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On a two-phase minmax method for parameter estimation of the Cox, Ingersoll, and Ross interest rate model

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Abstract In this paper we investigate a two-phase minmax optimization method for parameter estimation of the well known Cox, Ingersoll, and Ross one-factor interest rate model (CIR). In the first optimization phase we determine four CIR parameters by minimizing the sum of squares of differences of a theoretical CIR yield curve and real market yield curve data. We show that the minimum is attained on one dimensional curve in the four dimensional CIR parameter space. In order to find a global minimum we make use of a variant of an evolution strategy based minimization algorithm. Next we find a global maximum of the likelihood function restricted to this curve. We also introduce restricted maximum likelihood and nonlinear R^2 ratios measuring quality of estimation of the CIR parameters. The estimation procedure is extensively tested on the several term structures from various countries. We compare results of estimation for term structures of inter-bank offer rates for stable western Europe banking sector to those of transitional countries like e.g. Central European countries. Quality of prediction capability of term structures is also discussed.

1 Introduction

In the past decades, term structure models have attracted a lot of attention from both a theoretical as well as practical point of view. The term structure is a functional dependence between the time to maturity of a discount bond and its present price. Relevant interest rate models characterize the bond prices (or yields) as a function of time to maturity, state variables like e.g. instantaneous interest rate as well as several model parameters.

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Much effort is being spent to calibrate interest rate models. Soon after discovery of the Cox, Ingersoll, and Ross (1985) one-factor interest rate model (CIR) it has been applied in order to interpret nominal bond prices by Brown and Dybvig (1986). In the context of interest rate models, the Generalized Method of Moments for identification of model parameters has been studied by Chan, Karolyi, Longstaff and Sanders (1992). Pearson and Sun (1994) have shown failure of parameter estimation for a two-factor extension of the CIR model applied to portfolios of Treasury bills. Nowman, Saltoglu (2003) have applied Gaussian estimation methods of Nowman (1997) to continuous time interest rate models. Takahashi and Sato (2001) developed new methodology for estimation of general class of term structure models based on the Monte Carlo filtering approach. The method has been applied to LIBOR term structures and interest rates swaps in the Japanese market. There are other estimation techniques which become very popular e.g. QML method or the MCMC methods. Recall that the state space estimation approach is a combination of the Kalman filtering approach with the Quasi Maximum Likelihood (QML) estimation of parameters of the interest rate model under consideration. Geier and Pichler (1999) estimated and tested the state space estimation approach for the CIR multi-factor model. In this method the Maximum Likelihood estimation is based on the knowledge of the transition density of the state variable which is the instantaneous interest rate in the case of one-factor interest rate model. Chen and Scott (2002) applied a non-linear Kalman filter approach in combination with the QML method for estimation of the CIR model parameters. The approximate maximum likelihood estimator had a significant bias in the case of separated estimation of four CIR model parameters. Another important estimation procedure is based on the so-called MCMC (Markov Chain Monte Carlo) method. This method is an estimation tool for the non-normal and/or non-linear state space models. In the Bayesian framework the estimation of parameters starts with a conditional distribution of parameters (the Gibbs sampling method) or with a proposal density (the Metropolis-Hastings algorithm). Frühwirth-Schnatter and Geier studied the MCMC method for parameter estimation of the CIR multi-factor model. They estimate the CIR parameters by means of the MCMC method. Notice that the appropriate choice of the distribution of the parameters is one of the most important parts of this method. Moreover, the convergence to optimum values in the case of an inappropriate choice of initial distribution can be time consuming. Recently, other estimation methods for interest rate models have been proposed by Rebonato (1999). These methods are based on other interest rate derivatives like e.g. prices of caps and floors. However, such derivatives are still not available in some financial markets including most of transitional Central European countries.

Less attention is however put on possible applications of interest rate models to financial markets of transitional economies like e.g. Central European countries including Czech Republic, Slovakia, Poland and Hungary. Furthermore, a comparison to stable Western Europe financial markets has not been done yet. There is a partial progress in this direction made by Vojtek (2004). He estimated conditional volatilities by using the Brace, Gatarek, Musiela model (1997) and various types of GARCH models.

The purpose of this paper is to investigate a new method for estimation of model parameters of the CIR one-factor interest rate model. Recal that the CIR model is a general equilibrium model in which the term structure can be obtained from a solution to the CIR model. Moreover, a solution to the CIR model is given by an explicit expression. In order to estimate CIR model parameters we introduce a two-phase minmax optimization method. In the first optimization step we determine four CIR parameters by minimizing sum of squares of differences of theoretical CIR yield curve and real market yield curve data. To this end, we make use of a robust optimization method based on a variant of evolution strategy algorithm. It turns out that the minimum is attained on a one dimensional curve in the four dimensional CIR parameter space. Next we find a global maximum of the likelihood function computed over this curve. The argument of the maximum of such a restricted likelihood function represents the estimator of the CIR model parameters.

Our estimation method is extensively tested on real market term structures like e.g. London inter-bank offer rate (USD-LIBOR, EURO-LIBOR), Euro-zone term structure (EURIBOR) as well as Central European financial markets from transitional economies like Czech Republic (PRIBOR), Poland (WIBOR), Hungary (BUBOR), and Slovakia (BRIBOR). We also compare and discuss results of parameter estimation for stable western Europe markets to those of the above mentioned transitional economies. It turns out that relatively satisfactory results of estimation can be achieved for EURIBOR, USD-LIBOR, EURO-LIBOR as well as PRIBOR and WIBOR. On the other hand, the CIR model fails to provide a good estimation of BRIBOR and BUBOR term structures. In order to compare our method with existing methods (the state space method with Kalman's filtering or MCMC method) we note that both the state space methodology as well as the MCMC methodology when applied to the CIR model do not utilize the possibility of reduction of the parameter space dimension. Therefore it is not clear to what values of a one dimensional curve of optimal CIR parameters these methods converge to. Moreover, the MCMC method is sensitive to parameter transformations as the choice of proposal densities depends on the way we represent and/or aggregate model parameters. The novelty and possible advantage of our method consists in reduction of the parameter space and the two-phase optimization procedure based on a robust minimization method (evolution strategies algorithm). There is also no need for any assumptions made on the distribution of estimated parameters. In contrast to Chen and Scott (2002) non-linear Kalman filtering there was no evidence of a bias in the case of combined parameter estimation of three aggregated CIR parameters. It could be due our reduction of dimension of the parameter space.

The paper is organized as follows: in Section 2 we recall basic properties of the Cox, Ingersoll, and Ross model. Next we show how to transform four CIR model parameters into essential three aggregated parameters. In Section 3 we propose a two phase method for parameter estimation of the CIR model. First, we find a global minimum of the nonlinear cost function. Next we find a maximum of the likelihood function over a one dimensional curve consisting of global minimizers of the cost function. We introduce a notion of the restricted maximum likelihood and nonlinear R^2 ratios measuring the quality of the fit. Numerical methods for

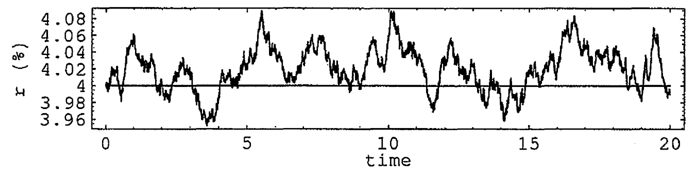


Fig. 1 Simulation of a mean reverting process driven by (1) with parameters: $\kappa = 1, \sigma = 0.2, \theta = 0.04$ within the time interval $(0, 20)$.

optimization steps are discussed in Section 4. A special attention is put on the algorithm for minimization of the cost function following a variant of the evolution strategy based optimization method. Section 5 is devoted to data description and parameter estimation results for several European financial markets. We also present results of a posteriori testing of prediction capability of the CIR model. Discussion and concluding remarks are presented in Section 6.

2 Cox, Ingersoll, and Ross interest rate model

The Cox, Ingersoll, and Ross interest rate model is derived from a basic assumption made on the form of a stochastic process driving the instantaneous interest rate $r_t, t \in [0, T]$. In the CIR model we assume the instantaneous interest rate (short rate) satisfies the following mean reverting process of the Orstein-Uhlenbeck type:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dw_t \tag{1}$$

where $\{w_t, t \geq 0\}$ denotes the standard Wiener process. Positive constants κ, θ and σ denote the adjustment speed of reversion, the long term interest rate and volatility factor of the process, respectively. The term $\sigma^2 r$ is then local variance of the real interest rate process. In Figure 1 we plot sample data obtained from a simulation of equation (1). In the CIR theory the price $P = P(t, T, r)$ of a zero coupon bond is assumed to be a function of the present time $t \in [0, T]$, expiration time $T > 0$ and the present value of the short interest rate $r = r_t$. Recall that the crucial step in derivation of any one-factor model, including CIR model in particular, consists in construction of a risk-less portfolio containing two bonds with different maturities. Next, as a consequence of the Itô lemma, one obtains a backward parabolic partial differential equation for the price of the zero coupon bond $P = P(t, T, r)$ of the form:

$$\frac{\partial P}{\partial t} + (\kappa(\theta - r) - \lambda r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} - rP = 0\,, \quad t \in (0, T)\,,\; r > 0\,. \tag{2}$$

The parameter $\lambda \in \mathbb{R}$ represents the so-called market price of risk (see Kwok (1998)). A solution P to (2) is subject to the terminal condition $P(T, T, r) = 1$ for any $r > 0$. It is well known (see e.g. Kwok (1998)) that PDE (2) with such a terminal condition admits an explicit solution in the form:

$$P(T - \tau, T, r) = A(\tau)e^{-B(\tau)r}, \quad \tau = T - t \in [0, T]\,, \tag{3}$$

where

$$B(\tau) = \frac{2(e^{\eta\tau}-1)}{(\kappa+\lambda+\eta)(e^{\eta\tau}-1)+2\eta}, \quad A(\tau) = \left(\frac{\eta e^{(\kappa+\lambda+\eta)\tau/2}}{e^{\eta\tau}-1}B(\tau)\right)^{\frac{2\kappa\theta}{\sigma^2}} \tag{4}$$

and $\eta = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$.

2.1 Essential CIR parameters. Parameter space reduction.

As it has been already pointed out by Pearson and Sun (1994) the adjustment speed κ and the risk premium λ appears in (4) only in summation $\kappa+\lambda$. This is why four CIR parameters can be reduced to three essential parameters fully describing the behavior of the functions A, B. In what follows, we present an idea how to reduce the four dimensional parameter space into essential three aggregated parameters. It consists in introducing the following new parameters:

$$\beta = e^{-\eta}, \quad \xi = \frac{\kappa+\lambda+\eta}{2\eta}, \quad \varrho = \frac{2\kappa\theta}{\sigma^2} \tag{5}$$

where $\eta = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$. Conversely, the original CIR parameters $\kappa, \sigma, \theta, \lambda$ can be expressed in terms of β, ξ, ϱ as follows:

$$\eta = -\ln\beta, \quad \kappa = \eta(2\xi - 1) - \lambda, \quad \sigma = \eta\sqrt{2\xi(1-\xi)}, \quad \theta = \frac{\varrho\sigma^2}{2\kappa}.$$

Let us denote $\mathcal{D} = (0,\infty)^3 \times \mathbb{R} \subset \mathbb{R}^4$ and $\Omega = (0,1)\times(0,1)\times\mathbb{R} \subset \mathbb{R}^3$. Then the transformation $T : \mathcal{D} \to \Omega$ defined as in (5), i.e. $T(\kappa,\sigma,\theta,\lambda) = (\beta,\xi,\varrho)$, is a smooth mapping and, for any $(\check\beta,\check\xi,\check\varrho) \in \Omega$, the preimage

$$T^{-1}(\check\beta,\check\xi,\check\varrho) = \{(\kappa_\lambda,\sigma_\lambda,\theta_\lambda,\lambda) \in \mathbb{R}^4,\ \lambda \in \check J\}, \quad \check J = (-\infty, -(2\check\xi - 1)\ln\check\beta),$$

is a smooth one-dimensional λ-parameterized curve in $\mathcal{D} \subset \mathbb{R}^4$. Here

$$\kappa_\lambda = -\lambda - (2\check\xi - 1)\ln\check\beta, \quad \sigma_\lambda = -\sqrt{2\check\xi(1-\check\xi)}\ln\check\beta, \quad \theta_\lambda = \check\varrho\sigma_\lambda^2/(2\kappa_\lambda). \tag{6}$$

It is worth to emphasize that the value of the volatility σ_λ is independent of λ. In terms of new variables β, ξ, ϱ, the functions $A(\tau), B(\tau)$ can be expressed as follows:

$$B(\tau) = -\frac{1}{\ln\beta}\frac{1-\beta^\tau}{\xi(1-\beta^\tau)+\beta^\tau}, \quad A(\tau) = \left(\frac{\beta^{(1-\xi)\tau}}{\xi(1-\beta^\tau)+\beta^\tau}\right)^{\varrho}. \tag{7}$$

In the sequel, we will use the notation $A_j = A(\tau_j)$ and $B_j = B(\tau_j)$ where $\tau_1 < \tau_2 < \ldots < \tau_m$ stand for maturities of bonds forming the yield curve. We put $\tau_0 = 0$.

Summarizing, in the CIR model the price of bonds and, consequently, the corresponding yield curve depends only on three transformed parameters β, ξ and ϱ defined as in (5).

Although the analysis of the Vasicek one-factor interest rate model (see e.g. Kwok (1998)) is not a subject of this paper it should be noted that a similar parameter reduction can be also done for this model. The reader is referred to Ševčovič and Urbánová (2004) for details.

3 Two-phase minmax optimization method for estimation of the CIR model parameters

In this section we discuss the core of the method for estimation of the CIR model parameters. The method consists of two steps. In the first step we identify one dimensional curve of the CIR parameters by minimizing the cost functional. Loosely speaking, it mimics least squares approach in linear regression methods. However, it is worthwhile to emphasize that the proposed minimization problem is highly nonlinear and as such it requires special treatment from both qualitative as well as numerical point of view. Having identified the curve of global minimizers of the cost functional we proceed by the second step - maximization of the likelihood function restricted to that curve. The global maximum is attained in a unique point - a desired estimator of the CIR model parameters. A short research announcement containing a brief description of this method and preliminary and incomplete calibration results have been presented by the authors in [27].

At the end of this section we furthermore introduce a notion of the restricted maximum likelihood ratio (MLR) and nonlinear R^2 ratio measuring quality of estimation of CIR parameters.

3.1 Nonlinear least square minimization

The goal of the first step of the two-phase estimation method consists in minimization of the weighted least square sum of differences between real market yield curve interest rates and those predicted by CIR model. To this end, we introduce the following cost functional:

$$U(\beta,\xi,\varrho) = \frac{1}{m}\sum_{j=1}^{m}\frac{1}{n}\sum_{i=1}^{n}(R_j^i - \bar{R}_j^i)^2\tau_j^2 \tag{8}$$

where $\{R_j^i, j = 1, ..., m\}$ represents the yield curve of the length m at time $i = 1, ..., n$. It consists of interest rates on zero coupon bonds with time τ_j to maturity. By $\{\bar{R}_j^i, j = 1, ..., m\}$ we have denoted the corresponding yield curve computed by using the CIR model with parameters β, ξ, ϱ. The instantaneous interest rate (short rate) at time $i = 1, ..., n$, is denoted by R_0^i. Notice that the values $\bar{R}_j^i$ can be calculated from the bond price - yield curve relationship:

$$A_j e^{-B_j R_0^i} = P = e^{-\bar{R}_j^i \tau_j}\,.$$

Hence $\bar{R}_j^i\tau_j = B_j R_0^i - \ln A_j$. After some straightforward calculations, the cost functional U can be rewritten in a simplified form

$$U(\beta,\xi,\varrho) = \frac{1}{m}\sum_{j=1}^{m}\left((\tau_j E(R_j) - B_j E(R_0) + \ln A_j)^2 + D(\tau_j R_j - B_j R_0)\right) \tag{9}$$

where $E(X_j)$ and $D(X_j)$ denote the mean value and dispersion of the random vector $X_j = \{X_j^i, i = 1, ..., n\}$, resp. Expression (9) for the cost functional is much more suitable for computational purposes because it contains aggregated time series information from the yield curve only. Indeed, $D(\tau_j R_j - B_j R_0) = \tau_j^2 D(R_j) - 2\tau_j B_j cov(R_j, R_0) + B_j^2 D(R_0)$ and cumulative statistical information regarding mean and covariance of term structure R_j series can be preprocessed prior to optimization.

Let us denote by P_j^i and $\bar{P}_j^i$ the market bond price and the bond price predicted by the CIR model, resp., at time instant i with time to maturity τ_j. By (3) $P_j^i = e^{-R_j^i \tau_j}$ and $\bar{P}_j^i = e^{-\bar{R}_j^i \tau_j}$. Thus $|P_j^i - \bar{P}_j^i| \le |R_j^i - \bar{R}_j^i| \tau_j$. Hence the value U of the cost function is an upper bound for the average value of squares of differences between observed market bond prices and those computed from the CIR model.

Concerning existence of a global minimum of U on Ω we note that the function U is coercive for $|\varrho| \to \infty$, i.e. $U \to \infty$ for $|\varrho| \to \infty$. Thus the global minimum of U is attained on a bounded subset Ω_b of Ω. Furthermore, data vectors $R_j, j = 0, ..., m$, enter expression for U in terms of their means and covariances. Now if a global minimum of U is attained at multiple minimizers, one can perturb input data vectors R_j slightly to achieve unique global minimizer of U. It means that, for a generic input vectors of term structures, there exists a unique global minimizer of U.

3.2 The restricted likelihood function and the maximum likelihood ratio

In this section we analyze how to measure a quality of estimation of CIR parameters. The idea is based on the maximum likelihood estimation combined with minimization of the cost functional U.

First, we focus on the estimation of parameters based on the analysis of the mean reverting process (1). According to Bergstrom [6, Theorem 2] (see also [5, 7]) the time discretized model corresponding to stochastic equation (1) reads as: $r_t - r_{t-1} = (\theta - r_{t-1})(1 - e^{-\kappa}) + \varepsilon_t$ where ε_t is a normally distributed random variable with zero mean and dispersion $\sigma^2 r_{t-1}$. Such a discretization can be used for the estimation of structural parameters κ, θ and σ. Furthermore, the logarithm of the Gaussian likelihood function for such a time discretization of (1) has the form:

$$\ln L(\kappa, \sigma, \theta) = -\frac{1}{2} \sum_{t=2}^{n} \left(\ln v_t^2 + \frac{\varepsilon_t^2}{v_t^2} \right) \tag{10}$$

where $v_t^2 = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa}\right) r_{t-1}$, $\varepsilon_t = r_t - e^{-\kappa} r_{t-1} - \theta\left(1 - e^{-\kappa}\right)$ (see [6]). Following the maximum likelihood approach, the estimator of the model parameters is the argument $(\kappa^u, \sigma^u, \theta^u)$ of the maximum of $\ln L(\kappa, \sigma, \theta)$ taken over the whole feasible set $\mathbb{R}_+^3$, i.e.

$$\ln L^u = \ln L(\kappa^u, \sigma^u, \theta^u) = \max_{\kappa,\sigma,\theta>0} \ln L(\kappa, \sigma, \theta)\,. \tag{11}$$

Hereafter, the upper index u indicates that the maximum is unrestricted and is taken over the whole feasible 3D parameter space $\mathbb{R}_+^3$.

Since the likelihood function $\ln L$ is determined from equation (1) both the value $\ln L^u$ as well as the estimated parameters $\kappa^u, \sigma^u, \theta^u$ are independent of the market price of risk λ. On the other hand, the rest of the yield curve $\{\bar{R}^i_j, j = 1, ..., m\}$ depends on λ and is not taken into account during calculation of the maximum of the likelihood function.

Combining information gained from minimization of the cost functional U we are yet able to introduce the concept of a restricted likelihood function. In this approach we optimize $\ln L$ over a one dimensional subset of parameter values. This restricted set consists of all triples $(\kappa_\lambda, \sigma_\lambda, \theta_\lambda) \subset (0,\infty)^3$, $\lambda \in \check{J}$, for which the cost function U (in corresponding transformed variables) attains the global minimum. More precisely, let $(\check{\beta}, \check{\xi}, \check{\varrho}) \subset \Omega$ be the unique global minimum of U on Ω, i.e.

$$U(\check{\beta}, \check{\xi}, \check{\varrho}) = \min_{(\beta,\xi,\varrho)\in\Omega} U(\beta, \xi, \varrho)\,.$$

Now we find a global maximum of the likelihood function $\ln L$ over the one dimensional curve $T^{-1}(\check{\beta}, \check{\xi}, \check{\varrho}) = \{(\kappa_\lambda, \sigma_\lambda, \theta_\lambda, \lambda) \in \mathbb{R}^4,\ \lambda \in \check{J}\}$ (see (6). Let

$$\ln L^r = \ln L(\kappa_{\bar{\lambda}}, \sigma_{\bar{\lambda}}, \theta_{\bar{\lambda}}) = \max_{\lambda\in\check{J}} \ln L(\kappa_\lambda, \sigma_\lambda, \theta_\lambda)\,. \tag{12}$$

Now we are in a position to identify the resulting optimal values $\bar{\kappa} = \kappa_{\bar{\lambda}}, \bar{\sigma} = \sigma_{\bar{\lambda}}, \bar{\theta} = \theta_{\bar{\lambda}}$ and $\bar{\lambda}$ as the estimator of the CIR parameters obtained by the two-phase minmax optimization method described above.

Next we introduce the maximum likelihood ratio as follows:

$$MLR = \frac{\ln L^r}{\ln L^u}\,. \tag{13}$$

We have $MLR \le 1$. Notice that the value of MLR close to 1 indicates that the restricted maximum likelihood value $\ln L^r$ is close to the unrestricted one. In this case one can therefore expect that the estimated values $(\bar{\kappa}, \bar{\sigma}, \bar{\theta})$ of CIR parameters are close to the argument $(\kappa^u, \sigma^u, \theta^u)$ of the unique global maximum of an unrestricted likelihood function. It means that the direct estimation of parameters of the mean reversion equation (1) is also suitable for estimation of the whole term structure. In other words, values of MLR close to 1 indicate that the CIR model can be accepted for estimation of the whole term structure.

3.3 The nonlinear R^2 ratio

The R^2 ratio plays an important role in the frame of linear regression methods based on the least square estimation techniques. The value of R^2 close to 1 indicates that the given data set can be regressed by a linear function. However, in nonlinear regression methods based on minimization of sum of squares of functions nonlinearly depending on estimated parameters there is no unique way how to define the equivalent concept of the R^2 ratio. It substantially depends on the choice of the reference value. Hence, by an appropriate choice of the reference value, we are able to introduce a notion of the so-called nonlinear R^2 ratio measuring quality of nonlinear regression based on minimization of the cost functional U defined as in (8).

We take the reference value of the cost functional U by taking the argument $(\beta, \xi, \varrho) = (1, 1, 1)$. Since $\lim_{\beta \to 1} B_j = \tau_j$ and $\ln A_j = 0$ for $\beta = 1$ it is easy to calculate that

$$U(1,1,1) = \frac{1}{m} \sum_{j=1}^{m} \tau_j^2 E((R_j - R_0)^2)$$

and, moreover, $U(1,1,1) = U(1, \xi, \varrho)$ for any $\xi \in [0,1]$ and $\varrho \in \mathbb{R}$. We define the nonlinear R^2 ratio as follows:

$$R^2 = 1 - \frac{U(\check{\beta}, \check{\xi}, \check{\rho})}{U(1,1,1)} \tag{14}$$

where $(\check{\beta}, \check{\xi}, \check{\rho})$ is the argument of the unique global minimum of the cost functional U. Then $0 \leq R^2 \leq 1$. The value of R^2 close to 1 indicates that $U(\check{\beta}, \check{\xi}, \check{\rho}) \ll U(1,1,1)$, i.e. the cost functional U is close to zero at $(\check{\beta}, \check{\xi}, \check{\rho})$. It means almost perfect matching of the CIR yield curve computed for parameters $(\check{\beta}, \check{\xi}, \check{\rho})$ and that of the given real market data set.

4 Numerical and optimization methods

In this section we discuss an optimization method for solving the minimization problem for the cost functional U on Ω. It is worthwhile to note that the function $U = U(\beta, \xi, \varrho)$ is nonlinear and it is not know (at least to the authors) whether U is convex or not. In such a case it is hard to find a global minimum by standard optimization techniques like e.g. steepest descent gradient minimization method of Newton-Kantorovich type (cf. [2]). In general, such methods are know to converge to a local minimum only or they may even converge to a global minimum but the speed of convergence is very slow. To solve the global minimization problem for the function U we need a robust and efficient numerical method unconditionally converging to a (unique) global minimum of U. Our optimization method is based on a variant of the evolution strategy (ES) algorithm. In each optimization step the approximation of a global minimum obtained by the evolution strategy is furthermore improved by the corrector step consisting of the Newton-Kantorovich steepest-descent gradient minimization method.

Evolution strategy (ES) is the one of the most successful stochastic algorithm which was invented to solve technical optimization problems. The reader is referred to papers by Schwefel (1977, 1995, 1998) for a comprehensive overview of modern ES optimization methods and their possible applications. Recall that many different versions and applications of ESs have been developed, but they are mostly evolution based processes by means of which we find an optimal value . Parameters are arranged in vectors of real numbers. The main concept of this strategy is based on the survival of the fittest calculated from the value of the function to be minimized. There exist many different types of this stochastical algorithm like the two membered $(1+1)$ ES, the multi-membered (p, c) ES, $(p+c)$ ES etc. The $(p+c)$ ES has p parents and c children (offsprings) per population, among which the p best individuals are selected to be next generations parents by their fitness value. The modification $(p + c + d)$ ES comprises selection from a wider set consisting of parents, children and d wild type individuals.

Recall that our task in the first optimization phase is to find the real valued vector $(\bar{\beta}, \bar{\xi}, \bar{p})$ for which the global minimum of the function U is attained on Ω. The proposed algorithmic description of the optimization method based on the ES is as follows:

The initial population of parent vectors $(\beta, \xi, \rho)_k$, $k = 1, ..., p$, is generated as uniformly distributed random numbers from the a bounded subset Ω_b of the domain Ω for every k. We take $\Omega_b = \{(\beta, \xi, \varrho) \in \Omega, 0 \leq \varrho \leq \varrho_{max}\}$ where ϱ_{max} is large enough.

1. Vector of children (offspring) $(\bar{\beta}, \bar{\xi}, \bar{\rho})_l$, $l = 1, ..., c$, is created from parents $(\beta, \xi, \rho)_k$, $k = 1, ..., p$, by mutation and recombination operators. The number of children must be greater or equal to the number of parents. Mutation is realized by perturbing the data by a Gaussian random variable with zero mean and preselected standard deviation to each component of the vector of parents $(\beta, \xi, \rho)_k, k = 1, ..., p$. Recombination is realized by crossing-over parts of randomly chosen vectors of children.
2. Vector of the wild type population $(\tilde{\beta}, \tilde{\xi}, \tilde{\rho})_o$, $o = 1, ..., d$, is generated randomly from Ω in the same way as the initial population.
3. Every member of the population (parents, children, wild population) is characterized by its fitness value, which is the value of the cost functional U.
4. Selection chooses the best p parent vectors by their fitness value U to be the next generation of parents. A set of p intermediate parents is obtained.
5. Each of intermediate parents is further improved by m steps of the gradient Newton-Kantorovich method. A set of p improved parents is obtained.
6. Finally, selection chooses the best p new parents by their fitness value from the set of intermediate and improved parents.

The process continues until termination criterion is fulfilled. In our case this criterion is the number of populations equal to a given number N. For optimization purposes we have used the parameters $p = c = d = 10^5, N = 300$ and the standard deviation for Gaussian mutation operator equal to 0.01. We did not up-date the the standard deviation according to Rechenberger's rule (see Rechenberger (1973)) as it turned to be ineffective in our case. The number of Newton-Kantorovich iterates was 30. It should be noticed that the time complexity of the two phase estimation method is very high. Computation of CIR parameters for a term structure with the length $m \approx 10$ takes 40 minutes in average. The overall computational time for all investigated examples of parameter estimation described in the next section was 12 hours of CPU of the Comenius University Linux parallel cluster containing twelve 3GHz processors.

5 European term structures

The aim of this section is to present results of the two-phase estimation method for terms structures for various European countries. Moreover, the aim is to make comparison of stable western European financial markets to those of transitional Central European economies. More precisely, we will compare European inter-bank offer rates EURIBOR, London inter-bank offer rates EURO-LIBOR and USD-LIBOR, and inter-bank offer rates in Slovakia (BRIBOR), Hungary (BUBOR), Czech Republic (PRIBOR) and Poland (WIBOR).

5.1 Term structure description and summary statistics

The instantaneous interest rate r_t is the yield of a discount bond with maturity in the subsequent time instant. As such is not observable and can not be used as a state variable r_t in (1). However, a good approximation for the instantaneous interest rate is the so-called overnight or short rate where available. It represents one day interest rate for inter-bank loans. The data for the overnight interest rate are available for all investigated Eastern Europe financial markets (BRIBOR, PRIBOR, WIBOR, BUBOR) as well as for London inter-bank offer rates (LIBOR). As far as the Euro-zone term structure EURIBOR is concerned we recall that there exists a commonly accepted substitute for the overnight rate which is referred to as EONIA. It is computed as a weighted average of all overnight unsecured lending transactions in the inter-bank market, initiated within the Euro-zone area by 48 European Panel Banks. We also remind ourselves that the number of Panel Banks is 16 for BUBOR, 10 for WIBOR and 7 for BRIBOR.

Our parameter estimation method for term structure data has been applied for the period from the year 2001 up to 2003. EURO-LIBOR and USD-LIBOR term structures contain bonds with the following maturities: 1 week, 1 up to 12 month, i.e. its length is $m = 13$. EURIBOR, in addition, contains 2 and 3 weeks maturities, i.e. its length is $m = 15$. BRIBOR contains 1, 2 weeks and 1, 2, 3, 6, 9, 12 months maturities with the length $m = 8$. BUBOR, PRIBOR, and WIBOR contain the following maturities: 1 and 2 weeks, 1, 2, 3, 6, 9, 12 months, and then each quarter up to 10 years maturity, i.e. their length is $m = 39$.

The sample mean and the standard deviation (STD), in each quarter, for different inter-bank offer rates is presented in Table 1 for year 2003. The mean of BRIBOR, WIBOR and BUBOR is higher than the mean of the last three data sample (PRIBOR, EURIBOR and EURO-LIBOR) during the whole year. The same is true concerning standard deviation. It is an indication that the Czech data could have similar qualitative properties as the western European term structures.

The key issue of the CIR model is to model the short-rate as the mean reverting process (1). In Figure 2 we present some examples of short-rate for specific samples of data (EURO-LIBOR, BRIBOR and PRIBOR). It can be seen from Figure 2 that the over-night is more volatile than the interest rates with longer maturity.

It is worthwile noting that parameters of various mean reverting diffusion processes with linear drift and non-constant variances were estimated in [14]. The parameters were obtained by ordinary least squares method for daily averaged loan rates. Fischer and Zechner (1984) showed that mean reverting diffusion processes are not capable of describing the long-run behavior of the instantaneous interest rate. Therefore one has to choose shorter time periods for estimating CIR parameters. We chose a period of quarter in our experiments.

Table 1 Descriptive statistics for various term structures. Mean and standard deviation (in %) are shown for the overnight rate and the rate on the longest bond with 1-year maturity.

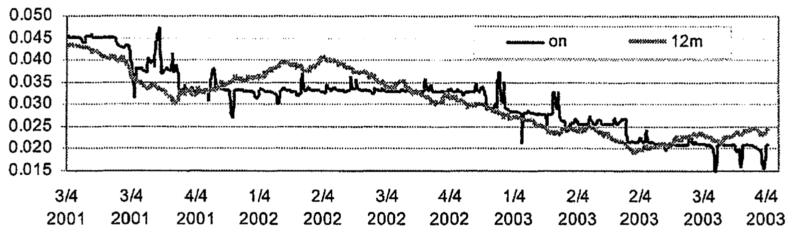
		1/4 2003		2/4 2003		3/4 2003		4/4 2003	
		Mean	STD	Mean	STD	Mean	STD	Mean	STD
BRIBOR	on	5.75	1.041	6.27	1.279	5.63	0.802	5.48	0.992
	1y	5.48	0.205	5.42	0.208	5.80	0.066	5.50	0.033
WIBOR	on	6.65	0.761	5.76	0.359	5.22	0.438	5.17	0.438
	1y	5.95	0.138	5.19	0.255	4.97	0.053	5.79	0.380
BUBOR	on	5.42	1.813	7.08	0.879	9.58	0.547	10.52	1.213
	1y	6.57	0.433	6.76	0.773	8.80	0.207	10.02	1.334
PRIBOR	on	2.52	0.107	2.44	0.045	2.06	0.135	1.94	0.032
	1y	2.43	0.130	2.33	0.084	2.13	0.063	2.19	0.061
EURIBOR	on	2.77	0.188	2.44	0.199	2.07	0.120	2.02	0.169
	1y	2.54	0.140	2.23	0.189	2.20	0.106	2.36	0.081
EURO-LIB	on	2.79	0.196	2.47	0.196	2.08	0.101	2.02	0.139
	1y	2.54	0.139	2.23	0.187	2.20	0.105	2.35	0.085

5.2 Results of parameter estimations

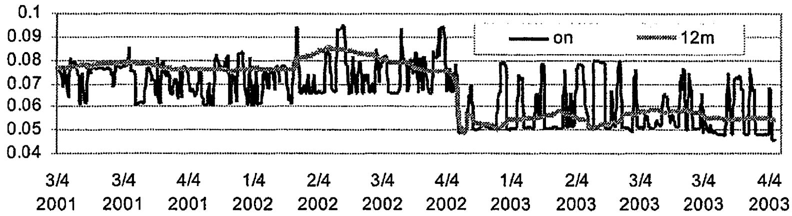
The results of estimation for CIR model parameters, corresponding maximum likelihood (MLR) and R^2 ratios are summarized in Table 2 for term structures with shorter maturities and in Table 3 for those having longer maturities.

Table 2 reports quarterly results for BRIBOR, WIBOR, BUBOR, PRIBOR, EURO-LIBOR and EURIBOR for the year 2003. Estimated parameters $\kappa, \sigma, \theta, \lambda$, the value of the cost functional ($U \times 10^{-6}$) and the nonlinear R^2 ratio together with the maximum likelihood ratio (MLR) are presented. Behavior of the long term average interest rate θ is in accordance to what is expected by the market in the long term run. It predicts long term interest rates close to 1.7% for EURO-LIBOR and EURIBOR as well as for PRIBOR. Other term structures also indicate decrease of interest rates in the future but these estimations of θ are quantitatively less convincing compared to EURO-LIBOR, EURIBOR and PRIBOR predictions. Results of estimation for κ show that the speed of adjustment for EURIBOR and EURO-LIBOR is comparable. The lowest values of estimated κ were achieved by the PRIBOR term structure. On the other hand, highest values of κ were achieved for BRIBOR which is in accordance to highly fluctuating character of BRIBOR interest rates (see Figure 2). The estimated volatility parameter σ of the mean reverting process has a similar behavior for WIBOR and BUBOR, and, it is very large for the Slovak BRIBOR data. On the other hand, results for the Czech data enables us to conclude that the volatility of PRIBOR quantitatively and qualitatively is very similar to that of EURIBOR and EURO-LIBOR. In terms of the maximum likelihood ratio (measuring appropriateness of the CIR model), the overall quality of estimation is better for PRIBOR, EURIBOR and EURO-LIBOR. The nonlinear R^2 ratio is mostly close to one for all data, but there are some exceptions.

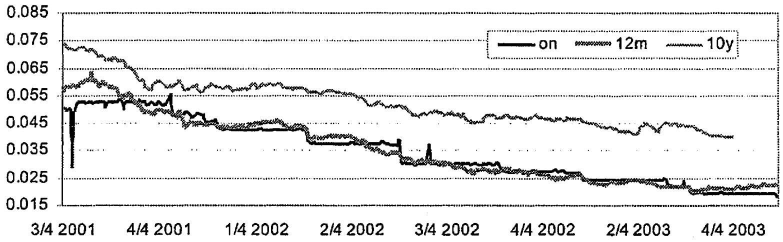
Table 3 presents quarterly results of parameter estimation for WIBOR, BUBOR and PRIBOR for years 2002-2003. In this case the term structures include interest rates on bonds with rather longer maturities, up to 10 years. Due to the



(a)



(b)



(c)

Fig. 2 Graphical description of overnight (short-rate) interest rates and those of bond with longer maturity. Daily data are plotted for EURO-LIBOR (a), BRIBOR (b) and PRIBOR (c)

increased length of the term structures, the quality of approximation expressed by MLR and R^2 has decreased. Estimated values of the parameter θ are again in accordance with market expectations on decrease of future interest rates. However, quantitatively these results are less convincing when compared to estimation of θ based on shorter term structures shown in Table 2. The reason is that bond with long maturities are considerably less traded and their price need not necessarily follow basic assumptions made in the CIR theory. In terms of R^2 ratio the accuracy of estimation is high for all term structures. Nevertheless, in terms of the MLR the CIR model seems to be more appropriate for the Czech PRIBOR, similarly as it was stated for shorter term structures.

In Figure 3, parts (a) and (b) we show comparison of the MLR and R^2 ratios for EURO-LIBOR an USD-LIBOR. As we can see they are strongly correlated.

Hence, in the next parts of this figure, we present a comparison of results of parameter estimation for EURO-LIBOR, BRIBOR, PRIBOR and WIBOR only. The graph (c) displays their MLR. In most quarters, this ratio is better for EURO-LIBOR and the worst for BRIBOR. For the last two samples (PRIBOR and WIBOR), it is varying. In the graph (d) we can see that the best R^2 is achieved for WIBOR and the worst for PRIBOR. R^2 is varying for BRIBOR and EURO-LIBOR. Parts (e) and (f) presents estimated parameters θ (long term average interest rate) and σ (volatility of the process (1)). For PRIBOR and EURO-LIBOR the results are similar not only for θ but also for σ. The volatility of the process for Slovak data is very high and also parameter θ is quite volatile.

Figure 4 presents the results of parameter estimation for term structures with longer maturity. The MLR depicted in part (a) is getting better for WIBOR and just opposite could be claimed for PRIBOR. The nonlinear R^2 ratio is comparable for these data, except of the beginning of the estimated period. Parts (c) and (d) compares the estimated parameters θ and σ, resp. The limiting interest rate is very different for these data, but the volatility of the process (1) of WIBOR is tending to the volatility of PRIBOR.

5.3 A posteriori prediction of future term structures

In this section we present results of a posteriori testing of term structures prediction. The idea of testing is rather simple and consists in comparison of the values of the optimal value of the cost functional U and its value computed by using CIR parameters calibrated a time period before.

Let us consider a term structure data for a given period p of time. Let us denote by $\check{\beta}_p, \check{\xi}_p, \check{\rho}_p$ optimal values of transformed parameters for which the cost functional $U = U_p$ (see (8)) corresponding to the period p attains its unique global minimum. Similarly, by $\check{\beta}_{p-1}, \check{\xi}_{p-1}, \check{\rho}_{p-1}$ we denote the minimizer of the cost functional U_{p-1} corresponding to the previous period $p-1$. Then the quality of prediction (QP) can be measured by the square root of the ratio of $U_p(\check{\beta}_p, \check{\xi}_p, \check{\rho}_p)$ and $U_p(\check{\beta}_{p-1}, \check{\xi}_{p-1}, \check{\rho}_{p-1})$, i.e.

$$QP = \sqrt{\frac{U_p(\check{\beta}_p, \check{\xi}_p, \check{\rho}_p)}{U_p(\check{\beta}_{p-1}, \check{\xi}_{p-1}, \check{\rho}_{p-1})}}\,.$$

Clearly, $0 \leq QP \leq 1$ and a value of QP close to 1 indicates very high level of prediction capability of the model. On the other hand, a value of QP close to zero enables us to conclude that one cannot use parameters calibrated from the previous period in order to predict the term structure in the present period. In Table 4 we present values of the QP ratio computed for various term structures (EURO-LIBOR, EURIBOR, WIBOR, PRIBOR, BUBOR and BRIBOR) for the three quarter periods in 2003. The QP ratio is qualitatively similar for EURO-LIBOR and EURIBOR as expected. High prediction ratios QP have been obtained for the second quarter of PRIBOR, BRIBOR and BUBOR when estimated CIR parameters were calibrated from the first quarter of 2003. Unsatisfactory QP ratios

Table 2 Numerical results of calibration for short term structures (up to one year) for BRIBOR, WIBOR, BUBOR, PRIBOR, EURIBOR and EURO-LIBOR. Results cover 4 quarters of 2003.

BRIBOR	κ	σ	θ	λ	U ($\times 10^{-6}$)	R^2	ML ratio
1/4 2003	688.298	8.960	0.0025	-658.657	1.629	0.947	0.528
2/4 2003	38.467	1.509	0.0458	-6.302	1.704	0.971	0.719
3/4 2003	598.875	8.276	0.0031	-568.839	0.487	0.971	0.536
4/4 2003	793.487	9.396	0.0022	-764.117	0.339	0.986	0.551
WIBOR							
1/4 2003	10.103	0.622	0.0564	-0.362	0.660	0.966	0.702
2/4 2003	7.459	0.877	0.0204	-4.457	1.075	0.897	0.519
3/4 2003	33.370	0.707	0.0483	-1.149	0.169	0.976	0.750
4/4 2003	40.925	0.972	0.0455	-8.718	3.766	0.820	0.709
BUBOR							
1/4 2003	14.233	0.576	0.0555	-2.425	6.118	0.965	0.964
2/4 2003	14.166	0.671	0.0626	-0.658	2.967	0.805	0.684
3/4 2003	134.256	5.064	0.0035	-130.883	1.037	0.940	0.363
4/4 2003	2.715	0.812	0.0111	-2.508	19.593	0.208	0.394
PRIBOR							
1/4 2003	0.098	0.007	0.0248	0.092	0.134	0.633	0.904
2/4 2003	36.934	0.728	0.0209	-4.714	0.238	0.428	0.514
3/4 2003	2.836	0.067	0.0200	-0.212	0.028	0.897	0.799
4/4 2003	3.385	0.097	0.0187	-0.626	0.088	0.924	0.685
EURIBOR							
1/4 2003	40.927	1.030	0.0202	-8.724	0.506	0.783	0.654
2/4 2003	0.818	0.028	0.0240	0.252	0.319	0.746	0.876
3/4 2003	39.209	0.644	0.0178	-6.986	0.143	0.807	0.735
4/4 2003	15.592	0.360	0.0180	-3.451	0.145	0.941	0.778
EURO-LIBOR							
1/4 2003	34.118	0.689	0.0241	-1.897	0.634	0.818	0.712
2/4 2003	0.734	0.024	0.0244	0.276	0.432	0.790	0.864
3/4 2003	40.018	0.699	0.0175	-7.797	0.221	0.714	0.703
4/4 2003	9.217	0.286	0.0178	-2.243	0.221	0.929	0.758

have been obtained in prediction of term structures of PRIBOR and BUBOR in the third quarter.

5.4 Risk premium analysis

In this section we discuss and analyze results of parameter estimation for the parameter λ representing the market price of risk in the CIR model. We remind ourselves that according to the CIR model the price of a zero coupon bond $P = P(t, T, r)$ satisfies the parabolic equation (2) and is given by the explicit formula

Table 3 Numerical results of calibration for ten years term structures of WIBOR, BUBOR, PRIBOR. Results cover 4 quarters of 2003.

WIBOR	κ	σ	θ	λ	U ($\times 10^{-6}$)	R^2	ML ratio
2/4 2002	32.496	2.604	0.0048	-32.244	423.220	0.973	0.408
3/4 2002	0.739	0.203	0.0625	-0.118	407.977	0.906	0.655
4/4 2002	1.506	0.060	0.0721	0.320	352.889	0.926	0.911
1/4 2003	4.614	0.334	0.0603	0.455	47.817	0.989	0.753
2/4 2003	3.773	0.080	0.0579	0.523	96.430	0.945	0.934
BUBOR							
2/4 2002	14.797	1.576	0.0049	-15.044	713.717	0.735	0.413
3/4 2002	4.380	0.928	0.0098	-4.453	216.067	0.966	0.412
4/4 2002	1.546	0.531	0.0289	-1.049	649.690	0.962	0.467
1/4 2003	23.237	1.231	0.0450	-6.870	64.998	0.996	0.902
2/4 2003	23.329	1.816	0.0066	-21.394	90.866	0.974	0.455
PRIBOR							
2/4 2002	0.686	0.174	0.0210	-0.524	54.246	0.991	0.520
3/4 2002	0.306	0.014	0.0319	-0.150	124.070	0.975	0.986
4/4 2002	0.274	0.012	0.0280	-0.177	62.261	0.989	0.876
1/4 2003	0.158	0.077	0.0075	-0.251	38.768	0.993	0.595
2/4 2003	0.119	0.075	0.0022	-0.239	97.038	0.981	0.538

Table 4 The quality of the prediction ratio QP evaluated for various term structures from Central European countries and three quarters of the year 2003.

	EURO-LIBOR	EURIBOR	PRIBOR	WIBOR	BRIBOR	BUBOR
2/4 2003	0.34	0.33	0.93	0.34	0.95	0.93
3/4 2003	0.39	0.37	0.13	0.45	0.38	0.09
4/4 2003	0.58	0.57	0.62	0.42	0.40	0.57

$P(t,T,r) = A(T-t)e^{-B(T-t)r}$. Thus $\partial_r P = -BP$. Hence equation (2) can be rewritten as

$$\frac{\partial P}{\partial t} + \kappa(\theta - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} - r^* P = 0\,, \quad t \in (0,T)\,,\ r > 0 \qquad (15)$$

where $r^* = (1-\lambda B)r$. According to [21] the multiplier $1-\lambda B$ can be interpreted as the risk premium factor and r^* as the expected rate of return on the bond. It is easy calculus to show $B(\tau) \geq \tau > 0$ and therefore we have $r^* > r$ iff $\lambda < 0$. On the other hand, if $\lambda > 0$ market bond return r^* is less than risk less return rate r.

It follows from Table 2 that the market price of risk λ is negative in most time periods. It implies that the expected rate of bond return r^* is greater than instantaneous rate r. There are however some short time periods in which the market price of risk is positive for EURIBOR and EURO-LIBOR (2nd quarter). In Figure 5 we plot the risk premium factor $1-\lambda B$ for 10 quarters since the third quarter of 2001. We chose $B = B_1$, i.e. we plotted the risk premium for bonds with one week maturity. Figure 5 (left) displays the risk premium of EURO-LIBOR and PRIBOR.

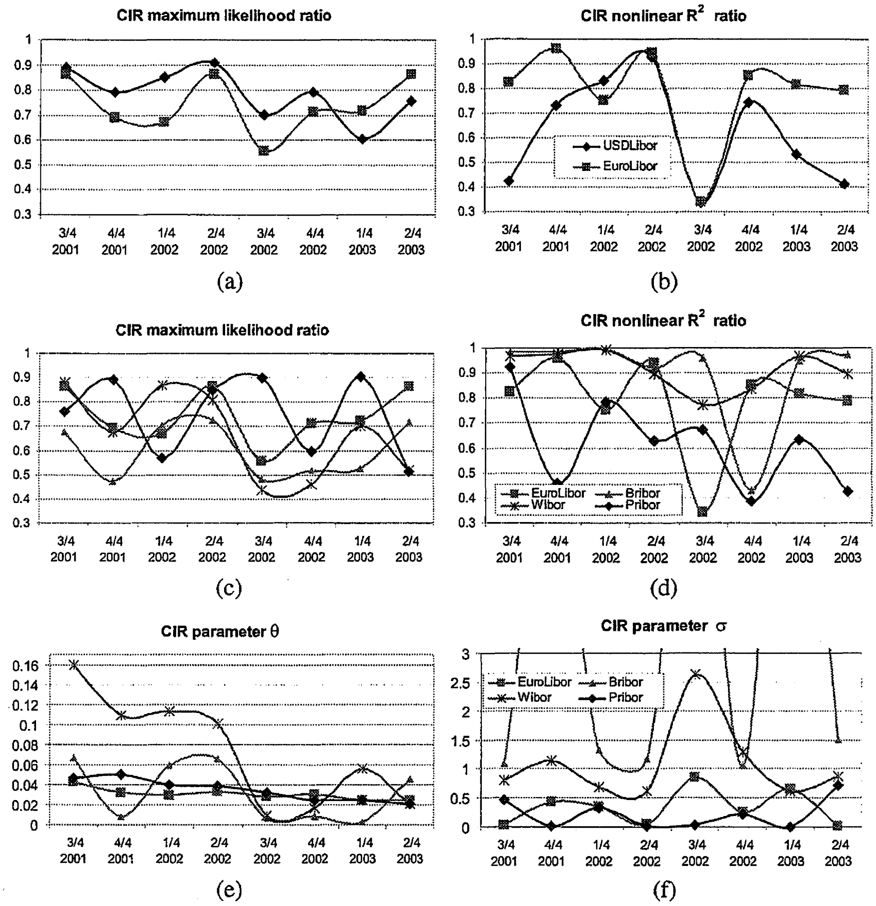


Fig. 3 Results of parameter estimation for various term structures. Maximum likelihood ratio (a) and R^2 ratio (b) for EURO-LIBOR and USD-LIBOR. Comparison of the same factors for EURO-LIBOR, BRIBOR, PRIBOR and WIBOR is presented in (c) and (d), resp. Estimated parameters θ and σ are shown in (e) and (f), resp.

They are comparable as far as behavior and range of values are concerned. The right figure presents a comparison for EURO-LIBOR with WIBOR and BRIBOR. The risk premium is quite similar for WIBOR and EURO-LIBOR except of the third quarter of 2003. However, this factor is extremely large and highly volatile for the Slovak BRIBOR term structure.

6 Discussion

We proposed a new two-phase minmax optimization method for parameter estimation of the CIR one-factor interest rate model. The advantage of our method consists in the reduction of the parameter space together with the two-phase op-

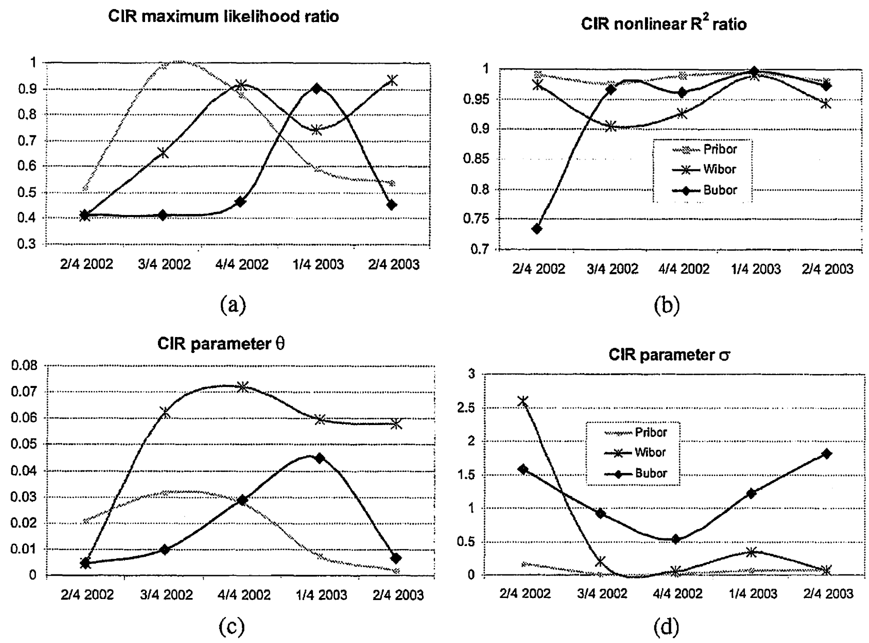


Fig. 4 Results of parameter estimation for term structures with maturities up to 10 years. Maximum likelihood ratio (a) and R^2 ratio (b) for BRIBOR, WIBOR and BUBOR. Estimated parameters θ and σ are shown in (c) and (d), resp.

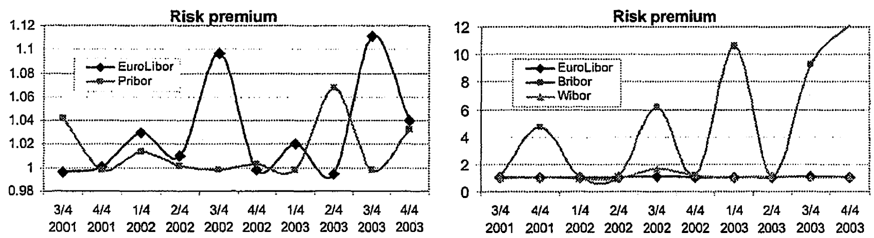


Fig. 5 Comparison of risk premium factors $1 - \lambda B_1$ for EURO-LIBOR and BRIBOR (left) and EURO-LIBOR, WIBOR and BRIBOR (right).

timization procedure based on minimization of the cost functional accompanied with maximization of the likelihood function restricted to the set of cost functional minimizers. We have tested our estimation method various term structures including stable western inter-bank offer rates as well as those of transitional countries. Based on our results of parameter estimation for the CIR one-factor model we can state that the western European term structure data are better described with CIR model compared to transitional economies represented by Central European countries. This model can be applied in order to estimate CIR parameters for EURO-LIBOR, USD-LIBOR and EURIBOR term structures. Interestingly, to some extent, it could be also applied for estimation of CIR parameters for the Czech PRIBOR term structure. On the other hand, we can observe, at least partial, quantitative failure of the CIR model for other Central European term structures.

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