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Dynamic Accumulation Model for the Second Pillar of the Slovak Pension System

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1. Introduction

Before January 2005, pensions in Slovakia were operated by the unfunded pay-as-you-go system. Mainly because of high unemployment and low contributions paid on behalf of the unemployed by the government as well as a high rate of contribution evasions, the system generated deficits. The demography crisis was supposed to generate further pressure on the balance of the pay-as-you-go system. In April 2003 the government passed the Principles of Pension Reform in the Slovak Republic. The goals of the pension reform were to secure a stable flow of high pensions to the beneficiaries, and sustainability and overall stability of the system. Corresponding legislation, as passed in December 2003, established a system based on three pillars:

1. the mandatory non-funded first pillar (pay-as-you-go pillar),
2. the mandatory fully funded second pillar,
3. the voluntary fully funded third pillar.

The contribution rates were set for the first pillar at 19.75 % (old age 9 %, disability and survival 6 % and reserve fund 4.75 %) and for the second pillar at 9 %. The total rate is about 0.75 % higher than the old one. A thorough description of the Slovak pension reform with calculations of the balance of the pension system and expected level of pensions in the new system can be found in (Goliaš, 2003), (Melicherčík – Ungvarský, 2004), (Thomay, 2002).

Compared to Poland and Hungary, the Slovak second pillar is more substantial. Contribution rates are higher in Slovakia – compared to 7.3 % in Poland and 6 % (with a possible future increase to 8 %) in Hungary. A thorough description of the pension reforms in Hungary and Poland can be found in (Benczúr, 1999), (Chlon et al., 1999), (Fultz, 2002), (Palacios – Rocha, 1998) and (Simonovits, 2000).

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TABLE 1 Limits for Investment for the Pension Funds

Fund type	Stocks	Bonds and money market instruments
Growth Fund	up to 80 %	at least 20 %
Balanced Fund	up to 50 %	at least 50 %
Conservative Fund	no stocks	100 %

The savings in the second pillar are managed by pension-asset administrators. Each pension-asset administrator manages three funds: Growth Fund, Balanced Fund and Conservative Fund, each of them with different limits for investment (see *Table 1*). At the same time instant savers may hold assets in one fund only. In the last 15 years preceding retirement, the saver may not hold assets in the Growth Fund and in the last seven years all assets must be in the Conservative Fund. Even with these restrictions the contributors have some space for individual decisions with regard to which fund is optimal in a specific situation (the age of the contributor, the saved amount, the past performance of the pension funds). The aim of this paper is to study whether the above restrictions for the funds can be illustrated by a mathematical model and to calculate optimal strategies for switching between the pension funds (Growth, Balanced and Conservative) keeping in mind the risk preferences of the contributors. Our model indicates that adopted pension-fund regulations can be supported by means of a dynamic accumulation model.

The paper is organized as follows: In Section 2 we present a simple example supporting the idea of gradual decreasing of the risk when saving for the future pension. We also give a motivation for studying the dynamic accumulation models. Section 3 contains the formulation of the dynamic stochastic programming accumulation model and the numerical scheme for finding a solution of this model. In Section 4 we present the calculated results and we discuss the sensitivity of fund-switching strategies with respect to various scenarios of development of financial markets, wage growth development as well as individual risk preferences. At the end of the section we compare dynamic and static strategies using the mean-variance framework. The last section contains final remarks and conclusions.

2. First Run a Risk, then Secure Savings

Pension funds usually hold portfolio consisting of bonds and equities. Limits for their weights in the portfolio may differ across countries. In Slovakia, each pension company manages three funds: Growth Fund, Balanced Fund and Conservative Fund, each of them with different limits for investment (see Table 1). As already mentioned in the Introduction, instant savers may hold assets in one fund only and they may not hold assets in the Growth Fund in the last 15 years preceding retirement. Moreover, all assets should be held in the Conservative Fund in the last seven years preceding retirement. The intention of these restrictions and government regulations was to lower the risk of the value of savings falling substantially shortly before retirement.

TABLE 2 Risk-Return Analysis of Different Strategies

No.	r_1	r_2	σ_1	σ_2	$E(M)$	σ_M	$E(M)-\sigma_M$	$E(M)-2\sigma_M$
1	5 %	5 %	0 %	0 %	2.1525	0.00 %	2.1525	2.1525
2	10 %	10 %	10 %	10 %	2.3100	23.73 %	2.0727	1.8354
3	10 %	5 %	10 %	0 %	2.2050	10.50 %	2.1000	1.9950
4	5 %	10 %	0 %	10 %	2.2550	20.50 %	2.0500	1.8450
5	10 %	cond.	10 %	cond.	2.2753	18.73 %	2.0880	1.9007

For the sake of simplicity let us consider a plain two-period model of saving (the length of each period is one year). At the beginning of each year the saver deposits an amount A. Suppose that the returns of the deposits are r_1 and r_2 in the first and second year, respectively. The saved amount M after two years is

$$M = A(1 + r_1)(1 + r_2) + A(1 + r_2) = A(2 + r_1)(1 + r_2)$$

Then for the sensitivity of M with respect to r_1 and r_2 we have

$$\frac{\partial M}{\partial r_1} = A(1 + r_2), \quad \frac{\partial M}{\partial r_2} = A(2 + r_1)$$

and therefore

$$\frac{\partial M}{\partial r_2} > \frac{\partial M}{\partial r_1} \tag{1}$$

for realistic asset returns r_1 and r_2. This is in accordance with intuition that the saved amount is more sensitive to later returns than to earlier ones. If the individual made just a single contribution at the start of his/her working career, the impact on his/her final pension wealth would be the same regardless of whether the asset-price fall occurred early in life or just before retirement. But if a series of contributions throughout one's life is made, a fall in assets value early in life does not affect the future contributions, i.e. only part of one's future pension wealth is affected, while if it occurs close to retirement it affects all past accumulated contributions and returns on them, i.e. most of one's pension wealth.

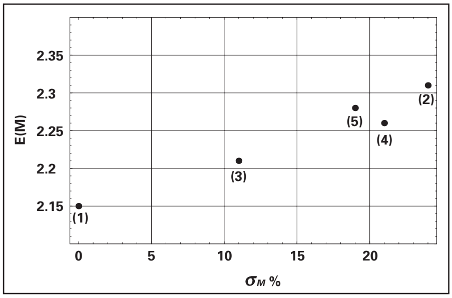
Let us consider two funds:

1. a risky fund with normally distributed return with average of 10 % and standard deviation of 10 %,
2. a secure fund with a certain return of 5 %.

Suppose that the saver deposits one unit in the first and one unit in the second period. *Table 2* and *Figure 1* demonstrate a risk-return analysis of five different strategies. Strategy 1 assumes that in both periods the savings are invested in the secure fund. Strategy 2 is the most risky one – in both periods the savings are invested in the risky fund. This strategy has the highest expected value of the savings at the end of the second period $E(M)$ but also the highest standard deviation σ_M.

To decrease the risk, Strategies 3 and 4 invest in the secure fund in one of the periods. According to (1) the level of final savings is more sensitive to the second-period asset returns. Therefore the risk (see the last three

FIGURE 1 Risk-Return Analysis of Different Strategies



columns of the Table 2) connected with Strategy 4 (first year secure fund, second year risky fund) is higher than the risk connected with Strategy 3 (first year risky fund, second year secure fund).

Strategy 5 is a dynamic strategy where in the first year the savings are deposited in the risky fund and the decision in the second year is conditional: if the return in the first year is more then 15 % then the secure fund is chosen in the second period; otherwise the risky fund is chosen again. Compared to Strategy 4, this strategy is more efficient (see Table 2 and Figure 1). Hence, by using closed-loop strategies, both the risk and return parameters could be improved.

3. The Dynamic Stochastic Programming Accumulation Model

Suppose that the future pensioner deposits once a year a τ-part of his/her yearly salary w_t in a pension fund $j \in \{1, 2, ..., m\}$. Denote by s_t, $t = 1, 2, ...T$ the accumulated sum at time t where T is the expected retirement time. Then the budget-constraint equations read as follows:

$$s_{t+1} = s_t(1 + r_t^j) + w_{t+1}\tau, \; t = 1, 2, ..., T-1 \qquad (2)$$
$$s_1 = w_1\tau$$

where r_t^j is the return of the fund j, in the time period $[t, t + 1)$. When retiring, the pensioner will strive to maintain his/her living standard at the level of the last salary. From this point of view, the saved sum s_T at the time of retirement T is not precisely what the future pensioner cares about. For a given life expectancy, the ratio of the cumulative sum s_T and the yearly salary w_T, i.e. $d_T = s_T/w_T$ is more important. Using the quantity $d_t = s_t/w_t$ one can reformulate the budget-constraint equation (2):

$$d_{t+1} = F_t(d_t, j), \; t = 1, 2, ..., T-1 \qquad (3)$$
$$d_1 = \tau$$

where

$$F_t(d, j) = d\,\frac{1 + r_t^j}{1 + \rho_t} + \tau, \; t = 1, 2, ..., T-1 \qquad (4)$$

and ρ_t denotes the wage growth defined by the equation

$$w_{t+1} = w_t(1 + \rho_t)$$

Suppose that each year the saver has the possibility to choose a fund $j\,(t, I_t) \in \{1, 2, ..., m\}$, where I_t denotes the information set consisting of the history of returns $r^j_{t'}$, $t' = 1, 2, ..., t-1, j \in \{1, 2, ..., m\}$ and the wage growth $\rho_{t'}$, $t' = 1, 2, ..., t-1$. Now suppose that the history of the wage growth ρ_t, $t = 1, 2, ..., T-1$ is deterministic and the returns r_t^j are assumed to be random and they are independent for different times $t = 1, 2, ..., T-1$. Then the only relevant information is the quantity d_t. Hence, $j(t, I_t) \equiv j(t, d_t)$. One can formulate a problem of dynamic stochastic programming:

$$\max_j E(U(d_T)) \tag{5}$$

with the following recurrent budget constraint:

$$d_{t+1} = F_t(d_t, j(t,d_t)) \; t = 1, 2, ..., T-1 \qquad d_1 = \tau \tag{6}$$

where the maximum is taken over all non-anticipative strategies $j = j(t,d_t)$. Here U stands for a given preferred utility function of wealth of the saver. Using the law of iterated expectations

$$E(U(d_T)) = E(E(U(d_T)|I_t)) = E(E(U(d_T)|d_t))$$

we conclude that $E(U(d_T)|d_t)$ should be maximal. Let us denote

$$V_t(d) = \max_j E(U(d_T)|d_t = d) \tag{7}$$

Then by using the law of iterated expectations

$$E(U(d_T)|d_t) = E(E(U(d_T)|d_{t+1})|d_t)$$

we obtain the Bellman equation

$$V_t(d) = \max_{j \in \{1,2,...,m\}} E[V_{t+1}(F_t(d,j))] = E[V_{t+1}(F_t(d,j(t, d)))] \tag{8}$$

For $t = 1, 2, ..., T-1$, where $V_T(d) = U(d)$. Using (8) the optimal feedback strategy $j(t,d_t)$ can be found backwards. This strategy gives the saver the decision for the optimal fund for each time t and level of savings d_t. Suppose that the stochastic returns r_t^j are represented by their densities f_t^j. Then equation (8) can be rewritten in the form

$$V_t(d) = \max_{j \in \{1,2,...,m\}} E[V_{t+1}(F_t(d,j))]$$

$$= \max_{j \in \{1,2,...,m\}} \int_{\mathbf{R}} V_{t+1}\left(d\,\frac{1 + r}{1 + \rho_t} + \tau\right) f_t^j(r)\mathrm{d}r$$

$$= \max_{j \in \{1,2,\dots,m\}} \int_{\mathbf{R}} V_{t+1}(y) f_t^j \left((y - \tau) \frac{1 + \rho_t}{d} - 1 \right) \frac{1 + \rho_t}{d} \mathrm{d}y$$

$$= \int_{\mathbf{R}} V_{t+1}(y) f_t^{j(t,d)} \left((y - \tau) \frac{1 + \rho_t}{d} - 1 \right) \frac{1 + \rho_t}{d} \, \mathrm{d}y \tag{9}$$

where the substitution $y = d(1+r)(1+\rho_t)^{-1} + \tau$ has been used and $\mathbf{R}$ denotes the set of real numbers.

3.1 The Constant Relative Risk Aversion (CRRA) Utility Function

An important part of the problem (5)–(6) is the choice of the utility function U. The utility function varies across the investors and represents their attitude to the risk. A key role in defining the utility function is played by the coefficient of relative risk aversion $C(x) = -xU''(x)/U'(x)$. Constant relative risk aversion implies that people hold a constant proportion of their wealth in any class of risky assets as their wealth varies – see e.g. (Friend – Blume, 1975), (Pratt, 1964) and (Young, 1990). In this case the utility function is of the form

$$U(x) = -Ax^{1-C} + B \quad \text{if } C > 1$$

$$U(x) = A \ln(x) + B \quad \text{if } C = 1$$

$$U(x) = Ax^{1-C} + B \quad \text{if } C < 1 \tag{10}$$

where A, B are constants and $A > 0$. One can easily prove that, concerning the problem (5)–(6), the utility function is invariant to positive affine transformations, i.e. U and $K.U + L$ are equivalent.

In our case, constant relative risk aversion implies that the utility functions $U(d)$ and $U(\kappa d)$ where κ is a constant lead to the same strategies. We use the constant relative risk aversion (CRRA) utility function

$$U(d) = \frac{1}{1-a} \left((\kappa d)^{1-a} - 1 \right) \tag{11}$$

where $a > 0$ is the constant coefficient of relative risk aversion. Using $\kappa = 1/12$ the utility function is "steeper" for reasonable values and the numerical procedure is more stable. Problem (5)–(6) then maximizes the expected utility of savings (compared to the last yearly salary) corresponding to 1/12 of the yearly benefits (i.e. the benefits for one month). It is clear that maximizing monthly benefits or yearly benefits should lead to the same strategy and therefore we can utilize the CRRA utility function.

The coefficient of relative risk aversion a plays an important role in many fields of economics. There is a consensus today that the value should be less than 10 – see e.g (Mehra – Prescott, 1985). In our typical results we considered values close to 9. It should probably be lower for lower equity premium. However, our goal was to formulate the mathematical model and to manage the numerical procedure. The reader can change the parameters and calibrate the model in his/her own way.

4. Pension Portfolio Simulations. Numerical Experiments

The purpose of this section is to present the results of pension portfolio simulations. The numerical approximation scheme is discussed in the *Appendix*. The output of the numerical code is a matrix of size (T = 40) × (k = 200) allowing us to "browse" between different years (rows) t and different levels of d (columns). At a given cell of the table we can read the name of the fund ($j = 1,\dots,m$) which has to be chosen. Plots of computed output matrices adjusted to the domain $\{(d,t),\ t \in (0,T),\ d \in (d_{min},t/2)\}$ are depicted in this section.

Our results will be summarized in graphical plots of the so-called optimal choice function $j = j(t,d)$ as well as several tables discussing computed results of optimization. The role of the optimal choice function $j = j(t,d)$ is to provide information when to switch between different funds for a given level of the ratio d of saved money and wages. We focus on two basic questions and problems: 1. what the regions of constant values of $j(t,d)$ are; 2. what the path of expected values of d_t is.

Before presenting results of the simulation we have to discuss input data such as, e.g., fund structures and characteristics, and the wage growth ρ. Concerning the structure of funds we consider the present situation in the Slovak Republic. According to the adopted government regulation, there are three funds (i.e. m = 3). Namely, the Growth, Balanced and Conservative Funds (see Table 1). Hereafter, we shall suppose that these three funds are constructed from stocks (S) and secure bonds (B) where stocks are represented by the S&P Index (Jan 1996–June 2002) with average return $r_s =$ = 0.1028 and standard deviation $\sigma_s = 0.1690$ whereas the secure bonds are represented by 10-year US government bonds (Jan 1996–June 2002) with the average return $r_b = 0.0516$ and standard deviation $\sigma_b = 0.0082$.[1] Using the historical data, the estimate of correlation between stocks and bonds is –0.1151. Stochastic asset returns are assumed to have normal distributions.[2]

We shall suppose that the structure of funds (F_1 = Growth Fund, F_2 = Balanced Fund, F_3 = Conservative Fund) of the second pension pillar in the Slovak Republic is as follows:

$$\begin{aligned} F_1 &= 0.8 \times S + 0.2 \times B \\ F_2 &= 0.5 \times S + 0.5 \times B \\ F_3 &= B \end{aligned} \tag{12}$$

[1] We considered the estimated asset returns only for illustration of the model capability. We do not have any ambition to estimate future asset returns. However, in Section 4.3 we present a sensitivity analysis for different asset returns.

[2] The normal distribution is a simplification. There is well-known empirical evidence that stock returns exhibit asymmetry and heavy tails. However, the model presented in Section 3 allows different distributions.

TABLE 3 Data Used for Computation. Fund Returns and Their Standard Deviations (left), Expected Wage Growth for the Period 2006–2050 (right)

Fund	Return	StdDev
F_1	$r_1 = 0.0926$	$\sigma_1 = 0.1350$
F_2	$r_2 = 0.0772$	$\sigma_2 = 0.0841$
F_3	$r_3 = 0.0516$	$\sigma_3 = 0.0082$

Period	Wage growth $(1 + \rho_t)$
2006–2008	1.075
2009–2014	1.070
2015–2021	1.065
2022–2024	1.060
2025–2050	1.050

Both returns r_i and standard deviations σ_i, $i = 1, 2, 3$ of the above funds can be easily calculated from returns r_s, r_b, standard deviations σ_s, σ_b and the estimated correlation (see *Table 3*).

According to the Slovak legislature, the percentage of salary transferred each year to a pension fund is 9 %. The law sets administrative costs of the second pillar at 1 % of the monthly contribution and 0.07 % of the monthly asset value (i.e. 0.84 % p.a.). Therefore, we considered effective contributions $\tau = 8.91\%$ (= 9%*0.99). The value 0.84 % was subtracted from the asset returns in Table 3. We assumed the period of saving to be $T = 40$ years. The data for the expected wage growth ρ are taken from the Slovak Savings Bank (SLSP).[3] The values are shown in Table 3.

4.1 Description of Computed Results and Simulations

In *Figure 2* we present a typical result of our analysis with the coefficient of relative risk aversion $a = 9$. It contains three distinct regions in the (d,t) plane determining the optimal choice $j = j(d,t)$ of a fund depending on time $t \in [1, T - 1]$ and the average saved-money-to-wage ratio $d \in [d_{min}, d^t_{max}]$. For practical purposes we chose[4] $d_{min} = 0.0891$ and $d^t_{max} = t/2$ for $t \geq 1$. In each year $t = 1, ..., T - 1$ we invest the saved amount of money s_t uniquely corresponding with d_t in one of the funds $j = 1, 2, 3$ depending on the computed optimal value $j = j(d, t)$. In the first year of saving we take $d_1 = d_{min}$.

The curvilinear solid line in Figure 2 represents the path of the mean wealth $E(d_t)$, obtained by 10,000 simulations and here we use $a = 9$. Notice that, for $t > 1$, the ratio d_t is a random variable depending on (in our case normally distributed) random returns of the funds and on the computed optimal fund choice matrix $j(d,t')$, $t' < t$. The dashed curvilinear lines correspond to $E(d_t) \pm \sigma_t$ intervals where σ_t is the standard deviation of the random variable d_t.

In *Table 4* we present the mean final wealth $E(d_T)$ as well as the so-called *switching-times* for mean path $E(d_t)$, $t \in [1, T - 1]$ and the intervals (in brackets) of switching times for one standard deviation of the mean path.

[3] The data were provided by the analysts of SLSP: Martin Barto and Juraj Kotian.

[4] $d_{min} = 0.0891$ because 8.91 % is the effective 2nd pillar contribution rate.

FIGURE 2 Regions of Optimal Choice and the Path of Average Saved-Money-to-Wage Ratio (a = 9)

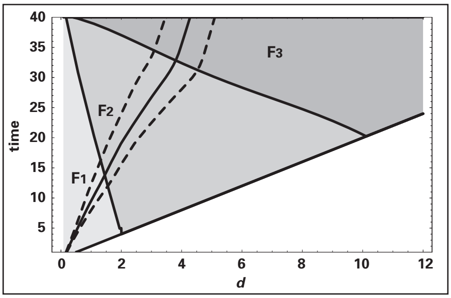


FIGURE 3 Cumulative Distribution Function 1 – F (left), Histogram of Simulations and Density Function (right)
Sample Mean $E(d_T)$ = 4.28 and Standard Deviation of d_T = 0.82

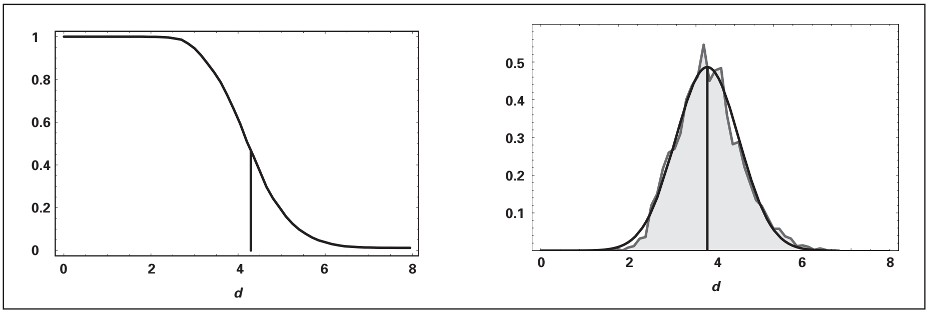


TABLE 4 Summary of Computation of the Mean Saved-Money-to-Wage Ratio d_t and Switching Times (a = 9)

Mean	Switch	Switch
$E(d_T)$	$F_1 - F_2$	$F_2 - F_3$
4.28	14 (12–16)	33 (32–35)

The normalized histogram of the simulated final wealth is very similar to a normal distribution, as can be seen in *Figure 3*.

In the next sections we focus on the sensitivity of results when some parameters change.

4.2 Sensitivity Analysis for Varying Risk Aversions

Let us consider different risk aversion parameters a in the utility function: a = 3, 5, 8, 9, 10. It should be obvious that increasing risk aversion leads to a choice of a less risky fund. Indeed, based on our computations, one can observe that increasing a (increasing risk aversion) causes that

FIGURE 4 Sensitivity of Regions of Optimal Choice with Respect to Different Risk-Aversion Values of the Parameter a = 3, 5, 8, 10

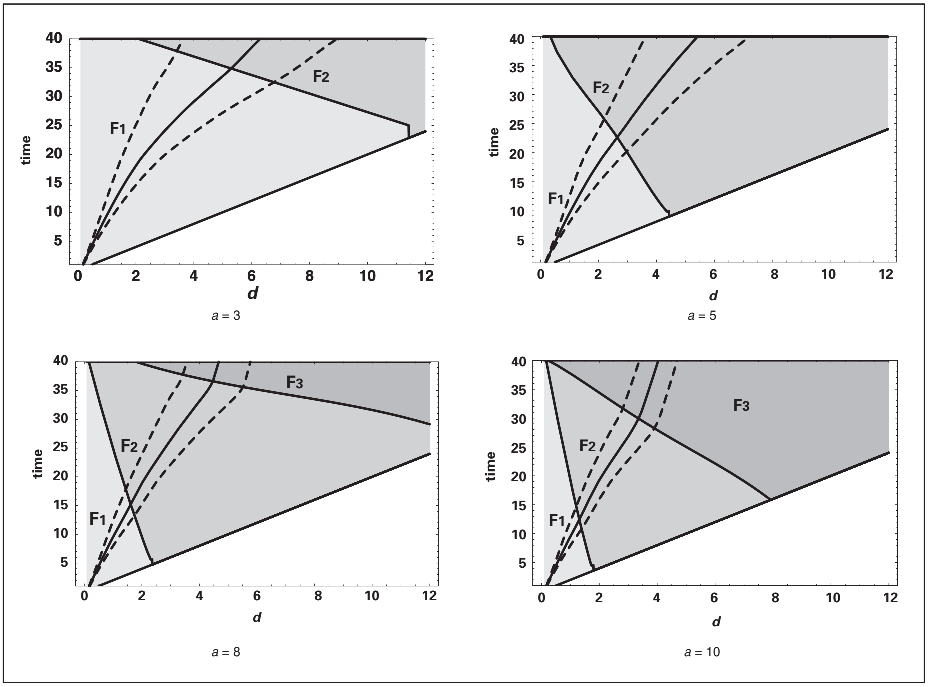


TABLE 5 Results for Fixed Wage Growths, Fixed Returns and Standard Deviations (see values in Table 3), Different Risk Aversion Parameter a

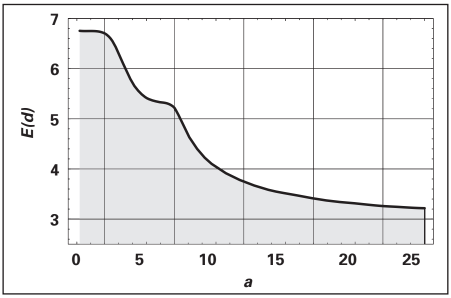
a	Mean $E(d_T)$	Switch $F_1 - F_2$	Switch $F_2 - F_3$
3	6.29	35 (32–38)	never
5	5.42	23 (20–26)	never
8	4.67	15 (13–18)	37 (36–38)
10	4.04	13 (11–15)	30 (29–32)

the switches between funds are shifted to an earlier time, i.e. we switch from F_1 to F_2 sooner, as well as from F_2 to F_3. Obviously, for higher values of the risk aversion parameter a we obtain lower levels of the final wealth. Results for the experiments are displayed in Figure 2, *Figure 4*, Table 4 and *Table 5*.

The relation between different values of risk aversion parameter $a \in (0,25)$ and the final mean wealth to last wage ratio is shown in *Figure 5*. We can see that the curve can be divided into three segments where the kinks separate ranges of the parameter a for which there are *no* switches, *one* switch, and *two* switches between funds in the optimal strategy.

One can see that results partially in accordance with legal regulations are reached for a = 9. This value is relatively high – see e.g. (Mehra –

FIGURE 5 Relation between the Risk Aversion Parameter a and the Level of Mean Saved Wealth $E(d_T)$



Prescott, 1985). In the next section we show that the results are highly sensitive to asset returns and for lower stock returns the "typical" value of a should be lower.

4.3 Sensitivity Analysis for Various Stock and Bond Returns

Now, let us examine the impact of the change in returns of funds on the optimal strategy and results. One can expect that if, for example, the return of stocks becomes higher, it will be more favorable to "stay" in F_1 or F_2 for a longer period. In our computations, we first fix the bond return and increase/decrease the stock return (a = 9 and other parameters are fixed). This change is mirrored in the returns of the funds F_1 and F_2. The results obtained show that a higher return of stocks implies a later switch from more risky to less risky funds. The wealth in the final period of savings is higher too. Secondly, we fix the stock return and increase/decrease the bond return. A higher return of bonds implies an earlier switch from more risky to less risky funds.[5] For an overview of all results, see *Figure 6* and *Table 6*.

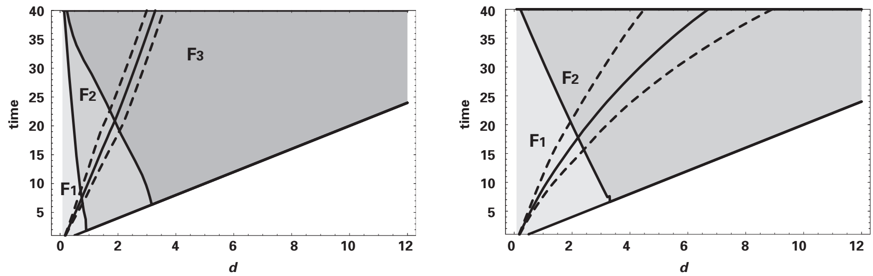
4.4 Sensitivity Analysis with Respect to Varying Wages

Finally, we consider different wage-growth rates. The intuition says that one can expect lower saved-money-to-wage ratio d_t for higher wage growth ρ.[6] To examine the influence of this parameter on results, we considered the wage growth being raised (uniformly for all time periods) and lowered by one percentage point. We denote by $\rho^{(+1pp)}$ ($\rho^{(-1pp)}$) the wage growth development derived from Table 3 where ρ_t has been increased by 1 pp (decreased by 1 pp) for each of five periods in Table 3. As we can see in *Figure 7* and *Table 7*, a higher wage growth leads to a lower wealth to last wage ratio, guided by shifting the switch-times to later moments.

[5] We have merely varied means of returns distributions up or down, but kept standard deviations unchanged. As a result, the coefficient of variation (= standard deviation/mean) changes. If riskiness changed proportionately with returns, the results would differ much less.

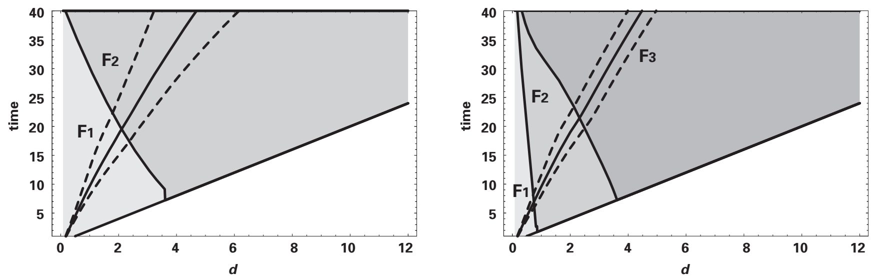
[6] Although this increases the contributions (contribution rate τ unchanged), there is a steeper wage profile and hence lower savings to last wage ratio.

FIGURE 6 Sensitivity of Regions of Optimal Choice for Various Expected Values of Stock and Bond Returns



a) lower stock return $r_s = 0.0828$ b) higher stock return $r_s = 0.1228$

and fixed bond return $r_b = 0.0516$



c) lower bond return $r_b = 0.0366$ d) higher bond return $r_b = 0.0666$

and fixed stock return $r_s = 0.1028$

TABLE 6 Results for Fixed Wage Growths, Fixed $a = 9$, Fixed Standard Deviations $\sigma_1 = 0.1350$, $\sigma_2 = 0.0841$, $\sigma_3 = 0.0082$ and Various Bond and Stocks Returns r_b and r_s, respectively

Stock & Bond returns	**Fund returns**	**Mean** $E(d_T)$	**Switch** $F_1 - F_2$	**Switch** $F_2 - F_3$
$r_s = 0.1028$ $r_b = 0.0516$	$r_1 = 0.0926$ $r_2 = 0.0772$ $r_3 = 0.0516$	4.28	14 (12–16)	33 (32–35)
$r_s = 0.0828$ $r_b = 0.0516$	$r_1 = 0.0766$ $r_2 = 0.0672$ $r_3 = 0.0516$	3.29	8 (7–9)	21 (19–23)
$r_s = 0.1228$ $r_b = 0.0516$	$r_1 = 0.1086$ $r_2 = 0.0872$ $r_3 = 0.0516$	6.70	18 (16–20)	never
$r_s = 0.1028$ $r_b = 0.0366$	$r_1 = 0.0896$ $r_2 = 0.0697$ $r_3 = 0.0366$	4.69	19 (17–22)	never
$r_s = 0.1028$ $r_b = 0.0666$	$r_1 = 0.0956$ $r_2 = 0.0847$ $r_3 = 0.0666$	4.48	7 (6–8)	22 (20–24)

FIGURE 7 Sensitivity of Regions of Optimal Choice for Various Wage Growth Scenarios

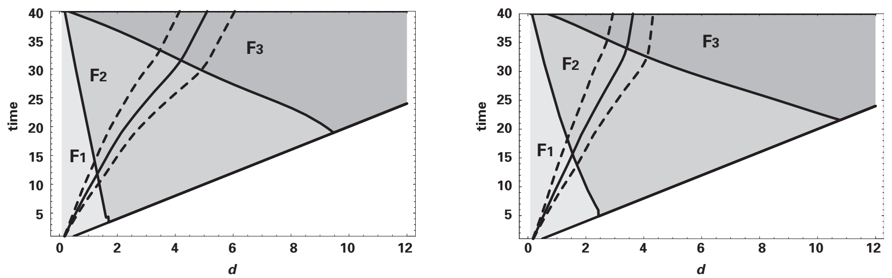


TABLE 7 Results for Fixed Returns and Standard Deviations (see values in Table 3), Fixed a = 9, and Different Wage-Growth Rates

Wage growth	Mean $E(d_T)$	Switch $F_1 - F_2$	Switch $F_2 - F_3$
$\rho^{(-1pp)}$	5.10	12 (10–14)	32 (30–34)
ρ	4.28	14 (12–16)	33 (32–34)
$\rho^{(+1pp)}$	3.63	16 (14–18)	34 (33–36)

4.5 The Comparison of Dynamic and Static Strategies

One can think about static strategies where the time instants when a contributor switches between the funds are determined at the beginning of the saving. The most risk-averse contributor always deposits the savings in the Conservative Fund. The least risk-averse investor contributes to the risky funds as long as the law permits it: in the first 25 years of saving to the Growth Fund (the total period of saving 40 years supposed), in the next eight years to the Balanced Fund and in the last seven years to the Conservative Fund.

To compare the performance of dynamic and static strategies we have chosen two representatives of the static ones:

1. The most risky (accepting the legal regulations) strategy with switching times 25 ($F_1 - F_2$) and 33 ($F_2 - F_3$).
2. The strategy with switching times 14 and 33 similar to a typical representative of dynamic strategies with the risk aversion parameter a = 9.

In *Figure 8* one can see the average d_t development and $E(d_t) \pm \sigma_t$ intervals for chosen static strategies. The strategy with switching times 14 and 33 has the same $E(d_T)$ = 4.67 comparing to a dynamic one with a = 8 but significantly higher the standard deviation of d_T, σ_T = 1.41 (compared to 1.10 for the dynamic strategy with a = 8). A mean-variance analysis of dynamic strategies with different risk aversion (represented by the curve-efficient frontier) and the two static ones is depicted in *Figure 9*. The static strategies are clearly inefficient.

FIGURE 8 Static Strategies – the Development of Savings

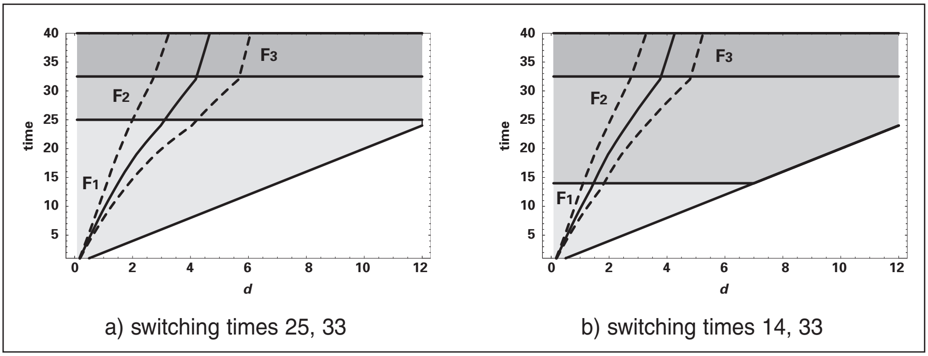
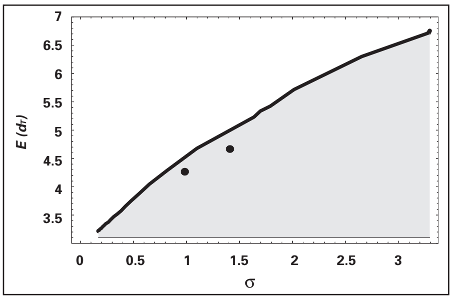


FIGURE 9 Mean-Variance Analysis of Dynamic and Strategic Strategies



5. Conclusions

We have presented a dynamic accumulation model for determining optimal switching strategies for choosing pension funds with different risk profiles. It turned out that dynamic strategies could be more efficient compared to static ones. The results of simulations of a mathematical model have illustrated that gradual decreasing of the risk (incorporated in the corresponding legislation) is reasonable and can be supported by means of a dynamic accumulation model. The resulting strategies depend on individual risk preferences of the future pensioners represented by their individual utility functions. In accordance with common intuition, higher wage growth implies lower performance of the funded pillar relative to the pay-as-you-go pillar. Since it is very difficult to predict the future asset returns, the results were calculated for various means of asset returns distributions.

Appendix

Numerical Approximation Scheme

In this section we discuss the numerical approximation scheme we used in our pension portfolio simulations. The principal difficulty in computing the Bellman integral (9) is due to significant oscillations in the integrand function. More precisely, it may attain both large values as well as low values of the order one. Therefore a scaling technique is needed when computing the integral (9). The idea of scaling is rather standard and is widely used in similar circumstances.

Let $H_t(d)$ be any bounded positive function for $t = 1, 2, ..., T$. We scale the function V_t by H_t, i.e. we define a new auxiliary function

$$W_t(d) = H_t(d)V_t(d)$$

Clearly, the original function $V_t(d)$ can be easily calculated from $W_t(d)$ as follows: $V_t(d) = W_t(d)/H_t(d)$. Then, for each time step t from $t = T$ down to $t = 2$ we have

$$W_T(d) = H_T(d)V_T(d) \quad \text{and}$$

$$W_{t-1}(d) = H_{t-1}(d)V_{t-1}(d)$$

$$= \max_{j\in\{1,2,...,m\}} \int_{\mathbf{R}} H_{t-1}(d)V_t\left(\frac{d}{1+\rho_t}(1+r)+\tau\right) f_t^j(r)\mathrm{d}r \tag{13}$$

$$= \max_{j\in\{1,2,...,m\}} \int_{\mathbf{R}} \frac{H_{t-1}(d)W_t\left(\frac{d}{1+\rho_t}(1+r)+\tau\right)}{H_t\left(\frac{d}{1+\rho_t}(1+r)+\tau\right)} f_t^j(r)\mathrm{d}r$$

$$= \max_{j\in\{1,2,...,m\}} \int_{\mathbf{R}} \frac{H_{t-1}(d)W_t(y)}{H_t(y)}\ f_t^j\left((y-t)\frac{1+\rho_t}{d}-1\right)\frac{1+\rho_t}{d}\,\mathrm{d}y$$

It is worthwhile to note that any choice of the family H_t, $t = 1, ..., T$, of positive bounded scaling function does not change the result. It may however significantly improve the stability of numerical computation.

In order to capture both large and small values of V_t we recursively define the scaling functions H_t, $t = T, T-1, ..., 2, 1$, depending on the previously computed solution V_{t+1} as follows:

$$H_T = \frac{1}{\sqrt{1+V_T^2}}, \quad \text{and} \quad H_t = \frac{1}{\sqrt{1+V_{t+1}^2}} \quad \text{for} \quad t= T-1, ..., 1 \tag{14}$$

In our algorithm we compute values of the function $W_t = W_t(d)$ for discrete values of d from the time dependent interval $d \in (d_{min}, t/2)$, where we use $d_{min} = 0.0891$. The upper bound $t/2$ has been chosen with respect to maximal expected values of the savings-to-yearly-salary ratio d. In each time level $t = T$ down to $t = 1$ we choose a uniform spatial discretization of the interval $(d_{min}, t/2)$ consisting of $k = 200$ mesh points. Stochastic fund r_t^j returns were assumed to have normal distributions with densities f_t^j having constant in-time means $\overline{r}^j$ and standard deviations σ^j, $j = 1, ..., m$. In order to compute numerically the Bellman type integral with normal distribution densities f_t^j we used the Simpson rule with 11 equidistant grid points covering the essential interval $(\overline{r}_j - 3\sigma_j, \overline{r}_j + 3\sigma_j)$.

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SUMMARY

JEL classification: C15, E27, G11, G23
Keywords: Bellman equation; dynamic stochastic programming; funded pillar; pension portfolio simulations; risk aversion; Slovak pension system; utility function

A Dynamic Accumulation Model for the Second Pillar of the Slovak Pension System

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Since January 2005, pensions in Slovakia are operated by a three-pillar system as proposed by the World Bank. This paper concentrates on the mandatory, fully funded second pillar. The authors present a dynamic accumulation model for determining the optimal switching strategy among pension funds with different risk profiles. The resulting strategies depend on the individual risk preferences of future pensioners. The authors' results illustrated that gradual decreasing risk while amassing savings for a pension is rational. Furthermore, the authors present several simulations of optimal fund-switching strategies for various model parameter settings.