^{-1}\partial_u\delta^\beta) + \bar{k}^2\delta^\beta, & \partial_t\delta^\nu &= \bar{g}^{-1}\partial_u\delta^\beta, \\ \partial_t\delta^g &= -\bar{g}\bar{k}\delta^\beta, & \partial_t\delta^x &= \delta^\beta\bar{\vec{N}} \end{aligned} \tag{15}$$

for $u \in S^1$, $t > 0$. Here $\delta^\beta = \beta(\bar{x}+\delta^x, \bar{k}+\delta^k, \bar{\nu}+\delta^\nu) - \beta(\bar{x}, \bar{k}, \bar{\nu}) = \nabla_x\bar{\beta}\cdot\delta^x + \bar{\beta}'_k\delta^k + \bar{\beta}'_\nu\delta^\nu +$ higher order terms. Clearly, all variations δ^k, δ^ν, δ^g, δ^x, δ^β are subject to periodic boundary conditions at $u = 0, 1$. As $\nabla_x\bar{\beta} = \nabla_x\beta(\bar{x}, \bar{k}, \bar{\nu})$, $\bar{\beta}'_k = \beta'_k(\bar{x}, \bar{k}, \bar{\nu})$, and $\bar{\beta}'_\nu = \beta'_\nu(\bar{x}, \bar{k}, \bar{\nu})$ do not depend on time the total variation δ^β satisfies the scalar parabolic equation

$$\begin{aligned} \partial_t\delta^\beta &= \nabla_x\bar{\beta}\cdot\partial_t\delta^x + \bar{\beta}'_k\partial_t\delta^k + \bar{\beta}'_\nu\partial_t\delta^\nu \\ &= \bar{\beta}'_k\bar{g}^{-1}\partial_u(\bar{g}^{-1}\partial_u\delta^\beta) + \bar{\beta}'_\nu\bar{g}^{-1}\partial_u\delta^\beta + (\bar{\beta}'_k\bar{k}^2 + \nabla_x\bar{\beta}\bar{\vec{N}})\delta^\beta, \end{aligned} \tag{16}$$

i.e. $\partial_t\delta^\beta = P\partial_u^2\delta^\beta + R\partial_u\delta^\beta + Q\delta^\beta$,
where

$$P = \bar{g}^{-2}\bar{\beta}'_k, \quad R = \bar{g}^{-1}\bar{\beta}'_\nu + \bar{g}^{-1}\bar{\beta}'_k\partial_u\bar{g}^{-1}, \quad Q = \bar{\beta}'_k\bar{k}^2 + \nabla_x\bar{\beta}\cdot\bar{\vec{N}}. \tag{17}$$

Functions P, Q and R are 1-periodic in u variable and depend on the stationary curve $\bar{\Gamma}$ only. A solution δ^β to (16) is subject to periodic boundary conditions at $u = 0, 1$.

Our concept of stability of stationary curves is based on the analysis of an infinitesimally small variation δ^β in the normal velocity. Roughly speaking, if the variation $\delta^\beta(.,t)$ decays to zero as $t \to \infty$, we say that $\bar{\Gamma}$ is stable. Otherwise $\bar{\Gamma}$ is unstable. More precisely,

Definition 3.2 A stationary curve $\bar{\Gamma} = \mathrm{Image}(\bar{x})$ is called linearly stable if the trivial solution to (15) is exponentially asymptotically stable in the space $L^2(S^1)$, i.e. there exist constants $M, \omega > 0$ such that $\|\delta^\beta(.,t)\|_{L^2(S^1)} \le Me^{-\omega t}\|\delta^\beta(.,0)\|_{L^2(S^1)}$ for any initial condition $\delta^\beta(.,0) \in L^2(S^1)$. A stationary curve $\bar{\Gamma}$ is called linearly unstable if the trivial solution to (15) is unstable in $L^2(S^1)$ norm.

In the next lemma we show that, under additional assumptions made on coefficients P, R, Q, the right-hand side of (16), i.e.

$$A\psi := P\psi'' + R\psi' + Q\psi \tag{18}$$

defines a selfadjoint second-order differential operator in a suitable weighted Lebesgue space.

Lemma 3.1 *Suppose $P, R, Q \in C^1(S^1), P > 0$. If $\int_0^1 (R(u)/P(u))\mathrm{d}u = 0$ then the linear operator A: $D(A) \subset L^2(S^1, w) \to L^2(S^1, w)$, $D(A) = W^{2,2}(S^1)$, is selfadjoint operator in the weighted Lebesgue space $L^2(S^1, w)$ with the weight defined as: $w(u) = P(u)^{-1}\exp(\int_0^u (R(v)/P(v))\mathrm{d}v)$.*

Proof Denote $[f, g] = \int_0^1 f(u)g(u)w(u)\,\mathrm{d}u$ the inner product in $L^2(S^1, w)$. Due to the assumptions made on P, R we have $w \in C^1(S^1)$. Therefore $[Af, g] - [f, Ag] = \int_0^1 (f''g - fg'')Pw + (f'g - fg')Rw\,\mathrm{d}u = \int_0^1 (f'g - fg')(Rw - (Pw)')\,\mathrm{d}u = 0$ because $Rw = (Pw)'$. Hence A: $D(A) \subset L^2(S^1, w) \to L^2(S^1, w)$ is selfadjoint. Moreover,

$$[A\psi, \psi] = \int_0^1 (\psi^2 Q - \psi'^2 P)w\,\mathrm{d}u. \tag{19}$$

Notice that the weight w associated with coefficients P, R from (16) is given by

$$w(u) = \frac{\bar{g}}{\bar{\beta}'_k}\exp\left(\int_0^u \frac{\bar{\beta}'_\nu}{\bar{\beta}'_k}\bar{g}\,\mathrm{d}u\right) \tag{20}$$

up to a multiplicative constant depending on $\bar{g}(0)$. It is worth noting that the proof of the previous lemma strongly relies on 1-periodicity of the weight function w. Therefore, in order to apply this result in the stability analysis, we have to assume the condition $\int_0^1 (\bar{\beta}'_\nu/\bar{\beta}'_k)\bar{g}\,\mathrm{d}u = \int_{\bar{\Gamma}} (\bar{\beta}'_\nu/\bar{\beta}'_k)\,\mathrm{d}s = 0$. In the next definition and lemma, we introduce the concept of the so-called *admissible normal velocity* and prove admissibility of a wide class of normal velocities including, in particular, the normal velocity β of vertically projected surface curves satisfying the geometric equation (1).

Definition 3.3 A C^1 smooth function $\beta = \beta(x, k, \nu)$ is called an admissible normal velocity if

$$\int_{\bar{\Gamma}} \frac{\bar{\beta}'_\nu}{\bar{\beta}'_k}\,\mathrm{d}s = 0 \tag{21}$$

for any closed stationary curve $\bar{\Gamma} = \mathrm{Image}(\bar{x})$ where $\bar{\beta} = \beta(\bar{x}, \bar{k}, \bar{\nu})$.

The aim of the next proposition is to prove admissibility of the normal velocity β defined as in (4) for vertically projected surface curves. Although we will prove admissibility for a slightly larger class of normal velocities the most important part of this proposition is contained in part (c) of Proposition 3.2.

Proposition 3.2 *The following functions are admissible normal velocities:*

(a) $\beta(x, k) = a(x)k + c(x)$ *where* $a(x) > 0, c(x)$ *are* C^1 *smooth functions*;
(b) $\beta(x, k, \nu) = a(\phi(x))k - b(\phi(x))\nabla\phi \cdot \vec{N}$ *where* $a(\phi) > 0$, $b(\phi)$ *are* C^1 *smooth functions and* $\phi(x)$ *is* C^2 *smooth*;

(c) $\beta(x,k,\nu) = a(x,\nu)k - b(x,\nu)\nabla\phi \cdot \vec{N}$ *where* a, b *are defined as in* (5) *and* $\phi(x)$ *is a* C^2 *smooth function.*

Proof The proof of the statement (a) is trivial because $\beta'_\nu = 0$. To prove (b) we note that $\vec{N} = (-\sin\nu, \cos\nu)$, $\partial_\nu \vec{N} = -\vec{T}$ and $\partial_s\phi = \nabla\phi \cdot \vec{T}$. For any stationary curve $\bar{\Gamma}$ we have

$$\int_{\bar{\Gamma}} \frac{\beta'_\nu}{\beta'_k}\,\mathrm{d}s = \int_{\bar{\Gamma}} \frac{b(\phi(x))\partial_s\phi(x)}{a(\phi(x))}\,\mathrm{d}s = \int_{\bar{\Gamma}} \partial_s\Phi\,\mathrm{d}s = 0$$

where Φ is a primitive function to b/a. In order to simplify the proof of (c) let us denote

$$d := \frac{\vec{T}^T\nabla^2\phi\,\vec{T}}{1+|\nabla\phi|^2}\nabla\phi\cdot\vec{N} \quad \text{and} \quad h := \partial_s\phi = \nabla\phi\cdot\vec{T}. \tag{22}$$

Let $\bar{\Gamma}$ be a stationary curve with respect to β. Then $\beta(x,k,\nu) = a(k+d) - (\gamma(\phi)/(1+|\nabla\phi|^2))\nabla\phi\cdot\vec{N}$ and thus

$$k + d = \frac{\gamma(\phi)}{1+|\nabla\phi|^2}\frac{\nabla\phi\cdot\vec{N}}{a} \tag{23}$$

on $\bar{\Gamma}$. Moreover,

$$\begin{aligned}
\frac{\beta'_\nu}{\beta'_k} &= \frac{a'_\nu}{a}(k+d) + d'_\nu + \frac{\gamma(\phi)}{1+|\nabla\phi|^2}\frac{\nabla\phi.\vec{T}}{a} \\
&= \frac{\gamma(\phi)}{1+|\nabla\phi|^2}\left(\frac{a'_\nu}{a^2}\nabla\phi.\vec{N} + \frac{h}{a}\right) + d'_\nu \\
&= \frac{\gamma(\phi)}{1+|\nabla\phi|^2}h\left(1 + h^2 - 2(\nabla\phi.\vec{N})^2\right) + d'_\nu
\end{aligned} \tag{24}$$

because $a'_\nu = -2a^2(\nabla\phi\cdot\vec{T})(\nabla\phi\cdot\vec{N}) = -2a^2h(\nabla\phi\cdot\vec{N})$ and $1/a = 1+h^2$. It follows from (22) and (23) and the identity $1+|\nabla\phi|^2 = 1+(\nabla\phi\cdot\vec{T})^2 + (\nabla\phi\cdot\vec{N})^2 = 1+h^2+(\nabla\phi\cdot\vec{N})^2$ that

$$\begin{aligned}
&(1+h^2)^{3/2}\partial_s\left(\frac{h}{(1+h^2)^{1/2}}\right) = \vec{T}^T\nabla^2\phi\,\vec{T} + k\nabla\phi\cdot\vec{N} \\
&= \vec{T}^T\nabla^2\phi\,\vec{T}\left(1 - \frac{(\nabla\phi\cdot\vec{N})^2}{1+|\nabla\phi|^2}\right) + \frac{\gamma(\phi)(1+h^2)}{1+|\nabla\phi|^2}(\nabla\phi\cdot\vec{N})^2 \\
&= \frac{1+h^2}{1+|\nabla\phi|^2}\left(T^T\nabla^2\phi\,\vec{T} + \gamma(\phi)(\nabla\phi\cdot\vec{N})^2\right)
\end{aligned} \tag{25}$$

on $\bar{\Gamma}$. Since

$$d'_\nu = \frac{1}{1+|\nabla\phi|^2}\left(2T^T\nabla^2\phi\,\vec{N}(\nabla\phi\cdot\vec{N}) - T^T\nabla^2\phi\,\vec{T}(\nabla\phi\cdot\vec{T})\right)$$

and

$$\partial_s \ln(1+|\nabla\phi|^2) = \frac{2\nabla\phi\nabla^2\phi\,\vec{T}}{1+|\nabla\phi|^2} = \frac{2}{1+|\nabla\phi|^2}\Big(T^T\nabla^2\phi\,\vec{T}(\nabla\phi\cdot\vec{T}) + T^T\nabla^2\phi\,\vec{N}(\nabla\phi\cdot\vec{N})\Big)$$

we obtain from (24) and (25) that the following identity is satisfied on any stationary curve $\bar{\Gamma}$:

$$\begin{aligned}\frac{\beta'_\nu}{\beta'_k} &= \partial_s \ln(1+|\nabla\phi|^2) - 3h\frac{T^T\nabla^2\phi\,\vec{T}}{1+|\nabla\phi|^2} + \frac{\gamma(\phi)}{1+|\nabla\phi|^2}h\Big(1+h^2-2(\nabla\phi\cdot\vec{N})^2\Big)\\ &= \partial_s \ln(1+|\nabla\phi|^2) - 3h(1+h^2)^{1/2}\partial_s\Big(\frac{h}{(1+h^2)^{1/2}}\Big)\\ &\quad + \frac{\gamma(\phi)}{1+|\nabla\phi|^2}h\Big(1+h^2+(\nabla\phi\cdot\vec{N})^2\Big)\\ &= \partial_s \ln(1+|\nabla\phi|^2) - 3\partial_s \ln\sqrt{1+h^2} + \gamma(\phi)h = \partial_s\Big(\ln\frac{1+|\nabla\phi|^2}{(1+h^2)^{3/2}} + \Omega(\phi)\Big)\end{aligned}$$

where $\Omega(\phi)$ is the primitive function to $\gamma(\phi)$, i.e. $\Omega'(\phi) = \gamma(\phi)$. Hence $\int_{\bar{\Gamma}}(\beta'_\nu/\beta'_k)\mathrm{d}s = 0$, as claimed. ■

As a consequence of the previous proposition and Lemma 3.1 we conclude:

THEOREM 3.1 *Suppose that $\bar{\Gamma}$ is a stationary curve with respect to the normal velocity β given by* (4), *i.e. $\bar{\Gamma}$ is the vertical projection of a stationary surface curve $\mathcal{G}$. Let λ_1 be the largest eigenvalue of the periodic Sturm-Liouville problem*

$$(p\psi')' + q\psi = \lambda\psi, \quad \psi(0) = \psi(1), \quad \psi'(0) = \psi'(1) \tag{26}$$

where $p := Pw,\ q := Qw$ and P, Q, w were defined as in (17) and (20). *Then*

(1) $\bar{\Gamma}$ *is linearly stable if* $\lambda_1 < 0$;
(2) $\bar{\Gamma}$ *is linearly unstable if* $\lambda_1 > 0$.

Proof To prove stability of a trivial steady state of equation (16) for the variation δ^β we have to investigate the spectral properties of the linear operator A defined as in (18). According to Lemma 3.1 and Proposition 2.1 the operator A is selfadjoint in the weighted Lebesgue space $L^2(S^1, w)$. By (19) we have $[A\psi, \psi] = \int_0^1(\psi^2 Q - \psi'^2 P)w\,\mathrm{d}u$ and the spectrum $\sigma(A) = \sigma_P(A)$ consists of eigenvalues to the Sturm-Liouville periodic boundary value problem (26). Now if we assume $\lambda_1 < 0$ then the trivial solution to (16) is exponentially asymptotically stable in $L^2(S^1)$ phase space. Hence $\bar{\Gamma}$ is linearly stable. On the other hand, if $\lambda_1 > 0$ the trivial solution to (16) is linearly unstable and so is the curve $\bar{\Gamma}$. ■

In order to determine the sign of the first eigenvalue λ_1 to the Sturm-Liouville problem (26) it might be useful to note that λ_1 is given by Rayleigh quotient $\lambda_1 = \sup_{\psi\in D(A)}[A\psi, \psi]/[\psi, \psi]$.

COROLLARY 3.2 *A stationary curve $\bar{\Gamma}$ is linearly stable if* $\sup_{\bar{\Gamma}} Q < 0$ *and it is linearly unstable if* $\int_0^1 Qw\,\mathrm{d}u > 0$ *where* $Q = \bar{\beta}'_k \bar{k}^2 + \nabla_x \bar{\beta} \cdot \vec{\bar{N}}$ *and* w *is the weight defined as in* (20).

Proof Since $[A\psi, \psi] = \int_0^1 (\psi^2 Q - \psi'^2 P) w\,\mathrm{d}u$ we have $\lambda_1 < 0$ in the case (1). On the other hand, in the case (2), we can choose a constant test function $\psi \equiv 1$ to show that $\lambda_1 > 0$. The statement now follows from Theorem 3.1. ■

3.4. *Radially symmetric solutions and their stability*

In this section we restrict our attention to the special solutions to the geometric equation (1). Throughout this section, we assume that the surface $\mathcal{M}$ is radially symmetric with respect to the origin, i.e. there exists a smooth function $f: \mathbb{R}_0^+ \to \mathbb{R}$, $f'(0) = 0$, such that

$$\phi(x) = f(|x|), \quad \mathcal{M} = \{(x, \phi(x)),\ x \in \mathbb{R}^2\}.$$

As already pointed out in the previous section, we can project surface curves into the plane and study evolution of planar curves satisfying (4) instead of the evolution of surface curves. Furthermore, if we assume that the initial curve is also radially symmetric, i.e. $\Gamma_0 = \{x, |x| = r_0\}$ then it follows from uniqueness of a solution that the evolving family of surface curve on a radially symmetric surface $\mathcal{M}$ consists of radially symmetric curves,

$$\Gamma_t = \{x, |x| = r(t)\}. \tag{27}$$

On any radially symmetric curve $\Gamma = \{x, |x| = r\}$ the following identities are satisfied:

$$\begin{aligned}
&\nabla r = \frac{x}{r}, \quad \vec{N} = -\frac{x}{r}, \quad \nabla\phi = \frac{f'(r)}{r} x, \quad |\nabla\phi|^2 = f'(r)^2, \quad k = \frac{1}{r},\\
&\nabla^2\phi = \frac{f'(r)}{r} I + \frac{1}{r}\left(\frac{f'(r)}{r}\right)' x \otimes x, \qquad \vec{T}^T \nabla^2\phi\, \vec{T} = \frac{f'(r)}{r}.\\
&\nabla\phi \cdot \vec{T} = 0, \quad \nabla\phi \cdot \vec{N} = -f'(r), \quad x \cdot \vec{T} = 0,
\end{aligned} \tag{28}$$

Using the above identities it easy easy to verify that the normal velocity β given by (4) on $\Gamma = \{x, |x| = r\}$ can be expressed as follows:

$$\beta(x, k, \nu) = F(r) \equiv \frac{1}{r} + \left(\gamma(f(r)) - \frac{f'(r)}{r}\right) \frac{f'(r)}{1 + f'(r)^2}. \tag{29}$$

Since $\partial_t x \cdot \vec{N} = -\mathrm{d}r/\mathrm{d}t$ the radius $r = r(t)$, $t > 0$, of the evolving family of planar curves (27) satisfying (4), is a solution to the ordinary differential equation

$$-\frac{\mathrm{d}r}{\mathrm{d}t} = F(r), \quad r(0) = r_0. \tag{30}$$

PROPOSITION 3.3 *A radially symmetric curve* $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ *is a stationary curve iff* $\bar{r} \in \mathbb{R}^+$ *is a solution to the equation* $\gamma(f(r))f'(r)r = -1$.

Example 3.1 If $\phi(x) = 1 - (1/2)|x|^2$ and $\gamma = \text{const} > 0$ then there exists a unique radially symmetric stationary curve $\bar{\Gamma}$ with the radius $\bar{r} = 1/\sqrt{\gamma}$.

Example 3.2 If $\phi(x) = (1 - |x|^2)^2$ and $\gamma = \text{const} > 1$ then there are exactly two radially symmetric stationary curves $\bar{\Gamma}^{\pm}$ with radii $\bar{r}^{\pm}$ given by $\bar{r}^{\pm} = ((1 \pm \sqrt{1 - 1/\gamma})/2)^{1/2}$. Moreover, $\bar{r}^{+} \to 1^{-}$ and $\bar{r}^{-} \to 0^{+}$ as $\gamma \to \infty$.

Equation (30) is an ODE with C^1 smooth right-hand side F. Hence stability of a stationary solution $\bar{r}$ can be deduced from the linearization $F'(\bar{r})$. Clearly, $\bar{r}$ is an exponentially asymptotically stable stationary solution if $F'(\bar{r}) > 0$, and, $\bar{r}$ is linearly unstable if $F'(\bar{r}) < 0$. It is worth to note that the sign condition for $F'(\bar{r})$ enables us to determine stability of a stationary curve $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ only in the phase-space consisting of all radially symmetric curves. In order to extend this result we need the following lemma.

LEMMA 3.2 *If* $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ *is a radially symmetric stationary curve then*

$$\bar{\beta}_k \bar{k}^2 + \nabla_x \bar{\beta} \cdot \vec{\bar{N}} = -F'(\bar{r}).$$

Proof The normal velocity β is given by $\beta = ak - b\nabla\phi \cdot \vec{N}$ with coefficients $a = a(x, \nu), b = b(x, \nu)$ defined as in (5). Long but straightforward calculations based on formulas (28) yield the following identities:

$$\nabla_x a \cdot \vec{N} = 0, \quad \nabla_x(\vec{T}^T \nabla^2\phi\, \vec{T}) \cdot \vec{N} = -\left(\frac{f'(r)}{r}\right)',$$

$$\vec{N}^T \nabla^2\phi\, \vec{N} = f''(r), \quad \nabla_x \frac{1}{1 + |\nabla\phi|^2} \cdot \vec{N} = \frac{2f'(r)f''(r)}{(1 + f'(r)^2)^2 0},$$

which are valid on any radially symmetric curve $\bar{\Gamma} = \{x, |x| = r\}$. Using the above identities we conclude, after some calculations, that

$$\nabla_x \beta \cdot \vec{N} = \frac{1}{1 + f'(r)^2}\left(\frac{f''(r)}{rf'(r)} - \frac{f'(r)^2}{r^2} - \gamma'(f(r))f'(r)^2\right)$$

and thus $\bar{\beta}_k \bar{k}^2 + \nabla_x \bar{\beta} \cdot \vec{\bar{N}} = -F'(\bar{r})$, as claimed. ■

Combining Lemma 3.2 and Theorem 3.1 we obtain

THEOREM 3.2 *Let* $\phi(x) = f(|x|)$ *where* $f: \mathbb{R}_0^+ \to R, f'(0) = 0$, *be a* C^2 *smooth function. A radially symmetric stationary curve* $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ *is linearly stable if* $F'(\bar{r}) > 0$ *and is linearly unstable if* $F'(\bar{r}) < 0$ *where* $F(r)$ *is defined as in* (29).

In Example 3.1 the unique stationary curve $\bar{\Gamma} = \{x, |x| = 1/\sqrt{\gamma}\}$ is always unstable. In Example 3.2 the stationary curve $\bar{\Gamma}^{+} = \{x, |x| = \bar{r}^{+}\}$ is linearly stable whereas $\bar{\Gamma}^{-} = \{x, |x| = \bar{r}^{-}\}$ is linearly unstable.

Example 3.3 If $\phi(x) = f(|x|)$ where $f(r) = \sin(r)/r$ and $\gamma = \text{const} > 1$ then there exist countably many stationary curves $\bar{\Gamma}^{(i)} = \{x, |x| = \bar{r}^{(i)}\}, i \in \mathbb{N}$, where $\bar{r}^{(1)} < \bar{r}^{(2)} < \cdots < \bar{r}^{(n)} < \infty$ are roots of the equation: $\gamma f'(r)r = -1$. Moreover, sgn $F'(\bar{r}^{(i)}) = (-1)^i$ and therefore $\bar{r}^{(2k)}, k \in \mathbb{N}$, are stable and $\bar{r}^{(2k-1)}$, $k \in \mathbb{N}$, are unstable solutions to (30). With regard to Theorem 3.2, stationary curves $\bar{\Gamma}^{(2k)}$, $k \in \mathbb{N}$, are linearly stable whereas $\bar{\Gamma}^{(2k-1)}$, $k \in \mathbb{N}$, are linearly unstable.

4. Examples

In this section we present various numerical experiments describing the flow of surface curves. We consider a flow of curves on a given surface $\mathcal{M} = \mathrm{Graph}(\phi)$ driven by (1). The flow of vertically projected planar curves is therefore driven by the geometric equation (4) with coefficients $a(x, \nu), b(x, \nu)$ defined as in (5). In all numerical experiments to follow, we make use of the numerical scheme for computing the evolution of plane curves satisfying (3) with the normal velocity having the form: $v = a(x, \nu)k + c(x, \nu)$ where $c(x, \nu) = -b(x, \nu)\nabla\phi(x) \cdot \vec{N}$. We refer to [22,23] for detailed derivation and discussion of the numerical scheme based on the so-called flowing finite volume method. It was also shown in [23] that the experimental order of convergence of this scheme is at least one which is often the case for finite volume approximations. Moreover, in [21–23] we have shown the importance of a suitable choice of a tangential velocity functional α entering the governing system of equations (6)–(9). Recall that if α is a solution to the equation:

$$\partial_s\alpha = k\beta - \langle k\beta\rangle_\Gamma + (L/g - 1)\omega, \quad \alpha(0, .) = 0, \tag{31}$$

where L is the length of the plane curve Γ and $\langle k\beta\rangle_\Gamma$ is the average of $k\beta$ over the curve Γ, i.e. $\langle k\beta\rangle_\Gamma = \frac{1}{L}\int_\Gamma k\beta\,\mathrm{d}s$, then we obtain asymptotically uniform parameterization: $g(u, t)/L_t \to 1$ as $t \to T_{\max}$ uniformly with respect to $u \in S^1$ provided that $\omega = \kappa_1 + \kappa_2\langle k\beta\rangle_\Gamma$ and $\kappa_1, \kappa_2 > 0$ are given constants. Here $T_{\max}$ denotes the maximal time of existence of a solution. It might be either finite or infinite. On the other hand, if $\omega = 0$ then tangential velocity preserves relative local length: $g(u, t)/L_t = g(u, 0)/L_0$ for any $u \in S^1$, $t \in (0, T_{\max})$. Construction of a suitable tangential velocity functional α leading to redistribution preserving relative local length has been discussed by Hou *et al.* [12,13]. It has been generalized to the case of asymptotically uniform parameterization by the authors in [21–23]. Notice that the tangential velocity functional α can be uniquely determined from (31) and satisfies the regularity condition (11).

In the example shown in figures 1 and 2 we present numerical results of simulations of a surface flow driven by the geodesic curvature and gravitational-like external force, $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$, on a wavelet surface given by the graph of the function $\phi(x) = f(|x|)$ where $f(r) = \sin(r)/r$ and $\gamma = 2$ (see Example 3.3). In the first example shown in figure 1 (left) we started with an initial surface curve having large variations in the geodesic curvature. The evolving family converges to the stable stationary curve $\bar{\Gamma}^{(4)} = \{x, |x| = \bar{r}^{(4)}\}$ with the second smallest stable radius $r^{(4)}$. Vertical projection of the evolving family to the plane driven by the normal velocity $v = \beta(x, k, \nu)$ is shown in figure 1 (right). In figure 2 we study a surface flow on the same surface as in figure 1 with the same external force. The initial curve is, however, smaller compared to that of figure 1. In this case the evolving family converges to the stable stationary curve $\bar{\Gamma}^{(2)} = \{x, |x| = \bar{r}^{(2)}\}$ with the smallest stable radius $r^{(2)}$. In both examples we chose 100 spatial grid points, the time step $\tau = 0.01$ and the time interval $t \in (0, 12)$ in the experiment depicted in figure 1 and $t \in (0, 5.4)$ for that of figure 2.

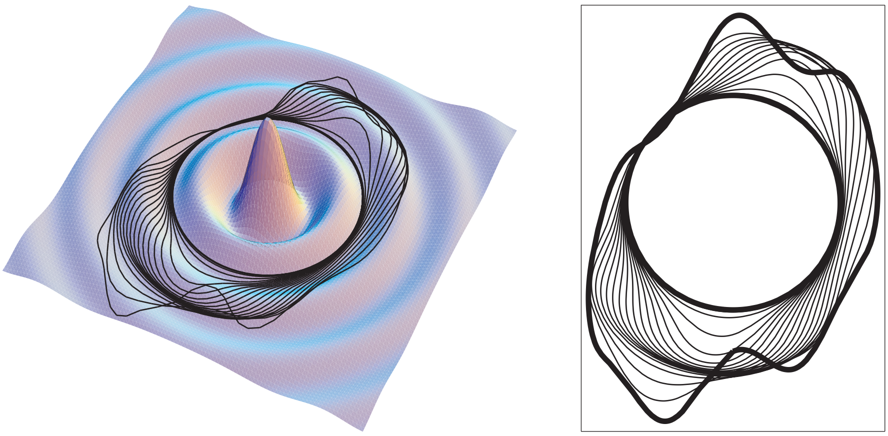

Figure 1. A surface flow on a wavelet like surface (left) and its vertical projection to the plane (right). Surface curves converge to the stable stationary circular curve $\bar{\Gamma}^{(4)} = \{x, |x| = \bar{r}^{(4)}\}$ with the second smallest radius $\bar{r}^{(4)}$.

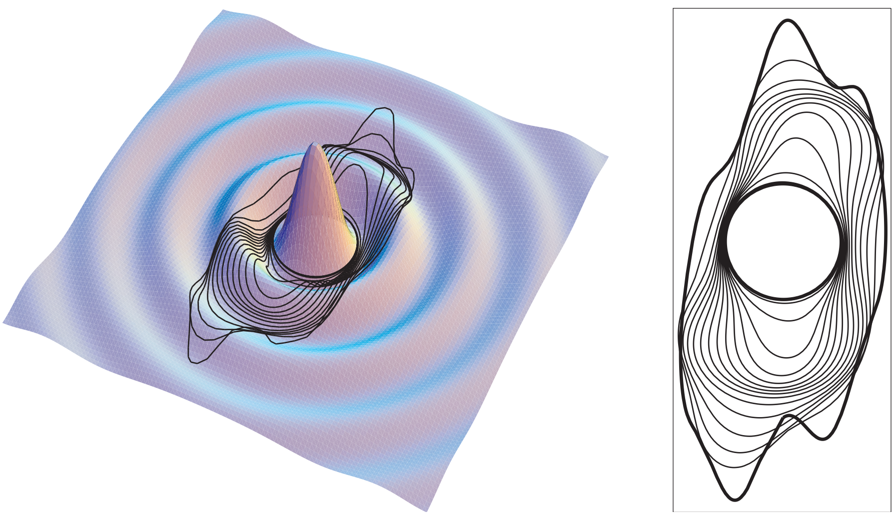

Figure 2. A surface flow on a wavelet like surface (left) and its vertical projection to the plane (right). Surface curves converge to the stable stationary circular curve $\bar{\Gamma}^{(2)} = \{x, |x| = \bar{r}^{(2)}\}$ with the smallest radius $\bar{r}^{(2)}$.

The next set of examples illustrates a geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps. In figure 3 (left) we considered a surface $\mathcal{M}$ defined as a graph of the function $\phi(x) = f(x_1 - 1, x_2) + 3f(x_1 + 1, x_2)$ where $f(x) = 2^{-1/(1-|x|^2)}$ for $|x| < 1$ and $f(x) = 0$ for $|x| \geq 1$ is a smooth bump function. In this example, the evolving family of surface curves shrinks to a point in finite time. On the other hand, in figure 3 (right) we considered the function $\phi(x) = 3(f(x_1 - 1, x_2) + f(x_1 + 1, x_2))$. We took the time step $\tau = 0.0002$. As an initial curve we chose an ellipse centered at the origin with axes 2 and $\sqrt{2}$. The spatial mesh contained 400 grid points. The initial curve was evolved until the time $T = 13$. As it can be seen from figure 3 the evolving family of surface

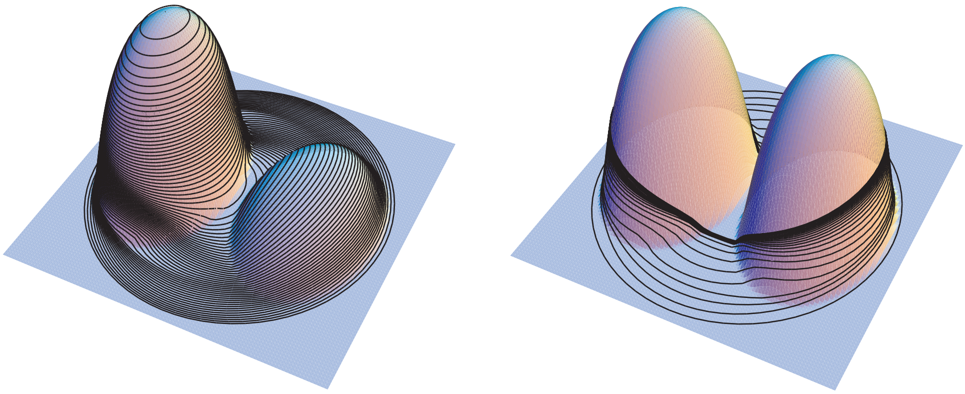

Figure 3. A geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps having different heights (left). The flow approaching a stable closed geodesic curve on a surface with two sufficiently high humps (right).

curves approaches a closed geodesic curve $\bar{\Gamma}$ as $t \to \infty$. It is worth noting that $\sup_{\bar{\Gamma}} Q = 0.000275 > 0$ and, therefore, the simple stability criterion contained in Corollary 3.2 cannot be used and we had to compute the first (largest) eigenvalue of the Sturm-Liouville problem (26). It turns out that $\lambda_1 \approx -0.095$, and, by Theorem 3.1, the stationary curve $\bar{\Gamma}$ is linearly stable.

Finally, we recall that a similar equation to (4) can also be found in the theory of image segmentation in which the goal is to to find object boundaries in the analyzed image. A given image can be represented by its intensity function $I\colon R^2 \to [0,1]$. Let us introduce an auxiliary function $\phi(x) = h(|\nabla I(x)|)$ where h is a smooth edge detector function like e.g. $h(s) = 1/(1+s^2)$ or $h(s) = e^{-s}$. Then the gradient $-\nabla\phi(x)$ has the important geometric property because it points towards the edge where the norm of the gradient ∇I is large. In the so-called *active contour models* one picks up an initial approximation of the closed edge and then constructs an evolving family of plane curves satisfying the geometric equation $v = \phi(x)k - \nabla\phi(x) \cdot \vec{N}$ and thus converging to the edge [14]. In the framework of level set methods, edge detection techniques based on this idea were first discussed by Caselles *et al.* and Malladi *et al.* in [3,17]. Later on, they have been revisited and improved in [4,15]. Our next aim is to show that the geodesic curvature driven flow of surface curves with an external force can be adopted to the edge detection problem. We will consider flow of surface curves with the normal velocity $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ on surface given by the function ϕ constructed as above. The surface $\mathcal{M} = \mathrm{Graph}(\phi)$ has a sharp narrow valley corresponding to points of the image in which the gradient $|\nabla I(x)|$ is very large representing thus an edge in the image. Choosing the gravitational force $\vec{G} = -(0,0,\gamma)$ sufficiently large, one may expect that the evolving family of surface curves 'falls' downward of the sharp narrow valley and hence its vertical projection to the plane converges to an edge of the image. We considered an artificial image with intensity function

$$I(x) = \frac{1}{2} + \frac{1}{\pi}\mathrm{arctg}\left(12.5 - 100\left(|x|\frac{2x_1^2 + |x|^2}{4x_1^2 + |x|^2}\right)^2\right).$$

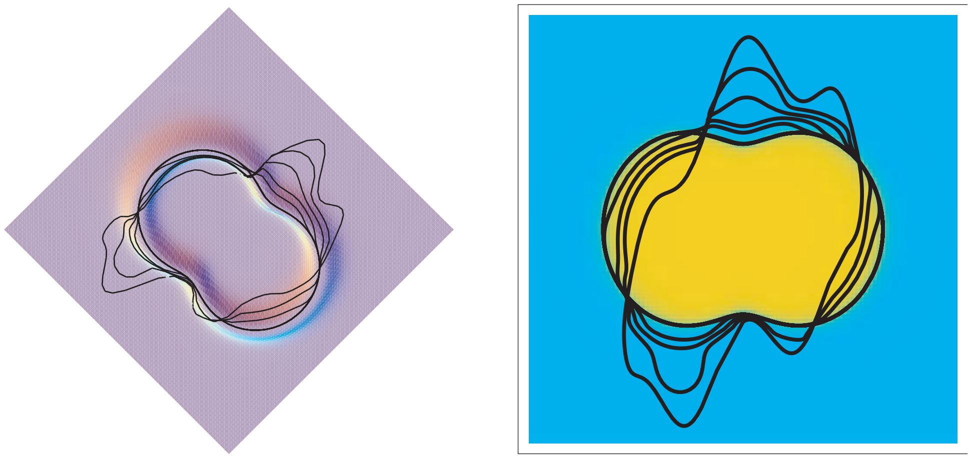

Figure 4. A geodesic flow on a flat surface with a sharp narrow valley (left) and its vertical projection to the plane with density plot of the image intensity function $I(x)$ (right).

If we take $\phi(x) = h(|\nabla I(x)|)$ where $h(\xi) = 1/(1+\xi^2)$ then the surface $\mathcal{M}$ defined as a graph of ϕ. Results of computation are presented in figure 4. The initial curve with large variations in the curvature is evolved according to the normal velocity $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ where the external force $\mathcal{F} = \vec{G} \cdot \vec{\mathcal{N}}$ is the normal projection of $\vec{G} = -(0, 0, \gamma)$. In the numerical experiment we considered a strong external force coefficient $\gamma = 30$. The evolving family of surface curves approaches a stationary curve $\bar{\Gamma}$ lying in the bottom of the sharp narrow valley defining thus a closed edge in the image. We also computed the largest eigenvalue of the Sturm-Liouville problem (26). It turns out that $\lambda_1 \approx -6.92943$. According to Theorem 3.1 the stationary curve $\bar{\Gamma}$ is linearly stable.

5. Discussion

We have analyzed a flow of closed surface curves driven by the geodesic curvature and external force. Its vertical projection to the plane represents a flow of planar curves driven by the normal velocity depending on the curvature, tangential angle as well as the position of the curve. Following the direct approach local and global existence of a classical solution to the governing system of parabolic–ordinary differential equations were shown. An important part of this article is devoted to the study of stability of stationary surface curves. We gave sufficient conditions for a stationary closed curve to be linearly stable with respect to small perturbations in the normal velocity. We presented various numerical examples of a flow of surface curves. We also presented an example how to implement a geodesic flow with external force in the context of the edge detection problem arising from the image segmentation theory.

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