KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
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4TH ACTUARIAL AND FINANCIAL MATHEMATICS DAY

February 10, 2006

Michèle Vanmaele, Ann De Schepper, Jan Dhaene,
Huguette Reynaerts, Wim Schoutens & Paul Van Goethem (Eds.)

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ON THE RISK ADJUSTED PRICING METHODOLOGY MODEL FOR PRICING DERIVATIVE SECURITIES

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Abstract

In this review paper we discuss a nonlinear model of Black-Scholes type for pricing derivative securities in the presence of both transaction costs as well as the risk from a volatile portfolio. The model is derived by following the Risk Adjusted Pricing Methodology approach proposed by Kratka (1998). It turns outs that prices of plain vanilla options can be computed from a solution to a fully nonlinear parabolic equation in which a diffusion coefficient representing volatility nonlinearly depends on the asset price and option's Gamma. It gives rise to explain several striking phenomena in option pricing analytically, including, in particular, the volatility smile behavior of the implied volatility.

1. INTRODUCTION

According to the classical theory due to Black, Scholes and Merton the price of an option in an idealized financial market can be computed from a solution to the well-known Black-Scholes linear parabolic equation (see e.g. Black and Scholes (1973), Kwok (1998), Dewynne et al. (1993), Hull (1989)). Assuming that the underlying asset follows a geometric Brownian motion one can derive a governing partial differential equation for the price of an option. We remind ourselves that the equation governing time evolution of the price $V(S,t)$ of an option is the following parabolic PDE:

$$\partial_t V + (r-q)S\partial_S V + \frac{1}{2}\hat{\sigma}^2 S^2 \partial_S^2 V - rV = 0 \tag{1}$$

where $\hat{\sigma}$ is a constant volatility of the underlying asset price process, $r > 0$ is the interest rate of a zero-coupon bond, $q \geq 0$ is the dividend yield rate. A solution $V = V(S,t)$ represents the price of an option at time $t \in [0,T]$ if the price of an underlying asset is $S > 0$. If the volatility $\hat{\sigma}$ is assumed to be constant the above equation is called the Black-Scholes equation derived by Black and Scholes (1973), and, independently by Merton (c.f. Kwok (1998)). The linear Black-Scholes equation has been derived under restrictive assumptions like e.g. perfect replication of a portfolio, frictionless, liquidity, complete markets, etc. Following this theory we can find a value of an option over moderate time intervals assuming transaction costs and the risk from a volatile portfolio are negligible. A solution to the linear Black-Scholes equation then provides a perfectly replicating hedging portfolio.

In recent years, some of these restrictive assumptions have been relaxed in order to model, for instance, the presence of transaction costs (Hoggard et al. (1994)), imperfect replication and investor's preferences (Barles and Soner (1998)), introduction of a given stock-trading strategy of a large trader (Frey and Patie (2002), Frey and Stremme (1997)), risk from unprotected portfolio (Kratka (1998), Jandačka and Ševčovič (2005)). These models lead to a generalized Black-Scholes equation for the price of an option in which the volatility need not be necessarily constant and it may depend on the asset price as well as the option price. More precisely, in these models the volatility has the general form:

$$\sigma^2 = \sigma^2(S^2\partial_S^2 V, S, T-t)\,. \tag{2}$$

For instance, if transaction costs are taken into account then the classical Black-Scholes theory is no longer applicable. In order to maintain the delta hedge one has to make frequent portfolio adjustments yielding thus a substantial increase in transaction costs. The effect of nontrivial transaction costs can be described by the so-called Leland model (cf. Hoggard et al. (1994)). In this model the volatility σ is given by $\sigma^2 = \hat\sigma^2(1 - \mathrm{Le}\,\mathrm{sgn}(\partial_S^2 V))$ where $\hat\sigma > 0$ is a constant historical volatility of the underlying asset price process and $\mathrm{Le} \ge 0$ is the so-called Leland constant given by $\mathrm{Le} = \sqrt{2/\pi}C/(\hat\sigma\sqrt{\Delta t})$. Here $C \ge 0$ is a constant round trip transaction cost per unit dollar of transaction in the assets market and $\Delta t > 0$ is the time-lag between portfolio adjustments. Since $S > 0$ we have

$$\sigma^2(S^2\partial_S^2 V, S, T-t) = \hat\sigma^2(1 - \mathrm{Le}\,\mathrm{sgn}(\partial_S^2 V))\,. \tag{3}$$

By assuming that investor's preferences are characterized by an exponential utility function, Barles and Soner (1998) derived a nonlinear Black-Scholes equation with the volatility σ given by

$$\sigma^2(S^2\partial_S^2 V, S, T-t) = \hat\sigma^2\left(1 + \Psi(a^2 e^{r(T-t)} S^2 \partial_S^2 V)\right)^2$$

where $a > 0$ is the risk-aversion coefficient and Ψ is a solution to the ODE: $\Psi'(x) = (\Psi(x) + 1)/(2\sqrt{x\Psi(x)} - x)$, $\Psi(0) = 0$. Another popular model has been derived for the case when the asset dynamics takes into account the presence of feedback effects. Frey and Stremme (1997) (see also Frey and Patie (2002)) introduced directly the asset price dynamics in the case when the large trader chooses a given stock-trading strategy. The volatility σ is nonconstant and it is given by:

$$\sigma^2(S^2\partial_S^2 V, S, T-t) = \hat\sigma^2\left(1 - \varrho S \partial_S^2 V\right)^{-2}$$

where $\hat\sigma, \varrho > 0$ are constants.

The last example of a nonlinear Black-Scholes equation is the so-called Risk Adjusted Pricing Methodology model proposed by Kratka (1998), revisited and modified by Jandačka and Ševčovič (2005). The idea of derivation of this model is simple: in order to maintain (imperfect) replication of a portfolio by the delta hedge one has to make frequent portfolio adjustments yielding thus a substantial increase in transaction costs. On the other hand, rare portfolio adjustments may lead to the increase of the risk from a volatile (unprotected) portfolio. Minimization of the sum of the measure of transaction costs and the risk from unprotected portfolio yields the optimal time lag between two consecutive portfolio adjustments. The resulting model is again a nonlinear Black-Scholes type equation with the volatility of the form

$$\sigma^2(S^2\partial_S^2 V, S, T-t) = \hat{\sigma}^2 \left(1 - \mu(S\partial_S^2 V)^{\frac{1}{3}}\right) \tag{4}$$

for $T - t > 0$ large enough where $\mu \geq 0$ is a coefficient proportional to the risk from volatile portfolio and transaction costs measures. In the next section we recall key steps and ideas of derivation of the Risk Adjusted Pricing Methodology (RAPM) model. We will furthermore present explanation of the volatility smile based on the RAPM model. We also discuss calibration of the RAPM model to real market data. We also introduce two new implied quantities: the implied RAPM volatility and implied RAPM risk coefficients. Finally, we will present results of calibration of these new implied quantities to real option and stock market data.

2. RISK ADJUSTED PRICING METHODOLOGY MODEL

In this section we recall key steps of derivation of the RAPM model. The original model was proposed by Kratka (1998). In Jandačka and Ševčovič (2005) we modified his approach (we chose a different measure for risk from unprotected portfolio) in order to construct a model which is scale invariant and mathematically well posed. These two important features were missing in the original model of Kratka. The model is based on the Black-Scholes parabolic PDE in which transaction costs are described by the Hoggard, Whalley and Wilmott extension of the Leland model (cf. Hoggard et al. (1994), Kwok (1998), Hull (1989)) whereas the risk from a volatile portfolio is described by the average value of the variance of the synthesized portfolio. Transaction costs as well as the volatile portfolio risk depend on the time-lag between two consecutive transactions. We define the total risk premium as a sum of transaction costs and the risk cost from the unprotected volatile portfolio. By minimizing the total risk premium functional we obtain the optimal length of the hedge interval. It also gives us a new strategy for hedging derivative securities based on option's Gamma parameter.

Concerning the dynamics of an underlying asset we will assume that the asset price $S = S(t), t \geq 0,$ follows a geometric Brownian motion with a drift ρ, standard deviation $\hat{\sigma} > 0$ and it may pay continuous dividends, i.e.

$$dS = (\rho - q)Sdt + \hat{\sigma}SdW \tag{5}$$

where dW denotes the differential of the standard Wiener process and $q \geq 0$ is a continuous dividend yield rate. This assumption is usually made when deriving the classical Black-Scholes equation (see e.g. Hull (1989), Kwok (1998)).

Similarly as in the derivation of the classical Black-Scholes equation we construct a synthesized portfolio Π consisting of a one option with a price V and δ assets with a price S per one asset:

$$\Pi = V + \delta S\,. \tag{6}$$

We recall that the key idea in the Black-Scholes theory is to examine the differential $\Delta\Pi$ of equation (6). The right-hand side of (6) can be differentiated by using Itô's formula whereas portfolio's increment $\Delta\Pi(t) = \Pi(t+\Delta t) - \Pi(t)$ of the left-hand side can be expressed as follows:

$$\Delta\Pi = r\Pi\Delta t + \delta q S\Delta t \tag{7}$$

where $r > 0$ is a risk-free interest rate of a zero-coupon bond. In the real world, such a simplified assumption is not satisfied and a new term measuring the total risk should be added to (7). More precisely, the change of the portfolio Π is composed of two parts: the risk-free interest rate part $r\Pi\Delta t$ and the total risk premium: $r_R S\Delta t$ where r_R is a risk premium per unit asset price. It means that $\Delta\Pi = r\Pi\Delta t + r_R S\Delta t$. The total risk premium r_R consists of the transaction risk premium r_{TC} and the portfolio volatility risk premium r_{VP}, i.e. $r_R = r_{TC} + r_{VP}$. Hence

$$\Delta\Pi = r\Pi\Delta t + \delta q S\Delta t + (r_{TC} + r_{VP})S\Delta t\,. \tag{8}$$

Our next goal is to show how these risk premium measures r_{TC}, r_{VP} depend on the time lag and other quantities, like e.g. $\hat\sigma, S, V$, and derivatives of V. The problem can be decomposed in two parts: modeling the transaction costs measure r_{TC} and volatile portfolio risk measure r_{VP}.

2.1. Modeling transaction costs and volatile portfolio risk measures

In practice, we have to adjust our portfolio by frequent buying and selling of assets. In the presence of nontrivial transaction costs, continuous portfolio adjustments may lead to infinite total transaction costs. A natural way how to consider transaction costs within the frame of the Black-Scholes theory is to follow the well known Leland approach extended by Hoggard, Whalley and Wilmott (cf. Hoggard et al. (1994), Kwok (1998)). In what follows, we recall crucial lines of the Hoggard, Whalley and Wilmott derivation of Leland's model in order to show how to incorporate the effect of transaction costs into the governing equation. More precisely, we will derive the coefficient of transaction costs r_{TC} occurring in (8).

Let us denote by C the round trip transaction cost per unit dollar of transaction. Then

$$C = (S_{ask} - S_{bid})/S \tag{9}$$

where S_{ask} and S_{bid} are the so-called Ask and Bid prices of the asset, i.e. the market price offers for selling and buying assets, respectively. Here $S = (S_{ask} + S_{bid})/2$ denotes the mid value.

In order to derive the term r_{TC} in (8) measuring transaction costs we will assume, for a moment, that there is no risk from the volatile portfolio, i.e. $r_{VP} = 0$. Then $\Delta V + \delta\Delta S = \Delta\Pi = r\Pi\Delta t + \delta q S\Delta t + r_{TC} S\Delta t$. Following Leland's approach (c.f. Hoggard et al. (1994)), using Itô's formula and assuming δ-hedging of a synthetised portfolio Π one can derive that the coefficient r_{TC} of transaction costs is given by the formula:

$$r_{TC} = \frac{C\hat\sigma S}{\sqrt{2\pi}} \left|\partial_S^2 V\right| \frac{1}{\sqrt{\Delta t}} \tag{10}$$

(see (Hoggard et al. 1994, Eq. (3)) and also formula (3)).

Next we focus our attention to the problem how to incorporate a risk from a volatile portfolio into the model. In the case when a portfolio consisting of options and assets is highly volatile an investor usually asks for a price compensation. Notice that exposure to risk is higher when the time-lag between portfolio adjustments is higher. We shall propose a measure of such a risk based on the volatility of a fluctuating portfolio. It can be measured by the variance of relative increments of the replicating portfolio $\Pi = V + \delta S$, i.e. by the term $var((\Delta\Pi)/S)$. Hence it is reasonable to define the measure r_{VP} of the portfolio volatility risk as follows:

$$r_{VP} = R\,\frac{var\left(\frac{\Delta\Pi}{S}\right)}{\Delta t}\,. \tag{11}$$

In other words, r_{VP} is proportional to the variance of a relative change of a portfolio per time interval Δt. A constant R is the so-called *risk premium coefficient*. It can be interpreted as the marginal value of investor's exposure to a risk. If we apply Itô's formula to the differential $\Delta\Pi = \Delta V + \delta\Delta S$ we obtain $\Delta\Pi = (\partial_S V + \delta)\,\hat\sigma S\Delta W + \frac{1}{2}\hat\sigma^2 S^2\Gamma(\Delta W)^2 + \mathcal{G}$ where $\Gamma = \partial_S^2 V$ and $\mathcal{G} = (\partial_S V + \delta)\rho S\Delta t + \partial_t V\Delta t$ is a deterministic term, i.e. $E(\mathcal{G}) = \mathcal{G}$ in the lowest order Δt-term approximation. Thus

$$\Delta\Pi - E(\Delta\Pi) = (\partial_S V + \delta)\,\hat\sigma S\phi\sqrt{\Delta t} + \frac{1}{2}\hat\sigma^2 S^2(\phi^2 - 1)\Gamma\Delta t$$

where ϕ is a random variable with the standard normal distribution such that $\Delta W = \phi\sqrt{\Delta t}$. Hence the variance of $\Delta\Pi$ can be computed as follows:

$$var(\Delta\Pi) = E\left([\Delta\Pi - E(\Delta\Pi)]^2\right) = E\left([(\partial_S V + \delta)\hat\sigma S\phi\sqrt{\Delta t} + \tfrac{1}{2}\hat\sigma^2 S^2\Gamma\left(\phi^2 - 1\right)\Delta t]^2\right)\,.$$

Similarly, as in the derivation of the transaction costs measure r_{TC} we assume δ-hedging of portfolio adjustments, i.e. we choose $\delta = -\partial_S V$. Since $E((\phi^2 - 1)^2) = 2$ we obtain an expression for the risk premium r_{VP} in the form:

$$r_{VP} = \frac{1}{2}R\hat\sigma^4 S^2\Gamma^2\Delta t\,. \tag{12}$$

Notice that in our approach the increase in the time-lag Δt between consecutive transactions leads to a linear increase of the risk from a volatile portfolio where the coefficient of proportionality depends on the asset price S, option's Gamma, $\Gamma = \partial_S^2 V$, as well as the constant historical volatility $\hat\sigma$ and the risk premium coefficient R.

2.2. Risk adjusted Black-Scholes equation

The total risk premium $r_R = r_{TC} + r_{VP}$ consists of two parts: transaction costs premium r_{TC} and the risk from a volatile portfolio r_{VP} premium defined as in (10) and (12), respectively. We assume that an investor is risk averse and he/she wants to minimize the value of the total risk premium r_R. For this purpose one has to choose the optimal time-lag Δt between two consecutive portfolio adjustments. As both r_{TC} as well as r_{VP} depend on the time-lag Δt so does the total risk premium r_R. In order to find the optimal value of Δt we have to minimize the following function:

$$\Delta t \mapsto r_R = r_{TC} + r_{VP} = \frac{C|\Gamma|\hat\sigma S}{\sqrt{2\pi}}\frac{1}{\sqrt{\Delta t}} + \frac{1}{2}R\hat\sigma^4 S^2\Gamma^2\Delta t\,.$$

The unique minimum of the function $\Delta t \mapsto r_R$ is attained at the time-lag $\Delta t_{opt} = K^2/(\hat{\sigma}^2|S\Gamma|^{\frac{2}{3}})$ where $K = (C/(R\sqrt{2\pi})^{\frac{1}{3}}$. For the minimal value of the function $\Delta t \mapsto r_R(\Delta t)$ we have

$$r_R(\Delta t_{opt}) = \frac{3}{2}\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}} \hat{\sigma}^2 |S\Gamma|^{\frac{4}{3}} . \tag{13}$$

Taking into account both transaction costs as well as risk effects from a volatile portfolio, we have shown that the equation for the change $\Delta\Pi$ of a portfolio Π reads as:

$$\Delta V + \delta \Delta S = \Delta\Pi\Delta t = r\Pi + \delta q S \Delta t + r_R S \Delta t$$

where r_R represents the total risk premium, $r_R = r_{TC} + r_{VP}$. On the other hand, by the no-arbitrage principle the change $\Delta\Pi$ in the portfolio Π is equal to the change $r\Pi\Delta t$ of secure bonds with the interest rate $r > 0$. Applying Itô's lemma to a smooth function $V = V(S,t)$ and assuming the δ-hedging strategy for the portfolio adjustments we finally obtain the following generalization of the Black-Scholes equation for valuing options:

$$\partial_t V + \frac{\hat{\sigma}^2}{2} S^2 \partial_S^2 V + (r-q) S \partial_S V - rV - r_R S = 0 .$$

By taking the optimal value of the total risk coefficient r_R derived as in (13), the option price V is a solution to the following nonlinear parabolic equation:

(*Risk adjusted Black-Scholes equation*)

$$\partial_t V + \frac{\hat{\sigma}^2}{2} S^2 \left(1 - \mu (S\partial_S^2 V)^{\frac{1}{3}}\right) \partial_S^2 V + (r-q) S \partial_S V - rV = 0 , \quad \text{where } \mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}} . \tag{14}$$

In the case there are neither transaction costs ($C = 0$) nor the risk from a volatile portfolio ($R = 0$) we have $\mu = 0$. Then equation (14) reduces to the original Black-Scholes linear parabolic equation (1). We note that equation (14) is a backward parabolic PDE if and only if the function $\beta(H) = \frac{\hat{\sigma}^2}{2}(1 - \mu H^{\frac{1}{3}})H$ is an increasing function in the variable $H := S\Gamma = S\partial_S^2 V$. Hence, in order to verify parabolicity of (14), we have to assume the following condition:

$$S\partial_S^2 V(S,t) < \kappa := \left(\frac{3}{4\mu}\right)^3 . \tag{15}$$

If we consider prices of either Call or Put options computed from a solution to the classical Black-Scholes equation (1) then the term $S\Gamma = S\partial_S^2 V(S,t)$ becomes infinite at $S = E$ for $t \to T^-$ and the (15) condition is violated. The same feature is present in the generalized equation (14) yielding thus the change of the sign of the diffusion coefficient of (14) close to expiration time T. This is why we have to modify the model equation (14) near the expiration time, i.e. for $0 < T - t \ll 1$. The idea of modified early exercise behavior was introduced by Jandačka and Ševčovič (2005). It consists in determining the so-called switching time $t_* < T$ such that the RAPM model is modified as follows: the price of an option is given by a solution $V(S,t)$ to the following problem:

1. $V(S,t)$ is a solution to equation (14) on the time interval $0 < t < t_*$; whereas

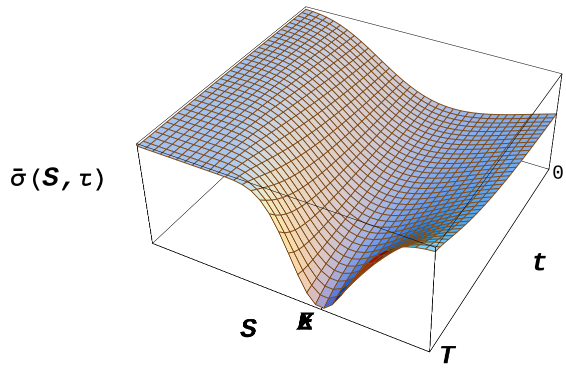

Figure 1: Explanation of the volatility smile based on RAPM. The implied volatility surface $(S,t) \mapsto \bar{\sigma}(S,t)$.

2. $V(S,t)$ is a solution to the linear Black-Scholes equation (1) on the time interval $t_* < t < T$ and satisfying the prescribed pay-off diagram at expiry $t = T$;

3. function $V(S,t)$ is continuous in $t = t_*$.

The switching time $t_* < T$ is chosen as nearest time to expiry T for which the value of $S\Gamma = S\partial_S^2 V$ is less or equal to the threshold value κ. Now if we compute the quantity $S\Gamma$ for plain Call or Put options by using the original Black-Scholes model (1) we obtain $\max_{S>0} S\Gamma(S,t_*) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2(T-t_*)}}$. Then we can deduce

$$T - t_* = \frac{C}{R\hat{\sigma}^2}\,. \tag{16}$$

As t_* must be positive we have $T - t_* < T$ it also turns out that we have to require the following structural condition

$$0 \le C < \hat{\sigma}^2 RT\,. \tag{17}$$

to be satisfied (see Jandačka and Ševčovič (2005) for details).

3. CALIBRATION OF THE RAPM MODEL TO REAL MARKET DATA

The purpose of this section is to discuss application of the RAPM model to real market option price data. We also introduce a concept of the so-called implied RAPM volatility σ_{RAPM} and the implied risk premium coefficient R. First we discuss capability of RAPM model to explain the so-called volatility smile analytically.

3.1. Volatility smile and RAPM model

One of the most striking phenomena in the Black-Scholes theory is the so-called *volatility smile* phenomenon. Notice that derivation of the classical Black-Scholes equation (1) relies on the assumption of a constant value of the volatility σ. On the other hand, as it might be documented by

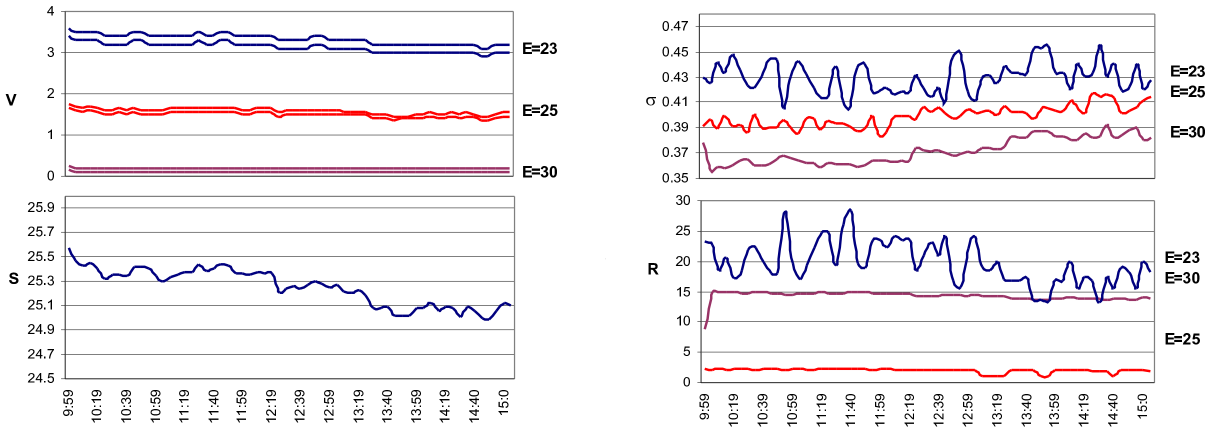

Figure 2: Intra-day behavior of Microsoft stocks (April 4, 2003) and shortly expiring Call options with expiry date April 19, 2003. Computed implied volatilities $\hat\sigma_{RAPM}$ and risk premium coefficients R.

many examples observed in market options data sets such an assumption is often violated. More precisely, the implied volatility σ_{impl} is no longer constant and it may depend on the asset price S, the strike price E as well as the time t.

In the RAPM approach we are able to explain the volatility smile analytically. The Risk adjusted Black-Scholes equation (14) can be viewed as an equation with a variable volatility coefficient, i.e. $\partial_t V + \frac{1}{2}\bar\sigma^2(S,t)\partial_S^2 V + (r-q)S\partial_S V - rV = 0$ where $\Gamma = \partial_S^2 V$ and the volatility $\bar\sigma^2(S,t)$ depends itself on a solution $V = V(S,t)$ as follows:

$$\bar\sigma^2(S,t) = \hat\sigma^2\left(1 - \mu(S\Gamma)^{1/3}\right) . \tag{18}$$

In Fig. 1 we show the dependence of the function $\bar\sigma(S,t)$ on the asset price S and time t. It should be obvious that the function $S \mapsto \bar\sigma(S,t)$ has a convex shape near the exercise price E. We have used the RAPM model in order to compute values of $\Gamma = \partial_S^2 V$. We chose $\mu = 0.2, \hat\sigma = 0.3, r = 0.011$, and $T = 0.5$. In Fig. 1 we show the dependence of the function $\bar\sigma(S,t)$ on the asset price S and time t. It should be obvious that the function $S \mapsto \bar\sigma(S,t)$ has a convex shape near the exercise price E.

3.2. Implied volatility and risk premium in RAPM model

Let us denote $V(S,t;C,\hat\sigma,R)$ the value of a solution to (14) with parameters $C,\hat\sigma,R$. Suppose that the coefficient of transaction costs C is known from and is given by (9). In real option market data we can observe different Bid and Ask prices for an option, $V_{bid} < V_{ask}$, respectively. Let us denote by V_{mid} the mid value, i.e. $V_{mid} = \frac{1}{2}(V_{bid} + V_{ask})$. By the RAPM model we are able to explain such a Bid-Ask spread in option prices. The lower Bid price corresponds to a solution to the RAPM model with some nontrivial risk premium R whereas the mid value V_{mid} corresponds to a solution $V(S,t)$ for vanishing risk premium $R = 0$, i.e. to a solution of the linear Black-Scholes equation (1).

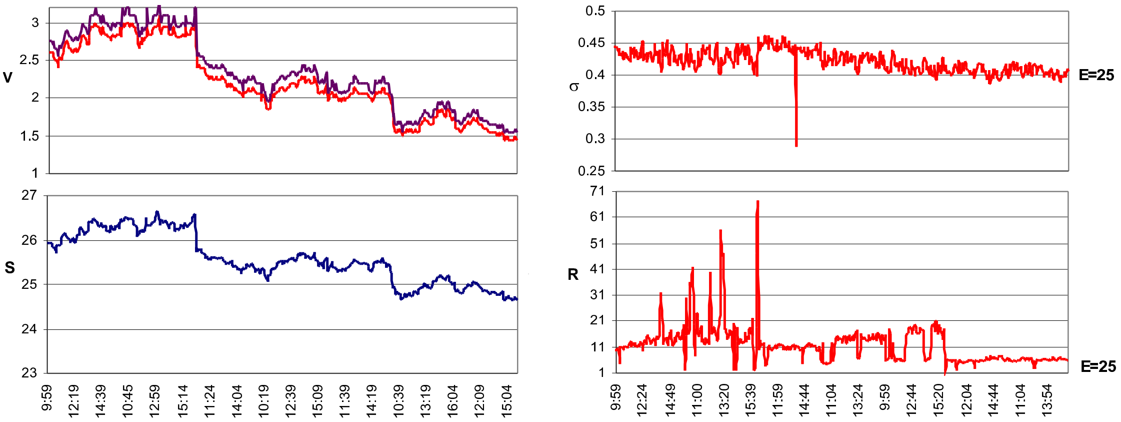

Figure 3: One week behavior of Microsoft stocks (March 20 - 27, 2003) and Call options with expiration date April 19, 2003. Computed implied volatilities $\hat{\sigma}_{RAPM}$ and risk premiums R.

In order to calibrate the RAPM model we are seeking for a couple $(\hat{\sigma}_{RAPM}, R)$ such that $V_{bid} = V(S, t; C, \hat{\sigma}_{RAPM}, R)$ and $V_{mid} = V(S, t; C, \hat{\sigma}_{RAPM}, 0)$. It means that we have to find a solution to a nonlinear problem:

$$F(\hat{\sigma}, R) = (V_{bid}, V_{mid}) \tag{19}$$

where the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as: $F(\hat{\sigma}, R) = (V(S, t; C, \hat{\sigma}, R), V(S, t; C, \hat{\sigma}, 0))$. It can be solved numerically by means of the Newton-Kantorovich iterative method for solving algebraic equations. A solution $V(S, t; C, \hat{\sigma}, R)$ can be computed from the Risk adjusted Black-Scholes equation by means of finite difference (see Jandačka and Ševčovič (2005) for details).

As an example we considered sample data sets for Call options on Microsoft stocks. We considered a flat interest rate $r = 0.02$, a constant transaction cost coefficient $C = 0.01$ estimated from (9), and we assumed that the underlying asset pays no dividends, i.e. $q = 0$. In Fig. 2 we present results of calibration of implied couple $(\hat{\sigma}_{RAPM}, R)$. Interestingly enough, two Call options with higher strike prices $E = 25, 30$ had almost constant implied risk premium R. On the other the risk premium of an option with lowest $E = 23$ was fluctuating and it had highest average of R.

Finally, in Fig. 3 we present one week behavior of implied volatilities and risk premium coefficients for the Microsoft Call option on $E = 25$ expiring at $T =$ April 19, 2003. In the beginning of the investigated period the risk premium coefficient R was rather high and fluctuating. On the other hand, it tends to a flat value of $R \approx 5$ at the end of the week. Interesting feature can be observed at the end of the second day when both stock and option prices went suddenly down. The time series analysis of the implied volatility $\hat{\sigma}_{RAPM}$ from first two days was unable to predict such a behavior. On the other, high fluctuation in the implied risk premium R during first two days can send a signal to an investor that sudden changes can be expected in the near future.

4. CONCLUSIONS

In this paper we discussed the Risk Adjusted Pricing Methodology model for for pricing derivative securities in the presence of both transaction costs as well as the risk from unprotected portfolio. We showed that the option price can be deduced from a solution to a nonlinear parabolic PDE. The governing equation extends the classical Black-Scholes equation and Leland's equation to the case when the risk from unprotected portfolio is taken into account. We have performed extensive numerical testing of the model and compared the results to real option market data. Furthermore, we introduced a concept of the so-called implied RAPM volatility and implied risk premium coefficients. We have computed these implied quantities for sample option data sets and we have indicated how these implied factors can be used in qualitative analysis of option market data sets.

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De Koninklijke Vlaamse Academie van België voor Wetenschappen en Kunsten coördineert jaarlijks tot 25 wetenschappelijke bijeenkomsten, ook contactfora genoemd, in de domeinen van de natuurwetenschappen (inclusief de biomedische wetenschappen), menswetenschappen en kunsten. De contactfora hebben tot doel Vlaamse wetenschappers of kunstenaars te verenigen rond specifieke thema's.

De handelingen van deze contactfora vormen een aparte publicatiereeks van de Academie.

Contactforum "4th Actuarial and Financial Mathematics Day" (10 februari 2006, Prof. M. Vanmaele)

De "4th Actuarial and Financial Mathematics Day" is een vaste waarde geworden als contactforum. Niet alleen academici maar ook heel wat collega's uit de bank- en verzekeringswereld blijven de weg vinden naar dit jaarlijkse evenement. Het is de gelegenheid bij uitstek om op de hoogte te blijven van het recente onderzoek op het vlak van financiële en actuariële wiskunde in België en van nieuwe uitdagingen die ons te wachten staan zoals in het kader van Basel II. Naast twee gastsprekers kwamen doctoraatsstudenten, postdocs en mensen uit de bedrijfswereld aan bod. In deze publicatie vindt u een neerslag van de voorgestelde onderwerpen. Alle onderwerpen kunnen gesitueerd worden in het ruime gebied van financiële en actuariële toepassingen van wiskunde, maar met een grote variatie: de bijdragen betreffen "capital allocation" problemen, modellen voor kredietrisico, voor stop-loss premies en voor basket- en spreadopties, risicomanagement van coupon bonds, etc.