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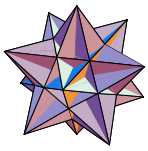

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METHOD OF LINES FOR THE LEVEL SET METHOD FOR SOLVING WILLMORE FLOW GEOMETRIC EQUATION

Michal Beneš, Karol Mikula, Daniel Ševčovič and Tomáš Oberhuber

Abstract

In this article we present an explicit in time numerical scheme for the level-set formulation for the Willmore flow of planar curves. The Willmore flow of planar curves can be described by a zero level set of a function which is a solution to a fourth order in space system of two evolutionary partial differential equations. We use an explicit method of lines for the discretization in time. We compare numerical results with known explicit solution and compute the experimental order of convergence of the method. We present a numerical experiment describing topological changes of evolved curves.

1. Willmore flow of planar curves

For a planar Jordan curve Γ we assume the elastic energy functional defined as:

$$\mathcal{E}(\Gamma) = \int_\Gamma k^2 \mathrm{d}s \tag{1}$$

where k is a curvature of Γ. In many applications like, e.g., physics of elasticity or image processing, we want to minimize the elastic energy functional (1). It is known [2] that evolution of the curve given by the normal velocity

$$\beta = -\partial_s^2 k - \frac{1}{2}k^3 \tag{2}$$

is a gradient flow for (1). This law generates a family of time dependent curves denoted in the sequel as $\Gamma(t)$. There are two main streams dealing with numerical solution of such a geometric problem. The direct Langrangean method parameterizes the curve $\Gamma(t) = \{\mathbf{x}(u) \mid u \in S^1\}$ by its position vector $\mathbf{x}(.,t)$ which is a solution of a position vector equation and equations for other relevant geometric quantities (see [2, 1, 3] for details). Their discretization yields a very efficient numerical scheme and it allows modeling of self-intersecting curves. However, this method cannot handle situations in which a topological changes like splitting and even merging of curves may occur. Such a change of topology is very difficult to handle with the parametric approach and in this case one should rather turn to implicit methods like the level-set method or the phase-field model. Both of them describe the curve implicitly and can handle the changes of topology automatically. The level-set method employs auxiliary function $u : \quad ^2 \to \quad$, describing the evolving curve by $\Gamma(t) = \{\mathbf{x} \in \quad ^2 \mid u(\mathbf{x},t) = 0\}$. Hence $\Gamma(t)$ has a meaning of interface between two regions where u is positive or negative. The drawback of the level-set method is its necessity to restore the level-set function time to time. Moreover, in comparison to the direct method, one needs to solve two dimensional PDEs.

2. Numerical scheme for the level-set method

We remind ourselves that the level set method treats an evolving family of planar curves $\Gamma_t, t \geq 0$, by their representation as zero level sets of the so-called shape

function $u : \Omega \times [0,T] \to$ where $\Omega \subset {}^2$ is a connected domain containing the whole family of evolving curves $\Gamma_t, t \in [0,T]$. Then the unit inward normal vector and signed curvature satisfy $\vec{N} = \nabla u/|\nabla u|$ and $k = -\mathrm{div}\,(\nabla u/|\nabla u|)$. Let us denote the following auxiliary functions:

$$H = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right),\ Q = |\nabla u|,\ w = QH\,.$$

According to [4] (see also [5]) the resulting system of two equations governing the evolution of the shape function is:

$$\partial_t u = -Q\,\mathrm{div}\left(\quad \nabla w - \frac{1}{2}\frac{w^2}{Q^3}\nabla u\right), \quad \text{where}\ \ w = Q\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \tag{3}$$

and $= \frac{1}{Q}\left(\quad - \frac{\nabla u}{Q} \otimes \frac{\nabla u}{Q}\right)$ is the orthogonal projection into a tangential space of the curve representing the zero level set of u. System of equations (3) is subject to the initial condition $u(x,0) = u^0(x)\,,\ x \in \Omega$ and clamped boundary conditions at $\partial\Omega$, i.e. $u(x,t) = 0,\ \partial_\nu u(x,t) = 0,\ x \in \partial\Omega$. The initial function u^0 is a signed distance function, i.e. $u^0(x) = \mathrm{dist}(x, \Gamma^0), x \in \Omega$.

Next we present a discretization scheme for solving the system of PDEs (3). For simplicity we consider the domain $\Omega \equiv \langle 0, L_1\rangle \times \langle 0, L_2\rangle$, $h_1 = \frac{L_1}{N_1}$ and $h_2 = \frac{L_2}{N_2}$ and we define numerical mesh and approximation of a solution u:

$$\begin{aligned}
\omega_h &= \{(ih_1, jh_2) \mid i = 1, \cdots, N_1 - 1, j = 1, \cdots, N_2 - 1\}\,,\\
\overline{\omega}_h &= \{(ih_1, jh_2) \mid i = 0, \cdots, N_1, j = 0, \cdots, N_2\}\,,\\
\partial\omega_h &= \overline{\omega_h} \setminus \omega_h.
\end{aligned}$$

$$\begin{aligned}
u^h_{ij} &= u\,(ih_1, jh_2)\,,\\
u^{rs}_{ij} &= \frac{1}{4}(u^h_{ij} + u^h_{i+r,j} + u^h_{i,j+s} + u^h_{i+r,j+s}),\\
&\quad i = 0, \cdots, N_1 - 1; j = 0, \cdots, N_2 - 1; |r| = |s| = 1,\\
\nabla^{r,0} u^h_{ij} &= \left(r\frac{u^h_{i+r,j} - u^h_{ij}}{h_1}, \frac{u^{r,1}_{ij} - u^{r,-1}_{ij}}{h_2}\right),\\
&\quad i + r = 0, \cdots, N_1; j = 1, \cdots, N_2 - 1; |r| = 1\\
\nabla^{0,s} u^h_{ij} &= \left(\frac{u^{1,s}_{ij} - u^{-1,s}_{ij}}{h_1}, s\frac{u^h_{i,j+s} - u^h_{ij}}{h_2}\right),\\
&\quad i = 1, \cdots, N_1 - 1; j + s = 0, \cdots, N_2; |s| = 1,
\end{aligned}$$

$$\begin{aligned}
Q^{rs}_{ij} &= \sqrt{\epsilon^2 + |\nabla^{rs} u^h_{ij}|^2},\\
&\quad i + r = 0, \cdots, N_1; j + s = 0, \cdots, N_2; |r| = |s| = 1,\\
Q^h_{ij} &= \frac{1}{4}\sum_{|r|+|s|=1} Q^{rs}_{ij},\\
&\quad i = 1, \cdots N_1 - 1; j = 1, \cdots, N_2 - 1,
\end{aligned}$$

$$
\begin{aligned}
w_{ij}^{rs} &= \frac{1}{4}(w_{ij}^h + w_{i+r,j}^h + w_{i,j+s}^h + w_{i+r,j+s}^h), \\
&\quad i = 0, \cdots, N_1 - 1; j = 0, \cdots, N_2 - 1; |r| = |s| = 1,
\end{aligned}
$$

$$
\begin{aligned}
{}_{ij}^{rs} &= \frac{1}{Q_{ij}^{rs}} \left(- \frac{\nabla^{rs} u_{ij}^h}{Q_{ij}^{rs}} \otimes \frac{\nabla^{rs} u_{ij}^h}{Q_{ij}^{rs}} \right), \\
&\quad i + r = 0, \cdots, N_1; j + s = 0, \cdots, N_2; |r| = |s| = 1 .
\end{aligned}
$$

Now the semi-discrete numerical scheme reads as:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} u_{ij}^h &= -Q_{ij}^h \sum_{|r|+|s|=1} \left[\frac{1}{h_{|r+2s|}} \; {}_{ij}^{rs} \nabla^{rs} w_{ij}^h \cdot \nu_{rs} - \frac{1}{2h_{|r+2s|}^2} \frac{\left(w_{ij}^{rs}\right)^2}{\left(Q_{ij}^{rs}\right)} \left(u_{i+r,j+s}^h - u_{ij}^h\right) \right], \\
w_{ij}^h &= Q_{ij}^h \sum_{|r|+|s|=1} \frac{1}{Q_{ij}^{rs} h_{|r+2s|}^2} \left(u_{i+r,j+s}^h - u_{ij}^h\right)
\end{aligned} \tag{4}
$$

for $i = 1, \cdots, N_1 - 1$, $j = 1, \cdots, N_2 - 1$, $\nu_{rs} = (r, s)$ and $u^h(0)_{ij} \,|_{\overline{\omega}_h} = \mathcal{P}\left(u_{ini}\right)_{ij}$. For the discretization in time one can consider either explicit or semi-implicit in time numerical methods. A semi-implicit method has been analyzed in details in [5]. In this paper we present the explicit method of lines for solving (4) which has been applied e.g. in numerical approximation of phase-field models (c.f. [9, 10]).

3. Explicit numerical scheme for the level-set formulation

Since the system of equations (4) is highly non-linear the choice of the explicit discretization in time is very natural. In many articles [6, 7, 8, 9, 10, 11] the method of lines together with the Merson alternative of the Runge-Kutta solver were successfully applied. Having a system of ordinary differential equations of the form

$$
\frac{\mathrm{d}u_{ij}^h}{\mathrm{d}t} = f\left(t, u^h\right)_{ij} \tag{5}
$$

the solver mentioned above consists of the following steps:

$$
\begin{aligned}
k_{ij}^1 &= \tau f\left(t, u^h\right), \qquad k_{ij}^2 = \tau f\left(t + \frac{1}{3}\tau, u^h + \frac{1}{3}k^1\right) \\
k_{ij}^3 &= \tau f\left(t + \frac{1}{2}\tau, u^h + \frac{1}{6}k^1 + \frac{1}{2}k^2\right), \qquad k_{ij}^4 = \tau f\left(t + \frac{1}{2}\tau, u^h + \frac{1}{8}k^1 + \frac{3}{8}k^3\right), \\
k_{ij}^5 &= \tau f\left(t + \tau, u^h + \frac{1}{2}k^1 - \frac{3}{2}k^3 + 2k^4\right) \qquad \text{for } \; i = 0, \cdots N_1, \;\; j = 0, \cdots, N_2.
\end{aligned}
$$

The error of the approximation with the current τ is given by

$$
e := \max_{\substack{i=0,\cdots,N_1 \\ j=0,\cdots,N_2}} \frac{\tau}{3} \left| \frac{1}{5}k_{ij}^1 - \frac{9}{10}k_{ij}^3 + \frac{4}{5}k_{ij}^3 - \frac{1}{10}k_{ij}^5 \right|. \tag{6}
$$

If this error is smaller than given tolerance ϵ we update u^h

$$
u_{ij}^h := u_{ij}^h + \frac{\tau}{6}\left(k_{ij}^1 + 4k_{ij}^4 + k_{ij}^5\right), \tag{7}
$$

and we set $t := t + \tau$. Independently on the previous condition we also update τ as $\tau = \min\left\{\tau \cdot \frac{4}{5}\left(\frac{\epsilon_{RK}}{e}\right)^{\frac{1}{5}}, T - t\right\}$. We repeat whole process with the new τ.

4. Experimental order of convergence

For a circle with the initial radius r_0 we can show that for the radius of its growth $r(t) = (2t + r_0^4)^{\frac{1}{4}}$ holds. With this analytical solution we can evaluate the experimental order of convergence for the scheme (4). We choose $r_0 = 1$, the problem domain $\Omega = \langle -2, 2\rangle \times \langle -2, 2\rangle$ and it is splitted subsequently into $n \times n$ meshes for $n = 10, 20, 40, 80$ with $h = 1/n$. The final time was chosen as $T = 0.5$ and again $\tau = h^2$. The regularization parameter ϵ was refined proportionally to the grid refinement as $\epsilon^2 = 2h$ and the redistancing period was $\tau_{redist} = 0.25h$. Errors in $L^p, p = 2, \infty$, norms and their EOC are presented in Table 1.

Error, EOC \ h	0.4	0.2	0.1	0.05
L^2 error	0.21497	0.06585	0.01699	0.00400
EOC		**1.707**	**1.954**	**2.086**
L^∞ error	0.71190	0.12286	0.03780	0.00973
EOC		**2.534**	**1.700**	**1.957**

Table 1: EOC for the level set scheme in $L^p, p = 2, \infty$ norms.

5. Numerical experiments

Finally, we present a numerical experiment showing the changes of topology (see Fig. 1) . As an initial condition we set four circles with radius $r_0 = 0.45$ on the domain $\Omega = \langle -1, 1\rangle^2$ with 200×200 mesh points and the restoration of the level set function $\tau_{redist} = 10^{-5}$. The parameter controlling the adaptivity in the Runge-Kutta method is $\epsilon_{RK} = 0.001$. The initial condition takes the form $u_0(\mathbf{x}) = \delta \operatorname{sgn}(r)(1 - \exp(-|r/\delta|))$ where $r = \operatorname{dist}(\mathbf{x}, \Gamma)$ and $\delta = 0.05$. We also set the zero Neumann b.c.

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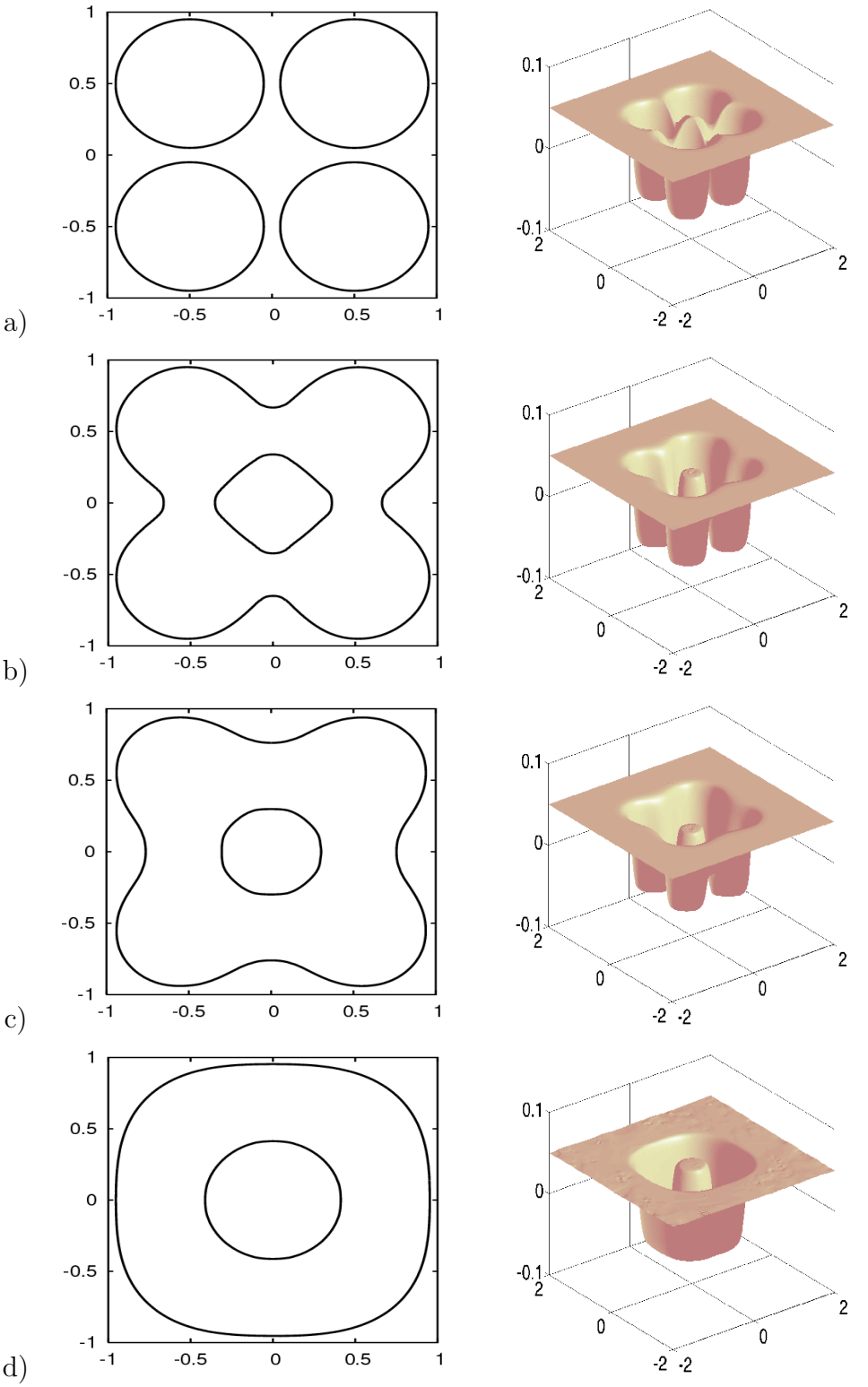

Figure 1: Example of the change of topology computed with the explicit scheme at times: a) $t = 0$, b) $t = 8 \cdot 10^{-5}$, c) $t = 0.0002$ and d) $t = 0.002$.

References

[1] Mikula, K., Ševčovič, D.: Tangentially stabilized Lagrangean algorithm for elastic curve evolution driven by intrinsic Laplacian of curvature, In: Proceedings of Algoritmy 2005, Conference on Scientific Computing, Eds. A.Handlovičová, Z.Krivá, K.Mikula, D.Ševčovič, (2005), pp. 32–41.

[2] Dziuk, G., Kuwert, E., Schätzle, R.: Evolution of elastic curves in ^{n}: existence and computation, SIAM J. Math. Anal., *33* (2002), pp. 1228–1245.

[3] Minárik, V., Kratochvíl, J., Dislocation Dynamics - Analytical Description of the Interaction Force between Dipolar Loops, to appear in Kybernetika (2007)

[4] Droske, M., Rumpf, M.: A level set formulation for Willmore flow, Interfaces Free Bound., Vol. 6, Num. 3 (2003), pp.361–378

[5] Beneš, M., Mikula, K., Oberhuber, T., Ševčovič, D.:Comparison study for Level set and Direct Lagrangian methods for computing Willmore flow of closed planar curves, Computing and Visualization in Science, accepted (`http://www.iam.fmph.uniba.sk/institute/sevcovic/research/index.html`).

[6] Oberhuber, T.: Numerical Solution for the Willmore flow of Graphs, Proc. of Czech-Japanese Seminar in Applied Mathematics 2005, *3*, (2005), pp. 126–138.

[7] Oberhuber, T.: Computational Study of the Willmore flow on Graphs, Proceedings of Equadiff 11, (2005), pp. 321–331.

[8] Beneš, M., Mikula, K.: Simulation of anisotropic motion by mean curvature - comparison of phase field and sharp interface approaches, Acta Mathematica Universitatis Comenianae, *67*, (1998), pp. 17–42.

[9] Beneš, M.: Mathematical and Computational Aspects of Solidification of Pure Substances, Acta Mathematica Univ. Comenianae, *70*, (2000), pp. 123–151.

[10] Beneš, M.: Mathematical analysis of Phase-field Equations with Numerically Efficient Coupling Terms, Interfaces and Free Bound., *3*, (2001), pp. 201–221.

[11] Beneš, M.: Diffuse-Interface Treatment of the Anisotropic Mean-Curvature Flow, Applications of Mathematics, *48*, No. 6, (2003), pp. 437–453.

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