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ANALYTICAL AND NUMERICAL METHODS FOR STOCK INDEX DERIVATIVE PRICING

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This paper deals with stock index derivative pricing. We recall the well-known Black-Scholes multi-asset derivative pricing theory. It leads to the generalized Black-Scholes PDE in very high space dimension. Although there is an explicit analytical solution to this problem, it should be noted that it is given in the form of a high-dimensional integral and as such is useless from the practical point of view. To overcome this difficulty we propose a new method for solving the generalized Black-Scholes PDE which is based on the so-called additive operator splitting (AOS) technique. The AOS method is applied to find an approximate solution. The AOS technique leads to decomposition of the multi-dimensional problem into several one-dimensional problems which can be solved very efficiently, even in parallel. The order of the approximative scheme is also investigated.

K e y w o r d s: multi-asset derivative pricing, Black-Scholes theory, additive operator splitting technique

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1 INTRODUCTION

In this paper we propose a new numerical method for pricing multi-asset derivatives like, e.g., basket options or stock indices. It is well-known that the price of such a financial derivative is a solution to a high space dimensional parabolic differential equation — the generalized Black-Scholes equation. For derivation of the generalized Black-Scholes equation we refer to a recent book by Kwok [4]. This equation governs the price of an option with multiple underlying assets. All asset prices are assumed to follow the lognormal distributions. The generalized B-S equation can be fully integrated and its analytical solution has the form of an n-dimensional integral where n is the number of underlying assets. In the case of stock indices, n ranges from 10 s to 100 s (e.g., $n = 500$ for Standard & Poor's 500 Index). Therefore, the analytical solution is useless for practical purposes. Several methods were applied to overcome this difficulty. Methods using Monte Carlo simulation [7], [5] are commonly used. We also mention an algebraic approach developed in [6].

In this paper we apply the additive operator splitting (AOS) method, known mainly from the image processing theory. A key idea of this approach is to decompose the multi-dimensional problem into several one-dimensional problems which can be solved very efficiently. We apply the AOS method to solve the Black-Scholes equation instead of solving a high-dimensional integral which is the analytical solution to this equation.

Finally, we deal with error estimates of the used method. Under the assumption on the ratio of time discretization step k and spatial step h we show that our numerical solution based on the AOS technique is of the accuracy order $O(h^2)$ when $h \to 0$.

2 MULTI–ASSET DERIVATIVES

We remind that the price of an index depends on n underlying asset prices S_i, $i = 1, \dots, n$. In general, the index value I can be expressed as

$$I = \sum_{i=1}^{n} w_i S_i \tag{1}$$

where w_i are suitable weights corresponding to the index definition. In general, index derivatives are contracts to buy or sell the index at a future time (expiry, T), with the price, quantity and other specifications defined today. Examples of derivatives include forwards, futures, Call and Put options, and many others. They differ by various types of the so-called pay-off diagrams corresponding to the terminal condition of a derivative.

3 PRICING MODEL: BLACK–SCHOLES EQUATION

The mathematical model describing the time evolution of the derivative value is well known as the Black-Scholes equation. The derivation process consists of two steps. The first step is to apply multidimensional Itô's lemma (see e.g., [4]) to find a stochastic equation governing the evolution of the derivative value V as a function of time t and prices S_i, $i = 1, \dots, n$, of assets building the index.

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Next we construct a self-financing portfolio comprising assets, options on these assets and risk-free bonds. The one-asset case is considered in many standard textbooks, in particular [4] or [8]. It is easy to extend it to a multi-asset case (cf [8]). In this derivation we always assume that the asset prices follow a stochastic differential equation representing the geometric Brownian motion

$$\frac{\mathrm{d}S_i}{S_i} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}Z_i\,, \quad i = 1, 2, \ldots, n\,,$$

where μ_i and σ_i denote the expected rate of return and the volatility of asset i, $\mathrm{d}Z_i$ is the Wiener process' differential for the i-th stock. Let ρ_{ij} denote the correlation coefficient of $\mathrm{d}Z_i$ and $\mathrm{d}Z_j$, i.e.,

$$E(\mathrm{d}Z_i \mathrm{d}Z_j) = \rho_{ij}\mathrm{d}t\,, \quad i, j = 1, 2, \ldots, n\,,\ i \neq j\,.$$

Then the equation describing the evolution of the price V of an index derivative in time has the form

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{n} S_i \frac{\partial V}{\partial S_i} - rV \tag{2}$$

where $0 < S_i < \infty$, $\tau > 0$ and $\tau = T - t$ is time to expiry. Equation (2) is referred to as *the generalized n-dimensional Black-Scholes partial differential equation.* Notice that the equation does not depend on the expected rates of return μ_i of the assets. For different types of derivatives different initial conditions at $\tau = 0$ (i.e., $t = T$) have to be added to equation (2). The initial condition at $\tau = 0$ corresponds to a terminal payoff condition $V(S,T) = V_0(S)$ at expiry $t = T$, where $S = (S_1, S_2, \ldots, S_n)^\top$. It is necessary to add also the boundary conditions to equation (2). Both initial as well as boundary conditions depend on the type of the derivative we deal with. For example, in the case of a Call option on an index constructed as in (1) we have $V_0(S) = V(S,T) = \max(\sum_{i=1}^n w_i S_i - E, 0)$ where E is the so-called exercise price. For comprehensive overview of initial and boundary conditions we refer to [4].

4 ANALYTICAL SOLUTION TO THE B–S PDE

In this section we focus on an analytical solution to the Black-Scholes partial differential equation (2) with an arbitrary initial condition. We seek a solution $V(S,t)$ in the form of a convolution of the initial condition V_0 and the fundamental solution ψ, i.e.,

$$V(S, T-\tau) = e^{-r\tau}\int_{R^n} V_0(\xi)\psi(\xi; S, \tau)\mathrm{d}\xi \tag{3}$$

where $\psi(\xi; S, \tau)$ is a function of an n-dimensional variable $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^\top$. According to [4] V is a solution to (2) iff a function ψ satisfies

$$\frac{\partial \psi}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 \psi}{\partial S_i \partial S_j} + r\sum_{i=1}^{n} S_i \frac{\partial \psi}{\partial S_i} \tag{4}$$

and the initial condition $\psi(\xi; S, 0) = \delta(\xi - S)$. Here $\delta(x)$ stands for the Dirac distribution.

In order to find an explicit form of the function ψ we apply a series of transformations of variables. By using transformations

$$y_i = \frac{1}{\sigma_i}\Big(r - \frac{\sigma_i^2}{2}\Big)\tau + \frac{1}{\sigma_i}\ln S_i\,, \quad i = 1, 2, \ldots, n\,, \tag{5}$$

$$x = \Lambda^{-\frac{1}{2}}Q^\top y \tag{6}$$

and putting $\tilde{\Phi}(x,\tau) = \Phi(y,\tau) = \psi(S,\tau)$ we obtain the following n-dimensional diffusion equation $\frac{\partial\tilde{\Phi}}{\partial\tau} = \frac{1}{2}\Delta\tilde{\Phi}$ where Δ is the so-called Laplace operator. A solution to this equation is known (cf. [9]) and is given by $\tilde{\Phi}(x,\tau) = \frac{1}{(2\pi\tau)^{\frac{n}{2}}}\exp(-\frac{1}{2\tau}|x|^2)$. By returning to original variables we finally obtain the solution $V(S, T-\tau)$ to (2) analytically given as in (3). Hence, if we want to calculate the value of a derivative on an index, we have to solve the high-dimensional integral (3). For example, if the index is comprised of $n = 500$ stocks, we should solve a 500-dimensional integral. This is a very difficult memory space and time consuming problem. In practice, it can be numerically solved in low space dimension $n \leq 5$ only.

5 ANOTHER APPROACH

Failure of practical usage of the analytical formula (3) was a motivation for development of a new method for finding a solution to the generalized Black-Scholes PDE (2). Following the analytical approach from the previous section we will seek a solution to (2) in the form $V(S, T-\tau) = e^{-r\tau}\tilde{\phi}(x,\tau)$ where $\tilde{\phi}$ is a solution to $\frac{\partial\tilde{\phi}}{\partial\tau} = \frac{1}{2}\Delta\tilde{\phi}$ satisfying the initial condition $\tilde{\Phi}(x) = V_0(S)$ where transformation of the variables $S \leftrightarrow x$ is exactly the same as the one defined in (5), (6). Without loss of generality, we will henceforth focus only on finding a solution to

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \Delta u = 0\,, \quad & x \in R^n\,,\ \tau \in [0,T] \\ u(x,0) = u^0(x)\,, \quad & x \in R^n\,. \end{aligned} \tag{7}$$

5.1 Full space-time discretization numerical scheme

First, we discretize (7) in time by dividing the time interval $[0,T]$ into m parts of equal length k and replace the time derivative by the time difference whereby we obtain the semidiscretization of the problem:

$$\frac{u^j(x) - u^{j-1}(x)}{k} - (Au^j)(x) = 0$$

where $Au = \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, and $u^j(x) \approx u(x, jk)$. After some rearrangement we obtain Rothe's scheme (cf. [2])

$$u^j = (I - kA)^{-1} u^{j-1}, \quad j = 1, 2, \ldots, m.$$

Next, we discretize also the spatial variable in each direction by the same number d of internal dividing points with the spatial step of size h. We cut-off the original unbounded domain R^n into a bounded n-dimensional cube $\Omega = (-L, L)^n$ where L is large enough. We use the central differences for the approximation of the second partial derivatives with respect to the space variables x_i, $i = 1, \ldots, n$. This way we obtain the full time-space discretization scheme

$$u^j = (I - kA)^{-1} u^{j-1}, \quad j = 1, 2, \ldots, m \tag{8}$$

where A represents the discretization of the Laplace operator, I is an identity matrix. Because A is a very large square matrix of type $N \times N = d^n \times d^n$ application of any classical numerical method for solving (8) would not be very efficient.

6 THE AOS TECHNIQUE

The additive operator splitting method (AOS) has been developed in [1], [11], [12]. Recall that this method is widely used in image processing, multidimensional signal processing and filtering, etc. Our new application of the additive operator splitting technique to index derivative pricing resides in finding an approximate solution to (8). By using backward transformations we subsequently find the value of a solution to (2) — the price of the multi-asset derivative.

The main idea of the additive operator splitting method is the following: replace the arithmetical mean of operators (or matrices) B_i, $i = 1, \ldots, n$, by the harmonic one, i.e.,

$$\frac{1}{n} \sum_{i=1}^{n} B_i \quad \longleftrightarrow \quad n\Big(\sum_{i=1}^{n} B_i^{-1}\Big)^{-1}.$$

Let $A = \sum_{i=1}^{n} A_i$ where A_i denotes the second partial derivative with respect to x_i. Then $I - kA = \frac{1}{n}\sum_{i=1}^{n} B_i$ where $B_i = I - knA_i$. The matrix from Rothe's scheme is then approximated as follows:

$$(I - kA)^{-1} \quad \longleftrightarrow \quad \frac{1}{n} \sum_{i=1}^{n} (I - knA_i)^{-1}.$$

Hereby, Rothe's scheme is approximated by another one:

$$\bar{u}^j = \frac{1}{n} \sum_{i=1}^{n} (I - knA_i)^{-1} \bar{u}^{j-1}, \quad \bar{u}^0 = u^0, \ j = 1, \ldots, m. \tag{9}$$

Matrices A_i are tridiagonal. From practical point of view, we could finish at this stage because the right hand side of (9) can be solved in a fast and efficient way. But, for the theoretical completeness, it is worth to recognize that each summand in (9) is nothing else but Rothe's approximation to an analytical solution to the one-dimensional partial differential equation

$$\frac{\partial v}{\partial \tau} - n\frac{\partial^2 v}{\partial x_i^2} = 0, \quad x_i \in R, \ \tau \in [0, T],$$

subject to the initial condition at k-th time level: $v(\tilde{x_1}, \ldots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n, 0) =$ $= \bar{u}^{j-1}(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$, where $x_i \in R$. Hence, our new numerical scheme based on the AOS technique can be rewritten in the following form:

$$\bar{\bar{u}}^j = \frac{1}{n} \sum_{i=1}^{n} \bar{\bar{v}}_i^j, \quad j = 1, 2, \ldots, m \tag{10}$$

where $\bar{\bar{v}}_i^j(x) = \int_R G(x_i - \xi_i, k) \bar{\bar{u}}^{j-1}(x_1, \ldots, \xi_i, \ldots, x_n) \mathrm{d}\xi_i$, G is the Green function.

7 THE ORDER OF THE AOS APPROXIMATION

We will examine the error estimate between the numerical Rothe scheme, the AOS scheme (9) based on tridiagonal systems, and the approximation (10) of the AOS scheme based on solving one-dimensional integrals only. We shall find the error estimate for the norm of the difference between solutions obtained by the first and the second scheme in the usual form

$$\|\bar{u}^j - u^j\| \le \mathrm{const} \cdot k^{\nu_1}$$

where ν_1 is an exponent and k is the time discretization step. It follows from [1] that for the difference between the corresponding matrices of the schemes the following statement holds.

Theorem ([1], Theorem 4.1). *Let $n \in N$, $k \ge 0$, and let $A_1, \ldots, A_n \in R^{N \times N}$ be simultaneously diagonalizable matrices with eigenvalues in the left half plane. Then there exists a constant C with*

$$\|M - \overline{M}\| \le Ck^2$$

where $M = (I - kA)^{-1}$ and $\overline{M} = \frac{1}{n}\sum_{i=1}^{n}(I - knA_i)^{-1}$.

The assumptions made in the previous theorem are fulfilled for matrices A_i representing the second partial derivatives with respect to individual variables.

Under the additional assumption $\|\bar{u}^j\| < \widetilde{C}, \widetilde{C} \in R$, we obtain the following error estimate:

$$\|u^j - \bar{u}^j\| \le \frac{1}{1-\theta} C\widetilde{C}k^2 \tag{11}$$

where $\theta = \max_{\lambda \in \sigma(M)} |\lambda| < 1$. Long but straightforward calculations show that constants $C, \widetilde{C}$ can be estimated as follows: $C = n^2 \frac{16}{h^4}$ and $\widetilde{C} = \|u^0\|$. For details we refer to [3]. It yields the estimate

$$\|\bar{u}^j - u^j\| \le \frac{1}{1-\theta} C\widetilde{C}k^2 \le O(\frac{k}{h^4}).$$

Furthermore, the estimate of the norm of the difference between the analytical solution to a one-dimensional parabolic equation and its numerical counterpart based on the implicit (in time) Euler scheme is known (see [10]) and reads as follows:

$$\|\bar{\bar{u}}^j - \bar{u}^j\| \le O(k + h^2),$$

which is also the error between our second and third scheme. Hence, the error estimate of our method is

$$\begin{aligned}\|\bar{\bar{u}}^j - u^j\| \le \|\bar{\bar{u}}^j - \bar{u}^j\| + \|\bar{u}^j - u^j\| &\le O(k + h^2) + O(\frac{k}{h^4}) \\ &\le O(k + h^2 + \frac{k}{h^4}).\end{aligned}$$

By choosing an appropriate ratio of the time step k and the spatial step h we can control this error. Especially, the choice $k = h^6$ implies $\|\bar{\bar{u}}^j - u^j\| \le O(h^2)$.

8 CONCLUSIONS

In this paper, we dealt with pricing of index options, i.e., options whose underlying is a set of assets. Instead of solving a high-dimensional integral representing the analytical solution to the Black-Scholes PDE we applied the AOS technique in order to find an approximate solution to the generalized Black-Scholes equation. By using this method, the complex multi-dimensional problem was replaced by several simple one-dimensional problems based on solving tridiagonal systems or one-dimensional integrals. We have shown that by choosing an appropriate relationship between the time-discretization step k and the spatial step h we can control the order of the error caused by the AOS method, and achieve the order of $O(h^2)$ as $h \to 0$. Another advantage of the proposed method in comparison with other approaches resides in possible sequential parallelization of the algorithm. Computer implementation of the method and its application to real data will be the subject of further study.

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