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ABSTRACT

We discuss the role of tangential stabilization in a curvature driven flow of planar curves. The governing system of nonlinear parabolic equations includes a nontrivial tangential velocity functional yielding a uniform redistribution of grid points along the evolving family of curves preventing numerically computed curves from forming various instabilities. (© 2008 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim)

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On tangential stabilization in curvature driven flows of planar curves

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We discuss the role of tangential stabilization in a curvature driven flow of planar curves. The governing system of nonlinear parabolic equations includes a nontrivial tangential velocity functional yielding a uniform redistribution of grid points along the evolving family of curves preventing numerically computed curves from forming various instabilities.

1 Introduction

In this paper we study evolution of a family of closed smooth plane curves $\Gamma_t : S^1 \to \mathbb{R}^2$, $t \geq 0$, driven by the normal velocity v which is assumed to be a function of the curvature k, tangential angle ν and position vector $x \in \Gamma_t$,

$$v = \beta(x, k, \nu)\,. \tag{1}$$

As a typical example one can consider a normal velocity of the form: $v = k$ (mean curvature driven flow), $v = k^{\frac{1}{3}}$ (affine invariant flow), $v = a(x,\nu)k + c(x,\nu)$ (Gibbs-Thomson law), etc. Geometric equations of the form (1) can often be found in variety of applied problems like e.g. the material science, dynamics of phase boundaries in thermomechanics, in modeling of flame front propagation, in combustion, in computations of first arrival times of seismic waves, in computational geometry, robotics, semiconductors industry, etc. They also have a special conceptual importance in image processing and computer vision. For an overview of important applications of (1) we refer to a book by Sethian [1].

An idea behind the direct (or Langrangean) approach consists in representing the family of immersed curves Γ_t by the position vector $x \in \mathbb{R}^2$, i.e. $\Gamma_t = \text{Image}(x(.,t)) = \{x(u,t),\ u \in S^1\}$ where x is a solution to the geometric equation

$$\partial_t x = \beta\vec{N} + \alpha\vec{T} \tag{2}$$

where $\beta = \beta(x,k,\nu)$, $\vec{N} = (-\sin\nu, \cos\nu)$ and $\vec{T} = (\cos\nu, \sin\nu)$ are the unit inward normal and tangent vectors, respectively. We chose the orientation of the tangent vector $\vec{T}$ such that $\det(\vec{T},\vec{N}) = 1$. Notice that the presence of arbitrary tangential velocity functional α has no impact on the shape of evolving curves and thus α can be viewed as free parameter to be suitably determined. The unit arc-length parameterization of a curve $\Gamma = \text{Image}(x)$ will be denoted by s. Then $ds = g\,du$ where $g = |\partial_u x|$.

According to [2, 3] (see also [4, 5]) the system of governing equations for the curvature k, tangent angle ν, local length g and the position vector x reads as follows:

$$\partial_t k = \partial_s^2\beta + \alpha\partial_s k + k^2\beta\,, \quad \partial_t\nu = \beta_k'\partial_s^2\nu + (\alpha+\beta_\nu')\partial_s\nu + \nabla_x\beta.\vec{T}\,, \quad \partial_t g = -gk\beta + \partial_u\alpha\,, \quad \partial_t x = \beta\vec{N} + \alpha\vec{T} \tag{3}$$

where $(u,t) \in S^1 \times (0,T)$, $\mathrm{d}s = g\,\mathrm{d}u$, A solution (k,ν,g,x) to (3) is subject to initial conditions and periodic boundary conditions in the u variable.

2 The role of the tangential velocity functional

Notice that the functional α is still undetermined and it may depend on variables k, ν, g, x in various ways including nonlocal dependence in particular. Suitable choices of the tangential velocity functional α are discussed in a more detail in this section. Although α plays an important role in the governing equations resulting in dependence of k, ν, g, x on α, the family of planar curves $\Gamma_t = \text{Image}(x(.,t)), t \in [0,T)$, is independent of a particular choice of α.

To motivate further discussion, we recall some of computational examples in which the usual choice $\alpha = 0$ fails and may lead to serious numerical instabilities like e.g. formation of so-called swallow tails. In Figure 1–a) we computed the mean curvature flow of an initial curve (bold faced curve). We chose $\alpha = 0$. It should be obvious that numerically computed grid points merge in some parts of the curve Γ_t preventing thus numerical approximation of $\Gamma_t, t \in [0,T)$, to be continued beyond some time T which is still far away from the maximal time of existence T_{max}. This and many other examples from [2, 3] showed that a suitable grid points redistribution governed by a nontrivial tangential velocity functional α is needed in order to compute the solution over its life-span.

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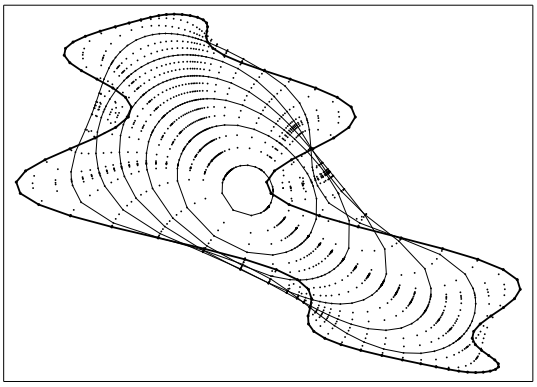
a)

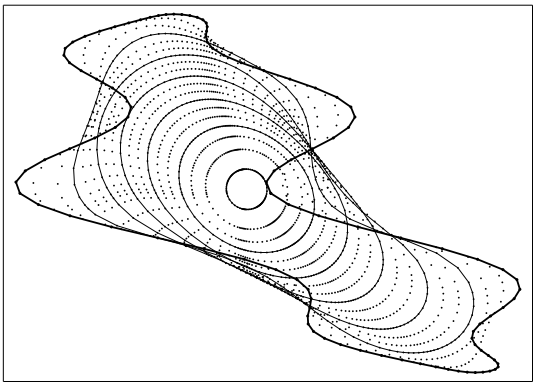
b)

Fig. 1 Numerically computed evolution of an initial curve (bold); a) merging of numerically computed grid points in the case of zero tangential velocity $\alpha = 0$; b) enhancement of grid point redistribution in the case of asymptotically uniform parametrization.

The idea behind construction of a suitable tangential velocity functional α is rather simple and consists in the analysis of the quantity θ defined as $\theta = \ln(g/L)$ where $g = |\partial_u x|$ is a local length and $L = L_t = \int_{\Gamma_t} \mathrm{d}s = \int_0^1 g(u,t)\,\mathrm{d}u$ is a total length of the curve $\Gamma_t = \mathrm{Image}(x(.,t))$. The quantity θ can be viewed as the logarithm of the relative local length ratio g/L. Taking into account equations (3) and the equation for the total length $\frac{d}{dt}L + \int_\Gamma k\beta \mathrm{d}s = 0$ (obtained again from (3) by integration) we have

$$\partial_t\theta + k\beta - \langle k\beta\rangle_\Gamma = \partial_s\alpha \tag{4}$$

where $\langle k\beta\rangle_\Gamma$ denotes the average of $k\beta$ over the curve Γ, i.e. $\langle k\beta\rangle_\Gamma = \frac{1}{L}\int_\Gamma k\beta\,\mathrm{d}s$ By an appropriate choice of $\partial_s\alpha$ in the right hand side of (4) appropriately we can therefore control the behavior of θ. Equation (4) can be also viewed as a kind of a constitutive relation determining redistribution of grid point along a curve. The simplest possible choice of $\partial_s\alpha$ is:

$$\partial_s\alpha = k\beta - \langle k\beta\rangle_\Gamma \tag{5}$$

yielding $\partial_t\theta = 0$ in (4). Consequently, $g(u,t)/L_t = g(u,0)/L_0$ for any $u \in S^1,\ t \in [0, T_{max})$. Notice that α can be uniquely computed from (5) under the additional renormalization constraint: $\alpha(0,t) = 0$. The tangential redistribution driven by a solution α to (5) is refereed to as *a parameterization preserving relative local length* (c.f. [2]). It has been first discovered and utilized by Hou et al. in [6, 7] and independently by the authors in [2, 3].

A more general choice of α is based on the following setup:

$$\partial_s\alpha = k\beta - \langle k\beta\rangle_\Gamma + \left(e^{-\theta} - 1\right)\omega(t) \tag{6}$$

where $\omega \in L^1_{loc}([0, T_{max}))$. If we additionally suppose $\int_0^{T_{max}} \omega(\tau)\,\mathrm{d}\tau = +\infty$ then, after insertion of (6) into (4) and solving the ODE $\partial_t\theta = \left(e^{-\theta} - 1\right)\omega(t)$, we obtain $\theta(u,t) \to 0$ as $t \to T_{max}$ and hence $g(u,t)/L_t \to 1 \quad$ as $t \to T_{max}$ uniformly w.r. to $u \in S^1$. In this case redistribution of grid points along a curve becomes uniform as t approaches the maximal time of existence T_{max}. We will refer to the parameterization based on (6) to as *an asymptotically uniform parameterization* (c.f. [3]). The impact of a tangential velocity functional defined as in (5) on enhancement of redistribution of grid points can be observed from two examples shown in Fig. 1–b) computed by the authors in [2]. It can be shown that the appropriate choice for the control function ω takes the form $\omega = \kappa_1 + \kappa_2\langle k\beta\rangle_\Gamma$ and $\kappa_1, \kappa_2 \geq 0$ are given constants. A detailed discussion on this topic can be found in [3, 4]. If we insert tangential velocity functional α computed from (6) into (3) the system of governing equations can be rewritten as follows:

$$\begin{aligned} &\partial_t k = \partial_s^2\beta + \partial_s(\alpha k) + k\langle k\beta\rangle_\Gamma + (1 - L/g)\,k\omega\,, \qquad \partial_t\nu = \beta'_k\partial_s^2\nu + (\alpha + \beta'_\nu)\partial_s\nu + \nabla_x\beta.\vec{T}\,, \\ &\partial_t g = -g\langle k\beta\rangle_\Gamma + (L - g)\omega\,, \qquad \partial_t x = \beta\vec{N} + \alpha\vec{T}\,. \end{aligned} \tag{7}$$

It is worth to note that the strong reaction term $k^2\beta$ in (3) has been replaced by the averaged term $k\langle k\beta\rangle_\Gamma$ in (7). This is a very important feature as it allows for construction of an efficient and stable numerical scheme discussed in more details in [3–5].

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