

## DYNAMIC STOCHASTIC ACCUMULATION MODEL WITH APPLICATION TO PENSION SAVINGS MANAGEMENT

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**Abstract:** We propose a dynamic stochastic accumulation model for determining optimal decision between stock and bond investments during accumulation of pension savings. Stock prices are assumed to be driven by the geometric Brownian motion. Interest rates are modeled by means of the Cox-Ingersoll-Ross model. The optimal decision as a solution to the corresponding dynamic stochastic program is a function of the duration of saving, the level of savings and the short rate. Qualitative and quantitative properties of the optimal solution are analyzed. The model is tested on the funded pillar of the Slovak pension system. The results are calculated for various risk preferences of a saver.

**Keywords:** Dynamic stochastic programming, funded pillar, utility function, Bellman equation, Slovak pension system, risk aversion, pension portfolio simulations.

**JEL classification:** C15, E27, G19, G11, G23

### 1. INTRODUCTION

Pension systems around the world are currently undergoing a shift from unfunded social security towards defined-contribution (DC) funded systems. The main reason of this development is the ongoing demographic change. Increasingly, therefore, capital market based private pension plans supplement the pay-as-you-go (PAYG) systems which are predominant in continental Europe. However, one should not

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dogmatically presume that funded systems are more efficient than PAYG ones. First, such a statement is based on the well-known Aaron condition (c.f. [1]) where the asset returns are supposed to be higher than the growth rate of the wage bill. This is questionable e.g. in the Central European countries (Hungary, Poland, Slovakia) where the shift from the PAYG to funded pension system was already performed. The transitional economies of these countries have quite high wage-growth forecasts. Secondly, even if the Aaron condition is fulfilled, still there is a question of transitional costs. Following ideas due to Orszag and Stiglitz [28] one can prove that present value of benefits of the funded system compared to the PAYG one is equal to the present value of the transitional costs.

In the case of the DC system, a future pensioner bears the risk of asset returns during the accumulation phase. A natural question arises whether such a system can deliver a sufficient pension. When a person retires, he/she strives to maintain the living standard at the level of the last income. The sufficiency should be therefore measured by the ratio of a pension to a preretirement income. However, no DC planner can guarantee any prescribed ratio of the first pension to the last wage. There are several models that can help (but need not guarantee) to reach a defined level of pension savings. Many financial advisers are based upon the Markowitz portfolio selection model [17]. The inputs to this model are returns of a set of risky assets characterized by their means, standard deviations and correlations. The outputs are in the form of risk-return choices placed on the efficient portfolio frontier. In [33] Tobin added a risk-free asset to the list of inputs. Merton in a series of papers [21,22,24,25] showed that hedging of a portfolio can be as important as its diversifying. Decades from 1970's to 90's saw major market innovations and the rise of the new field of financial engineering. For a review and discussion of these innovations we refer to Bodie [5].

It is commonly assumed that stock returns should outperform bond returns in the long term run. Historical data confirm this conventional wisdom. Therefore investors with a long time horizon should prefer stocks to bonds. One popular rule is to invest 100% minus one's age in the stocks. Therefore, a future pensioner with the age of 50 should invest 50% in stocks. Merton and Samuelson have written several papers showing the fallacy of such statements (see e.g. [32,23]). Their counterarguments are based on the theory of expected utility maximization. Samuelson in [32] showed that for the so-called constant relative risk aversion utility functions the proportion of the total wealth to invest in stocks is independent of investor's age. Bodie's counterargument is based on the option pricing theory. If the stocks would perform better than bonds in the long run, the cost of insuring against earning less than risk-free rate should decline with increasing time horizon. In [4] he has shown that just the opposite is true. However, these models did not take into account future contributions. This is a typical situation of the pension saving system where one should take into account not only the actual level of savings but also future contributions in the investment decision. If a series of contributions throughout a lifespan is made, a fall in assets value in the beginning of saving does not affect the future contributions, i.e. only part of his/her future pension wealth is affected. On the other hand, if it occurs close to the retirement it affects all past accumulated contributions and returns on them, i.e. most of savers pension wealth. Therefore, the investment decision should depend on the time to the maturity of saving. A similar argument was used in [3] in which a lifetime consumption-portfolio choice model with a labor/leisure decision was developed and analyzed. The authors concluded that pension

saving becomes more conservative as retirement approaches. The model was extended in [6] by incorporating habit information, two distinct periods during the life cycle (accumulation and retirement periods) and a more general financial market with multiple assets.

In [14] Kilianová et al. developed a simplified dynamic stochastic model of pension fund management with regular yearly contributions. In this model the investment decision depends on the level of savings and duration of saving. Future pensioner can choose from many finite funds with different risk profiles. The major drawbacks of the model [14] consisted in the assumptions that the bond investments are supposed to have independent in time and normally distributed returns. In the present paper we improve the simplified model proposed in [14]. In reality the bond returns are not independent in time. We describe bond returns by means of the Cox-Ingersoll-Ross model where the interest rates have the mean-reversion property. Furthermore, instead of choosing from a finite number of funds, the decision variable is the proportion of the portfolio invested in stocks. The saver can subsequently choose a pension fund which is the closest to his/her investment decision. In contrast to the earlier model [14] now we have to solve a higher dimensional optimization problem making thus computations more complex and time consuming. The numerical results have confirmed that investment decisions really depend on the level of savings and the duration of saving. Furthermore, it turned out that gradual decreasing of risk is an optimal strategy. The model has been tested on the funded pillar of the pension system in Slovak republic.

The paper is organized as follows. In Section 2 we present the dynamic stochastic accumulation model of saving with regular contributions. Section 3 contains numerical scheme for the calculation of the corresponding Bellman equation. In Section 4 we present application of the model to the funded pillar of the Slovak pension system. We have calculated numerical results for different levels of risk aversion. The question whether the governmental regulations make sense is also discussed. The last section contains final remarks and conclusions.

## 2. THE DYNAMIC STOCHASTIC PROGRAMMING ACCUMULATION MODEL

Suppose that a future pensioner deposits once a year a  $\tau$ -part of his/her yearly salary  $w_t$  to a pension fund with a  $\delta$ -part of assets in stocks and a  $(1-\delta)$ -part of assets in bonds where  $\delta \in [0,1]$ . Denote by  $\gamma_t$ ,  $t=1,2,\dots,T$ , the accumulated sum at time  $t$  where  $T$  is the expected retirement time. Then the budget-constraint equations read as follows:

$$\begin{aligned} \gamma_{t+1} &= \delta \gamma_t \exp(R^s(t,t+1)) + (1-\delta) \gamma_t \exp(R^b(t,t+1)) + w_{t+1} \tau, \quad t=1,2,\dots,T-1, \\ \gamma_1 &= w_1 \tau, \end{aligned} \quad (1)$$

where  $R^s(t,t+1)$  and  $R^b(t,t+1)$  are the annual expected returns of stocks and bonds in the time interval  $[t,t+1)$ , resp. When retiring, a pensioner will strive to maintain his/her living standards in the level of the last salary. From this point of view, the saved sum  $\gamma_T$  at the time of retirement  $T$  is not precisely what a future pensioner takes care about. For

a given life expectancy, a ratio of the cumulative sum  $\gamma_T$  and the yearly salary  $w_T$ , i.e.  $d_T = \gamma_T/w_T$  is of practical importance to a pensioner. Using the quantity  $d_t = \gamma_t/w_t$  one can reformulate the budget-constraint equation (1) as follows:

$$d_{t+1} = d_t \frac{\delta \exp(R^s(t, t+1)) + (1-\delta) \exp(R^b(t, t+1))}{1 + \beta_t} + \tau, \quad (2)$$

$$d_1 = \tau,$$

for  $t = 1, 2, \dots, T-1$ , where  $\beta_t$  denotes the wage growth defined by the equation

$$w_{t+1} = w_t(1 + \beta_t).$$

We shall assume that the term structure of the wage growth  $\beta_t, t = 1, \dots, T$ , is known and can be externally estimated from an econometric model. We refer the reader to [15] for details of estimation of the wage growth term structure in Slovakia.

## 2.1 Modeling of bond and stock returns

The main goal of this section is to present key ideas of modeling stock and bond returns. We follow standard models in this field. Concerning interest rates and their term structure we make use of the one factor Cox-Ingersoll-Ross model [7]. The price of a zero coupon bond is then computed from the overnight rate and several model parameters through an explicit formula. The reason for considering the CIR term structure model is twofold: there are just four parameters in this model and there are recent studies on how to estimate these parameters in the case of Central European term structures (see e.g. [31]). Stock prices are supposed to be driven by the geometric Brownian motion.

### 2.1.1 Arbitrage free modeling of bond returns and their term structures

We suppose that the short rate  $r = r_t$  (overnight rate) is driven by a stochastic differential equation:

$$dr_t = \mu(r_t, t) dt + \omega(r_t, t) dZ_t \quad (3)$$

where  $r_t$  is the short rate at time  $t$  and  $Z_t$  is the Wiener process (c.f. [16]). Concerning the drift and volatility terms  $\mu$  and  $\omega$  we shall henceforth consider the following mean reverting one factor interest rate model:

$$dr_t = \kappa(\theta - r_t) dt + \sigma^b |r_t|^\gamma dZ_t \quad (4)$$

where  $\theta > 0$  is the long term interest rate,  $\kappa > 0$  is the rate of reversion,  $\sigma^b > 0$  is the volatility of the process and  $\gamma \geq 0$ . Within a variety of one factor term structure models having the short rate process of the form (4) there are, in particular, the Vasicek model ( $\gamma = 0$ ) and the Cox-Ingersoll-Ross model ( $\gamma = 1/2$ ). In these two models the term structure of zero coupon bonds can be expressed by explicit formulae. We shall consider the CIR model for description of term structures. Suppose that the bond part of the fund consists of 1-year zero coupon bonds. If we denote by  $R^b(t, t+1)$  the return on a one year maturing zero coupon bond at time  $t$  then it can be expressed as an affine function of the short rate  $r_t$ ,

$$R^b(t, t+1) = B(1)r_t - \ln A(1).$$

Recall that for the CIR model ( $\gamma = 1/2$ ) the terms  $A, B$  can be explicitly expressed by explicit formulae:

$$B(\xi) = \frac{2(e^{\eta\xi} - 1)}{(\kappa + \lambda + \eta)(e^{\eta\xi} - 1) + 2\eta}, \quad \ln A(\xi) = \frac{2\kappa\theta}{(\sigma^b)^2} \ln \left( \frac{\eta e^{(\kappa + \lambda + \eta)\xi/2}}{e^{\eta\xi} - 1} B(\xi) \right), \quad (5)$$

where  $\eta = \sqrt{(\kappa + \lambda)^2 + 2(\sigma^b)^2}$ . The parameter  $\lambda \in R$  stands for the so-called market price of risk (c.f. [16]). Using a standard discretization of the short rate process (4)  $r_{t+1} = g(r_t, \Phi)$  where

$$g(r, x) = \theta + e^{-\kappa} (r - \theta) + \sigma^b |r|^{\frac{1}{2}} e^{-\kappa} \left( \frac{1}{2\kappa} (e^{2\kappa} - 1) \right)^{\frac{1}{2}} x \quad (6)$$

and  $\Phi \square N(0,1)$  is a normally distributed random variable (see e.g. [2,27]).

**Remark 1.** We chose the CIR model for brevity of presentation. One can also make use of other one factor and multi-factors interest rate models like e.g. Vasicek equilibrium model or Hull and White noarbitrage model (c.f. Kwok [16]).

### 2.1.1. Normal modeling of stock returns

Next we propose a stochastic differential equation for modeling stock assets returns. Following a standard assumption on the lognormal behavior of stock prices [9,16] we shall assume the stock prices  $S_t$  are driven by the geometric Brownian motion

$$dS_t = (\mu^s + \frac{1}{2}(\sigma^s)^2)S_t dt + \sigma^s S_t dZ_t. \quad (7)$$

Using Itô's formula (c.f. [9,16]) we obtain  $d(\ln S_t) = \mu^s dt + \sigma^s dZ_t$ . The annual stock return  $R^s(t, t+1) = \ln(S_{t+1}/S_t)$  can be therefore expressed as:

$$R^s(t, t+1) = \mu^s + \sigma^s \Psi \quad (8)$$

where  $\mu^s$  and  $\sigma^s$  are the mean value and volatility of annual stock returns in the time interval  $[t, t+1)$ ,  $\Psi \square N(0,1)$  is a normally distributed random variable. We shall assume the correlation structure between the bond and stock returns of the form  $\text{corr}(\Phi, \Psi) = E(\Phi\Psi) = \rho$  where  $-1 \leq \rho \leq 1$  represents the correlation coefficient.

**Remark 2.** For the sake of simplicity, the parameters  $\mu^s, \sigma^s$  were assumed to be constant with respect to the time  $t \in [1, T]$ . It is straightforward to generalize the model by assuming the time dependence of these parameters, i.e.  $\mu^s = \mu_t^s, \sigma^s = \sigma_t^s, t = 1, \dots, T-1$ . Similarly, parameters  $\kappa, \theta, \lambda, \sigma^b$  can be assumed to be time dependent. Results of a simplified dynamic accumulation model [14] for various time profiles of parameters  $\mu_t^s, \sigma_t^s$  have been discussed by Kilianová in [12]. For comparison of results of optimal

pension planning obtained by the dynamic stochastic accumulation model and the risk measure based approach we refer to a recent paper [13].

## 2.2. Formulation of the problem

Suppose that a saver has a possibility to re-hedge a level  $\delta_t(I_t)$  of stocks included in the portfolio every year. Here  $I_t$  denotes the information set consisting of the history of bond and stock returns  $R^b(t', t'+1)$ ,  $R^s(t', t'+1)$ , and wage growths  $\beta_{t'}$ ,  $t' = 1, 2, \dots, t-1$ . Now we suppose that the forecast of the wage growth  $\beta_t$ ,  $t = 1, 2, \dots, T-1$ , is deterministic, the stock returns  $R^s(t, t+1)$  are random and independent for different times  $t = 1, 2, \dots, T-1$ , and the interest rates are driven by the Markov process (4). Then the only relevant information are the quantities  $d_t$  and the short rate  $r_t$ . Hence  $\delta_t(I_t) \equiv \delta_t(d_t, r_t)$ . One can formulate a problem of dynamic stochastic programming:

$$\max_{\delta} E(U(d_T)) \quad (9)$$

subject to the following recurrent budget constraints:

$$\begin{aligned} d_{t+1} &= F_t(d_t, r_t, \delta_t(d_t, r_t), \Psi), t = 1, 2, \dots, T-1, \\ d_1 &= \tau \end{aligned} \quad (10)$$

where

$$F_t(d, r, \delta, y) = d \frac{\delta \exp[\mu_t^s + \sigma_t^s y] + (1 - \delta) \exp[B(1)r - \ln A(1)]}{1 + \beta_t} + \tau \quad (11)$$

and the short rate process is driven by (4):

$$\begin{aligned} r_{t+1} &= g(r_t, \Phi), t = 1, 2, \dots, T-1, \\ r_1 &= r_{init}. \end{aligned} \quad (12)$$

The vector of random variables  $(\Phi, \Psi)$  is assumed to be normally distributed,  $(\Phi, \Psi) \square N(0, 1)$ , with correlation  $E(\Phi\Psi) = \rho$ . In the dynamic stochastic optimization problem (9) the maximum is taken over all non-anticipative strategies  $\delta = \delta_t(d_t, r_t)$ . We assume the stock part of the portfolio is bounded by a given upper barrier function  $\Delta_t$ :

$$0 \leq \delta_t(d_t, r_t) \leq \Delta_t. \quad (13)$$

The function  $\Delta_t : \{1, \dots, T-1\} \mapsto [0, 1]$  is subject to governmental regulations. In Section 4 we shall discuss an example of governmental regulations imposed in the Slovak pension system.

The function  $U$  stands for a given preferred utility function of wealth of a saver. Using the law of iterated expectations

$$E(U(d_T)) = E(E(U(d_T)|I_t)) = E(E(U(d_T)|d_t, r_t))$$

we conclude that  $E(U(d_T)|d_t, r_t)$  should be maximal. Let us denote by  $V_t(d, r)$  saver's intermediate utility function at time  $t$  defined as:

$$V_t(d, r) = \max_{0 \leq \delta \leq \Delta_t} E(U(d_T) | d_t = d, r_t = r). \quad (14)$$

Then, by using the tower law of iterated expectations

$$E(U(d_T) | d_t, r_t) = E(E(U(d_T) | d_{t+1}, r_{t+1}) | d_t, r_t),$$

we obtain the Bellman equation

$$\begin{aligned} V_t(d, r) &= U(d) \\ V_t(d, r) &= \max_{0 \leq \delta \leq \Delta_t} E[V_{t+1}(F_t(d, r, \delta, \Psi), g(r, \Phi))] \end{aligned} \quad (15)$$

for every  $d, r > 0$  and  $t = 1, 2, \dots, T-1$ . Let us denote by  $\hat{\delta}_t = \hat{\delta}_t(d, r)$  the unique argument of the maximum in (15), i.e.

$$V_t(d, r) = E[V_{t+1}(F_t(d, r, \hat{\delta}_t(d, r), \Psi), g(r, \Phi))]. \quad (16)$$

Uniqueness of  $\hat{\delta}_t(d, r)$  is discussed in the forthcoming Proposition 1. The optimal feedback strategy  $\hat{\delta}_t(d, r)$  can be found backwards. This strategy gives a saver decision hint what is the optimal fund choice for time  $t$ , actual level of savings  $d_t$  and the short rate  $r_t$ . Using (8) and (4) the Bellman equation (15) can be rewritten in the form

$$V_t(d, r) = \max_{0 \leq \delta \leq \Delta_t} \int_{\mathbf{R}^2} V_{t+1}(F_t(d, r, \delta, y), g(r, x)) f_\varrho(x, y) dx dy \quad (17)$$

where  $f_\varrho(x, y)$  is the joint distribution of the normal variables  $\Phi$  and  $\Psi$ :

$$f_\varrho(x, y) = \frac{1}{2\pi\sqrt{1-\varrho^2}} \exp\left[-\frac{1}{2(1-\varrho^2)}(x^2 - 2\varrho xy + y^2)\right]$$

having correlation  $E(\Phi\Psi) = \varrho \in (-1, 1)$ . After the change of variables  $x = \xi\sqrt{1-\varrho^2} + \varrho y$  we have  $f_\varrho(x, y) dx dy = f_0(\xi, y) d\xi dy$ . With this we have

$$V_t(d, r) = \max_{0 \leq \delta \leq \Delta_t} \int_{\mathbf{R}^2} V_{t+1}\left(F_t(d, r, \delta, y), g(r, \xi\sqrt{1-\varrho^2} + \varrho y)\right) f_0(\xi, y) d\xi dy. \quad (18)$$

Formula (18) will be used in numerical approximation of  $V_t(d, r)$  described in Section 3.

**Remark 3** In the simplified dynamic accumulation model proposed by Kilianová *et al.* [14] we did not take into account serial dependence of bond returns. The bond returns were assumed to be independent in time, i.e.  $r_{t+1} = g(\Phi)$  where the shape function  $g$  is independent of the short rate  $r = r_t$ , i.e.  $g(x) = \tilde{\mu}^b + \tilde{\sigma}^b x$ . By contrast to (18) for the

simplified model [14] we have to compute the intermediate utility function  $V_t(d)$  depending on the  $d$  variable only making thus computations of the optimal strategy faster.

**Remark 4** In our model we assumed the wage growth  $\beta_t$  to be deterministic and given exogenously. Notice that the deflator  $1/(1+\beta_t)$  used in (11) is an approximation for the continuous deflator  $\exp(-\beta_t)$ . Therefore we can reformulate the dynamic stochastic accumulation model with the function  $F_t$  given by

$$F_t(d, r, \delta, y) = d \left( \delta \exp[\mu_t^s - \beta_t + \sigma_t^s y] + (1 - \delta) \exp[B(1)r - \beta_t - \ln A(1)] \right) + \tau.$$

This way we can generalize our model to include a wage growth  $\beta_t$  driven by another stochastic process. The resulting recursive Bellman formula (18) will contain integration over three-dimensional space  $R^3$ . It is also worth to note that only the difference of stochastic asset returns and the wage growth is important for modeling levels of savings.

### 2.3 The constant relative risk aversion (CRRA) utility function

An important part of the problem (9)-(10) is a proper choice of the utility function  $U$ . The utility function varies across investors and represents their attitude to risk. A key role in determining the utility function is played by the coefficient of relative risk aversion  $C(d) = -dU''(d)/U'(d)$ . A constant relative risk aversion  $C(d) \equiv a > 0$  for every  $d > 0$  implies that an investor has a tendency to hold a constant proportion of his/her wealth in any class of risky assets as the wealth varies (see e.g. Friend & Blume [8], Pratt [29] and Young [34]). In this case the utility function is uniquely given by

$$\begin{aligned} U(d) &= -Ad^{1-a} + B & \text{if } a > 1, \\ U(d) &= A \ln(d) + B & \text{if } a = 1, \\ U(d) &= Ad^{1-a} + B & \text{if } a < 1, \end{aligned} \tag{19}$$

where  $A, B$  are constants and  $A > 0$ . The coefficient  $a$  of relative risk aversion plays an important role in many fields of economics. There is a consensus today, that the value should be less than 10 (see e.g. Mehra and Prescott [18]). In our numerical experiments we considered values of  $a$  close to 9. It could be lower for lower equity premium. It is worth to note that the CRRA function is a smooth, increasing and strictly concave function for  $d > 0$ .

### 2.4 Qualitative properties of intermediate utility functions

The aim of this section is to analyze qualitative properties of intermediate utility functions  $V_t(d, r), t = 1, \dots, T-1$ . The next proposition enables us to conclude that properties of the utility function  $U$  are inherited by functions  $V_t$  for each  $t = 1, \dots, T$ . Furthermore, we shall prove that the optimal feedback function  $\hat{\delta}_t$  is well defined, i.e. there exists a unique argument of the maximum in (15). In order not to interrupt



presentation of results we postpone the proofs of the following three propositions to Appendix.

**Proposition 1.** Let  $U(d)$  be an increasing, strictly concave,  $C^2$  smooth function for  $d > 0$ . Then for any  $t = 1, \dots, T$ ,

1. the function  $V_t(d, r)$  is increasing, strictly concave in the  $d$ -variable;
2. there exists the unique argument  $\hat{\delta}_t(d, r)$  of the maximum in the right-hand side of (15);
3. the functions  $V_t(d, r)$  and  $\hat{\delta}_t(d, r)$  are continuous in both variables  $d, r > 0$  and they are  $C^\infty$  smooth in the  $d$ -variable in any bounded interval  $(0, d_{max})$  except at most finite number of isolated points.

The next result is focused on qualitative properties of intermediate utility functions  $V_t$ . It enables us to conclude that the coefficient of the relative risk aversion increases with time to maturity. More precisely, we have the following result:

**Proposition 2** Let  $U(d)$  be an increasing, strictly concave,  $C^2$  smooth function for  $d > 0$ . Denote by

$$C_t = \sup_{d, r > 0} \frac{-d \frac{\partial^2 V_t}{\partial d^2}(d, r)}{\frac{\partial V_t}{\partial d}(d, r)}$$

the maximal relative risk aversion coefficient of the intermediate utility function  $V_t$  for  $t = 1, \dots, T$ . If  $C_T < \infty$  then

$$0 < C_1 \leq C_2 \leq \dots \leq C_{T-1} \leq C_T.$$

Recall that the CRRA utility function  $U$  described in the previous section satisfies

$$-dU''(d)/U'(d) = a$$

for every  $d > 0$  where  $a > 0$  is the coefficient of the relative risk aversion. By Proposition 2 we have  $C_t \leq a$  for every  $t$ . The result is in accord with a financial intuition: for  $t$  close to the retirement  $T$  the decision affects nearly all pension savings whereas for earlier times only a part of the total savings is affected. Therefore, the saver should be more conservative as the retirement approaches.

If the performance of the stock part of the fund portfolio is lower than its bond part it is natural to expect that  $\hat{\delta}_t = 0$ . The following result justifies such an expectation. More precisely, we have the following result:

**Proposition 3.** Let  $U(d)$  be an increasing, strictly concave,  $C^2$  smooth function for  $d > 0$ . Suppose that  $\rho = 0$ . Then the following statements are equivalent:

1. the bond returns outperform the average stock returns, i.e.

$$\mu^s + \frac{(\sigma^s)^2}{2} \leq B(1)r - \ln A(1);$$

2. there are no stocks in the optimal fund portfolio, i.e.  $\hat{\delta}_t(d, r) = 0$  for every  $d > 0$ .

### 3. NUMERICAL APPROXIMATION OF THE BELLMAN EQUATION

In this section we present a numerical approximation procedure for solving the iterative problem (18). Our algorithm is as follows: The function  $V_T(d, r) = U(d)$  is given. Then we compute the functional  $V_t$  recurrently from  $t = T - 1$  down to  $t = 1$ . In each time step  $t$  we compute approximate values of the function  $V_t(d_i, r_j)$  in a discrete two dimensional mesh  $\{(d_i, r_j), i = 1, \dots, n_d, j = 1, \dots, n_r\}$  where  $d_i$  and  $r_j$  represent equidistant division of the intervals  $(d_{min}, d_{max})$  and  $(r_{min}, r_{max})$ . Given a mesh point  $(d_i, r_j)$  we make use of a "brute force" algorithm for computing the maximum value of the  $\delta$ -parameterized integral

$$\int_{\mathbb{R}^2} V_{t+1} \left( F_t(d, r, \delta, y), g(r, \xi \sqrt{1 - \varrho^2} + \varrho y) \right) f_0(\xi, y) d\xi dy. \quad (20)$$

We construct an equidistant division  $\{\delta_k, k = 1, \dots, n_\delta\}$  of the interval  $[0, \Delta_t]$  where  $n_\delta$  is sufficiently large. By Proposition 1 there exists a unique argument of maximum  $\hat{\delta}_t(d, r)$  of (20). Hence we can find a unique  $\delta_k$  such that the value of the integral (20) is maximal. We then set  $\hat{\delta}_t(d_i, r_j) = \delta_k$  and  $V_t(d_i, r_j)$  is equal to this maximal value of the integral. We repeat this computation for all mesh points  $\{(d_i, r_j), i = 1, \dots, n_d, j = 1, \dots, n_r\}$ .

The difficulty in computation of the integral (20) is due to large variance of values of the integrated function. More precisely, it may attain both large values as well as low values of the order one. Therefore a scaling technique is needed when computing the integral (18). The idea of scaling is rather standard and is widely used in similar circumstances. Let  $H_t(d, r)$  be any bounded positive function for  $t = 1, \dots, T$ . We scale the function  $V_t$  by  $H_t$ , i.e. we define a new auxiliary function

$$W_t(d, r) = H_t(d, r) V_t(d, r).$$

The original intermediate utility function  $V_t(d, r)$  can be easily calculated from  $W_t(d, r)$  as follows:  $V_t(d, r) = W_t(d, r) / H_t(d, r)$ . It is worth to note that the functions  $V_t, t = 1, \dots, T$ , are independent of a particular choice of scaling functions  $H_t$ . Other suitable choices of  $H_t$  are also possible. For example, one can take  $H_t(d, r) = |U(d + \tau(T - t))|$ .

For each time step  $t$  from  $t = T$  down to  $t = 1$  we have

$$\begin{aligned}
W_T(d, r) &= H_T(d, r)V_T(d, r) = H_T(d, r)U(d) \quad \text{and} \\
W_i(d, r) &= H_i(d, r)V_i(d, r) \\
&= \max_{0 \leq \delta \leq \Delta_i} \int_{\mathbf{R}^2} \frac{H_i(d, r)W_{i+1}(F_i(d, r, \delta, y), g(r, \xi \sqrt{1-\varrho^2} + \varrho y))}{H_{i+1}(F_i(d, r, \delta, y), g(r, \xi \sqrt{1-\varrho^2} + \varrho y))} f_0(\xi, y) d\xi dy.
\end{aligned}$$

As the initial scaling function we choose  $H_T(d, r) = (1 + V_T(d, r)^2)^{-\frac{1}{2}} = (1 + U(d)^2)^{-\frac{1}{2}}$  and then  $H_i$  is defined recursively as:

$$H_i(d, r) = (1 + V_{i+1}(d, r)^2)^{-\frac{1}{2}} = H_{i+1}(d, r) \left( H_{i+1}(d, r)^2 + W_{i+1}(d, r)^2 \right)^{-\frac{1}{2}}$$

for  $t = T-1, \dots, 1$ . The core of computation consists in evaluation of the two dimensional integral over the infinite domain  $\mathbf{R}^2$ . As it is usual in such circumstances we restrict the computational domain  $\mathbf{R}^2$  to  $\{(\xi, y), -L < \xi < L, -L < y < L\}$  where  $L \gg 1$  is a sufficiently large constant. For practical purposes it is sufficient to take  $L \approx 3$  as the domain  $(-3, 3) \times (-3, 3)$  captures more than 98% of significant values of the density function  $f_0(\xi, y)$ . The numerical approximation of the integral (20) over a domain  $(-L, L) \times (-L, L)$  is then realized by a simple quadrature rule (the Simpson or trapezoidal rule) with  $n_\xi = n_y$  grid points in each direction.

#### 4. TESTING THE MODEL ON THE SLOVAK PENSION SYSTEM

In this section we present an example of a possible application of the dynamic stochastic accumulation model to the Slovak pension system. Notice that the dynamic stochastic accumulation model derived in Section 2.2 can be utilized in other pension saving system having constant proportion of contributions to salary.

##### 4.1 Slovak pension system and calibration of the model parameters

Since January 2005, pensions in Slovakia are operated by a three-pillar system proposed by the World Bank<sup>2</sup>:

1. the mandatory non-funded first pillar (pay-as-you-go pillar);
2. the mandatory fully funded second pillar;
3. the voluntary fully funded third pillar.

Contribution rates were set for the first pillar at 19.75% (old age 9%, disability and survival 6% and reserve fund 4.75%) and for the second pillar 9%. The savings in the second pillar are managed by pension asset administrators. Each pension administrator manages three funds: Growth Fund, Balanced Fund and Conservative fund, each of them with different limits for investment (see Tab. 1). At the same time instant savers may hold assets in one fund only. In the last 15 years preceding retirement, a saver may not hold assets in the Growth Fund and in the last 7 years all assets must be deposited in the

<sup>2</sup>For comprehensive information about Slovak pension system and the pension reform in 2005 see e.g. Melicherčík and Ungvarský [20].

Conservative Fund.<sup>3</sup> Even with these restrictions contributors have some space for individual decisions which fund is optimal in a specific situation (the age of the contributor, the saved amount, the past performance of the pension funds, etc.).

The proposed model has been tested on the second pillar of the Slovak pension system. According to Slovak legislature the percentage of salary transferred each year to a pension fund is 9%.<sup>4</sup> We have assumed the period  $T = 40$  of saving. The forecast for the expected wage growth  $\beta_t$  in Slovakia has been taken from a recent paper by Kvetan et al. [15]. The term structure  $\{\beta_t, t = 1, \dots, T\}$  from 2007 to 2050 is shown in Fig. 1. Stocks have been represented by the S&P500 Index. The stock returns have been modeled according to (8). For the calibration we have taken the same time period (Jan 1996-June 2002) as in Kilianová et al. [14] with average return  $\mu^s = 10.28\%$  and standard deviation  $\sigma^s = 16.90\%$ . The model parameters describing the term structure of the zero coupon BRIBOR<sup>5</sup> bonds have been adopted from the paper by Ševčovič and Urbánová Csajková [31]. We assumed the long term interest rate  $\theta = 0.029$ ,  $\sigma^b = 0.15$ ,  $\kappa = 1$  and  $\lambda = 0$ . The correlation between stock and bond returns was set to  $\varrho = -0.1151$  (the same as in [14]). We chose the following numerical parameters for computations:  $(d_{min}, d_{max}) = (0.09, 12)$ ,  $(r_{min}, r_{max}) = (0.005, 0.09)$ ,  $n_d = 100$ ,  $n_r = 15$ ,  $n_\delta = 30$ ,  $n_\xi = n_y = 16$ .

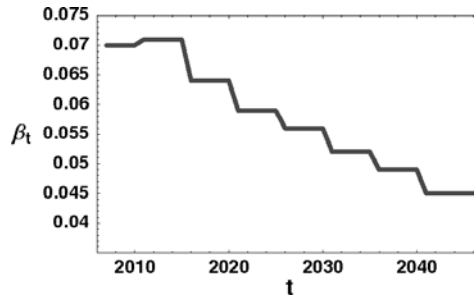
**Table 1.** Governmental limits for investment for the pension funds

<b>Fund type</b>	<b>Stocks</b>	<b>Bonds and money market instruments</b>
Growth Fund	up to 80%	at least 20%
Balanced Fund	up to 50%	at least 50%
Conservative Fund	no stocks	100%

<sup>3</sup>The real distribution of savings to the Growth, Balanced and Conservative funds and the performance analysis of the funds in first years after the pension reform can be found in Mit'ková et al. [26].

<sup>4</sup>The law sets administrative costs of the second pillar at 1% of monthly contribution and 0.07% of the monthly asset value (i.e. 0.84% p.a.). Therefore, the effective contribution rate is  $\tau = 8.91\%$  ( $= 9\% * 0.99$ ). The value 0.84% should be subtracted from both stock and bond returns.

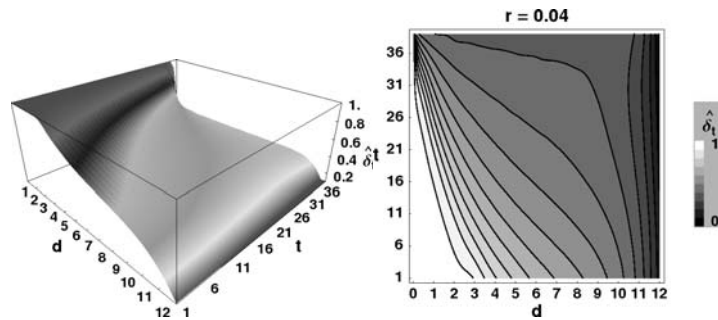
<sup>5</sup>BRIBOR (Bratislava Interbank Offering Rate) is the yield curve term structure in Slovakia

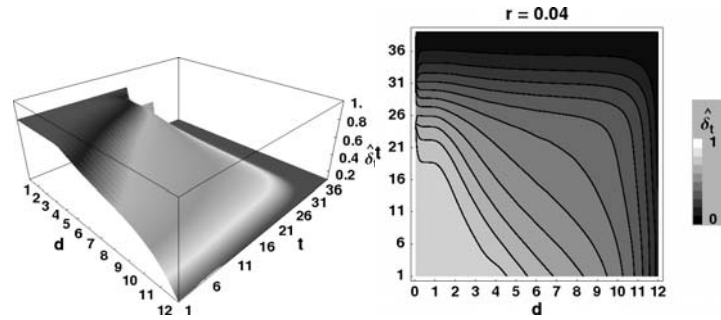


**Figure 1.** The expected term structure of the wage growth  $\beta_t$  for years 2007-2050 in Slovakia

**4.2 Results**

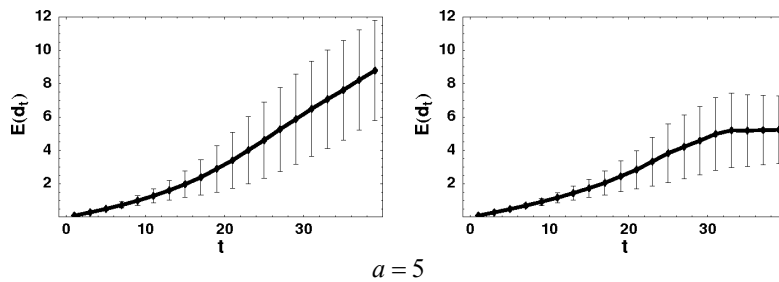
In Fig. 2 we present a typical result of our analysis with the coefficient of risk aversion  $a = 9$ . It depicts the function  $\hat{\delta}_t(d, r)$  of optimal decisions with fixed interest rate  $r = 4\%$ . The results are calculated with and without the governmental regulations (see Tab. 1). One can see that in both cases the saver starts with the most possible risky investment. Later on the risk (expressed in terms  $\hat{\delta}_t$ ) is gradually decreased. The reason for such a behavior is that more contributions are accumulated and higher part of the future pension is affected by asset returns. One can also observe gradual decrease of the dependence of the decision on the level of savings. This is due to the fact that less amount of forthcoming contributions is expected. In the case of no future contributions, a decision based on a CRRA utility function is independent of the level of savings (see e.g. Samuelson [32]). One can also observe that the governmental regulations have essential impact on the decision process.



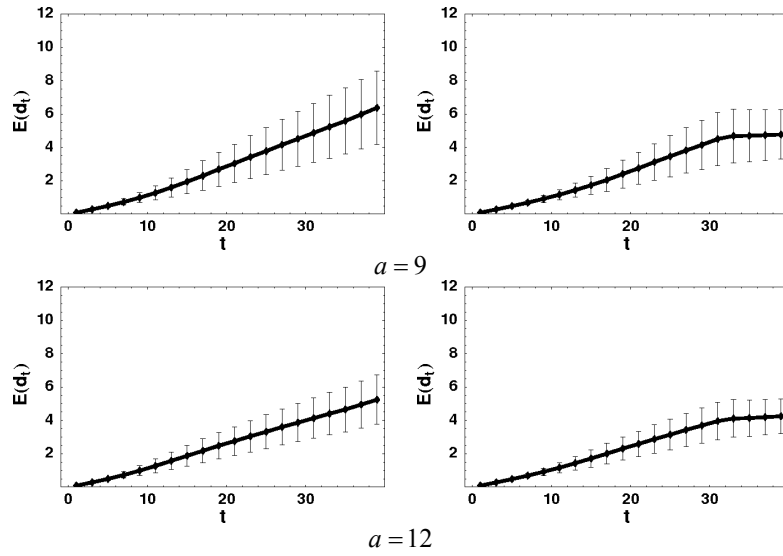


**Figure 2.** The 3D and contour plots of the function  $\hat{\delta}_t(d, r)$  for  $r = 4\%$  with no governmental limits imposed (above) and governmental limitations (below). The risk aversion parameter  $a = 9$  has been considered

Plots in Fig. 3 represent the mean wealth  $E(d_t)$  obtained by 10 000 simulations of random paths  $\{(d_t, r_t), t = 1, \dots, T\}$  calculated according to (10) and (12). The results with and without the governmental regulations are calculated for different risk aversion parameters  $a = 5, 9, 12$ . In accordance with financial intuition, for a higher aversion to risk we obtained lower levels of the expected wealth and lower standard deviations. Not surprisingly, the average wealth achieved is higher without governmental regulations. The regulations diminish standard deviations of the wealth achieved. The values of the average final wealth and standard deviations could be found in Tab. 2. The optimal decision  $\hat{\delta}_t(d_t, r_t)$  depends on the level of savings  $d_t$  and the short rate  $r_t$  at a time  $t$ . The development of the average decision  $E(\hat{\delta}_t)$  for different risk aversion parameters  $a = 5, 9, 12$  obtained by Monte-Carlo simulations is presented in Fig. 4. A gradual decreasing of the stock investment could be seen in all cases (even in the case of low risk aversion without governmental regulations). Except of an obvious impact in the first years of saving, for all risk aversion coefficients one can observe a significant impact of the governmental regulations in the case of a low aversion to risk.



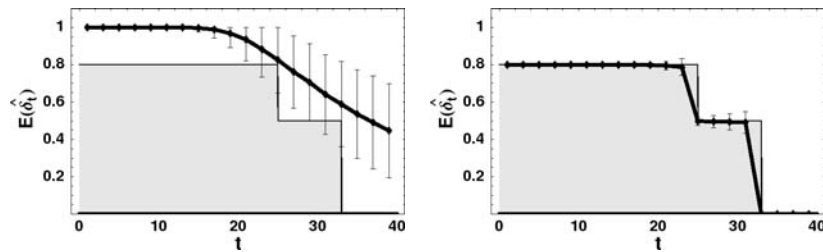
$a = 5$

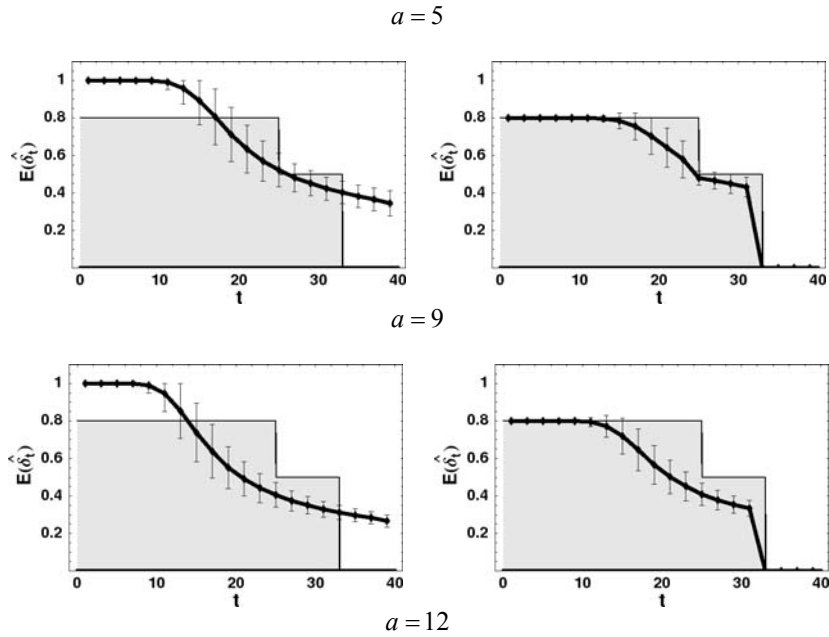


**Figure 3.** The average value  $E(d_t)$  for various values of the risk aversion parameter  $a = 5, 9, 12$ . No governmental limitations on the optimal choice of  $\hat{\delta}_t$  (left); governmental limitations imposed (right). The error bars show the standard deviation of  $d_t$

**Table 2.** The average value  $E(d_T)$  of  $d_T$  and its standard deviation  $\sigma(d_T)$  for various risk aversion parameters  $a$

$a$	3	4	5	6	7	8	9	10	11	12
Governmental limitations										
$E(d_T)$	5.264	5.261	5.247	5.203	5.109	4.966	4.791	4.6	4.427	4.275
$\sigma(d_T)$	2.033	2.026	1.997	1.928	1.809	1.644	1.462	1.288	1.143	1.023
No limits										
$E(d_T)$	9.871	9.574	9.04	8.402	7.738	7.112	6.561	6.089	5.697	5.375
$\sigma(d_T)$	3.075	3.024	3.002	2.912	2.736	2.496	2.233	1.968	1.718	1.505
Cautious investment										
$E(d_T)$	3.818	3.818	3.818	3.818	3.818	3.817	3.814	3.806	3.793	3.774
$\sigma(d_T)$	0.848	0.848	0.848	0.848	0.848	0.846	0.839	0.825	0.805	0.78

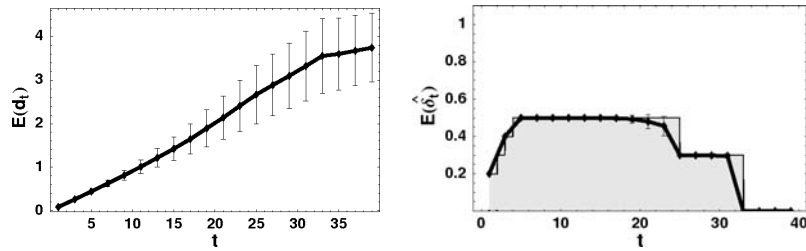




**Figure 4.** The average value of  $\hat{\delta}_t(d_t, r_t)$  for various values of the risk aversion parameter  $a = 5, 9, 12$ . No limitations on the optimal choice of  $\hat{\delta}_t$  (left) and governmental limitations (right). Error bars show the standard deviation of  $\hat{\delta}_t(d_t, r_t)$

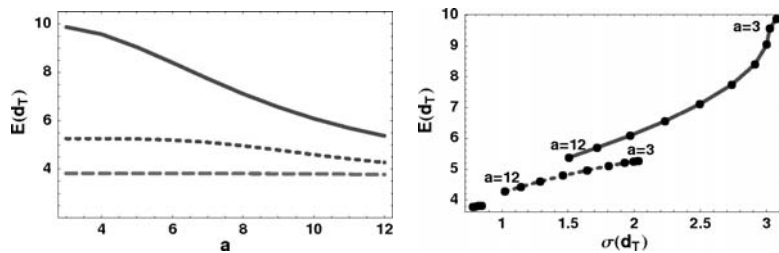
Pension assets managers have very cautious investment strategies in the first years after the pension reform. In March 2007 growth funds contained only up to 20% of stock investments. Suppose that this proportion will be linearly increased up to 50% in the next 3 years. After that the proportion of the stock investment in the balanced fund will be 30%. The development of the average level of savings and average proportion of the stock investment with standard deviations for such a cautious investment strategies can be found in Fig. 5 and Tab. 2. One can make a conclusion that, even in the case of very high risk aversion coefficient  $a = 12$ , it is optimal to stay in the growth and balanced funds as long as possible. Furthermore, comparing to the situation where the funds just undergo the governmental regulations (Fig. 3 right and Tab. 2), the level of savings is significantly lower. Since at least a half of pension is supposed to be paid from the second funded pillar (old age contributions to the first pillar and contributions to the second pillar are equal), the expected level  $E(d_T)$  of terminal savings less than 4 yearly salaries does not seem to be sufficient. One can therefore conclude that the ongoing cautious investment of pension asset administrators in Slovakia could lead to insufficient pensions in the future. Therefore, the assets managers should soon increase the stock investments to higher levels.





**Figure 5.** The average values  $E(d_t)$  (left) and  $E(\hat{d}_t)$  (right). Error bars depicts standard deviations for the cautious investment strategy. The risk aversion coefficient  $a = 12$  has been considered

Fig. 6 contains a risk-return analysis of all considered strategies. The solid lines represent investment strategies with no limitations whereas the dashed lines stand for strategies with governmental limitations and the cautious strategies with different risk aversion coefficients. The dependence of the expectation of the level of final wealth  $d_T$  on the risk aversion coefficient  $a$  is depicted in the left side. In the case of the cautious strategies (the bottom dashed line), one can observe that the expected level of the final savings is almost independent of the risk aversion coefficient. This is because of the fact that the pension asset administrators are so cautious that all savers (regardless of their aversion to risk) stay in the growth and balanced funds as long as possible. In the case when the strategies are restricted by the governmental regulations only (the upper dashed line) a moderate dependence of the final expected level of savings on the risk aversion coefficient could be seen. A significant dependence can be seen in the case of strategies with no governmental limitations. A mean-variance analysis is depicted in the right part of Fig. 6. The points in left bottom part of the mean-variance diagram represent the cautious strategies. They have low risk  $\sigma(d_T)$  connected with low expected level  $E(d_T)$  of savings. The points connected with a dashed line represent the strategies with the governmental regulations only. Using the mean-variance framework analysis, it is seen that the comparable strategies with no limitations (the points connected with a solid line) are better for use than the ones with governmental limitations.



**Figure 6.** Dependence of the terminal expected value  $E(d_T)$  as a function of the utility function parameter  $a$  (left). The mean-variance diagram for the terminal value  $d_T$  (right)

## 5. CONCLUSION

We have proposed a dynamic stochastic accumulation model for determining the optimal decision strategy between stock and bond investments. Stock prices were assumed to be driven by the geometric Brownian motion. Interest rates were modeled by the CIR model. The optimal decision (represented by a part of the savings to be invested in stocks)  $\hat{\delta}_t(d_t, r_t)$  is a function of duration of saving  $t$ , the level of savings  $d_t$  and the short rate  $r_t$ . The model was tested on the funded pillar of the Slovak pension system.

We also analyzed qualitative properties of intermediate utility functions. In particular, we proved existence and uniqueness of the optimal stock investment proportion  $\hat{\delta}_t(d_t, r_t)$ . Furthermore, we showed that the relative risk aversion of intermediate utility functions increases when approaching the retirement time.

The experimental results confirmed that pension saving becomes more conservative as retirement approaches. The dependence of the decision on the level of savings gradually decreases. The resulting strategies depend on individual risk preferences of a future pensioner represented by his/her individual utility function. Higher risk aversion implies lower level of the expected wealth which is associated with lower standard deviations. The average wealth achieved is higher without governmental regulations. The regulations lower standard deviations of the wealth achieved. It also turned out that cautious investment strategies of pension asset managers in the first years after the Slovak pension reform could lead to insufficient pensions.

## 6. APPENDIX

**Proof of Proposition 1.** First we prove that the function  $V_t(d, r)$  is strictly increasing for all  $t=1, \dots, T$ . We proceed by mathematical induction from  $t=T$  down to  $t=1$ . For  $t=T$  we have  $V_T(d, r) = U(d)$  and therefore  $V_T$  is an increasing function. Now suppose that  $V_{t+1}$  is increasing in  $d$  variable. Then, for any  $\delta \in [0, \Delta_t]$ , we have  $F_t^1 < F_t^2$  where  $F_t^i = F_t(d^i, r, \delta, y)$ ,  $i=1, 2$ . Hence  $\int_{R^2} V_{t+1}(F_t^1, g) f_g(x, y) dx dy < \int_{R^2} V_{t+1}(F_t^2, g) f_g(x, y) dx dy$ ,  $g = g(r, x)$ . By taking the maximum over the compact interval  $[0, \Delta_t]$  we obtain  $V^t(d^1, r) < V^t(d^2, r)$ , as claimed.

In order to prove that the function  $d \mapsto V_t(d, r)$  is strictly concave we again proceed by induction. For  $t=T$  the statement follows from concavity of the utility function  $U$ . Now suppose that  $V_{t+1}$  is strictly concave in the  $d$ -variable, i.e.  $\frac{\partial^2 V_{t+1}}{\partial d^2} < 0$  for almost every  $d > 0$ . Let us denote

$$\phi(d, r, \delta) = \int_{R^2} V_{t+1}(F_t(d, r, \delta, y), g(r, x)) f_g(x, y) dx dy.$$

Notice that the function  $\phi$  is real analytic in  $d$  and  $\delta$  variables. Indeed, by the change of variables  $\eta = F_t(d, r, \delta, y)$  we have

$\phi(d, r, \delta) = \int_{\Omega_t} V_{t+1}(\eta, g(r, x)) f_\varrho(x, G_t(d, r, \delta, \eta)) \frac{\partial G_t}{\partial \eta} dx d\eta$  where  $y = G_t(d, r, \delta, \eta)$  is the inverse function of  $F_t$  and  $\Omega_t = \Omega_t(d, r, \delta) = \{(x, \eta), \eta = F_t(d, r, \delta, y), (x, y) \in \mathbb{R}^2\}$ . Hence the real analyticity of the function  $\phi$  is inherited from the analyticity of the Gaussian kernel  $f_\varrho$ .

Since the function  $F_t(d, r, \delta, y)$  is affine in the  $\delta$  variable we have  $\frac{\partial^2 F_t}{\partial \delta^2} = 0$  and therefore

$$\frac{\partial^2 \phi}{\partial \delta^2} = \int_{\mathbb{R}^2} \frac{\partial^2 V_{t+1}}{\partial d^2}(F_t, g) \left( \frac{\partial F_t}{\partial \delta} \right)^2 f_\varrho(x, y) dx dy < 0.$$

Hence the function  $\phi$  is strictly concave in  $\delta$  and so there exists a unique  $\hat{\delta} = \hat{\delta}_t(d, r)$  - the argument of a maximum of  $\phi$  over the interval  $[0, \Delta_t]$ , i.e.

$$V_t(d, r) = \phi(d, r, \hat{\delta}_t(d, r)). \quad (21)$$

Similarly, the function  $F_t(d, r, \delta, y)$  is affine also in the  $d$  variable and therefore

$$\frac{\partial^2 \phi}{\partial d^2} = \int_{\mathbb{R}^2} \frac{\partial^2 V_{t+1}}{\partial d^2}(F_t, g) \left( \frac{\partial F_t}{\partial d} \right)^2 f_\varrho(x, y) dx dy < 0.$$

Moreover,

$$\frac{\partial \phi}{\partial \delta} = \int_{\mathbb{R}^2} \frac{\partial V_{t+1}}{\partial d}(F_t, g) \frac{\partial F_t}{\partial \delta} f_\varrho(x, y) dx dy \quad (22)$$

and

$$\frac{\partial^2 \phi}{\partial d \partial \delta} = \int_{\mathbb{R}^2} \frac{\partial^2 V_{t+1}}{\partial d^2}(F_t, g) \frac{\partial F_t}{\partial d} \frac{\partial F_t}{\partial \delta} f_\varrho(x, y) dx dy + \int_{\mathbb{R}^2} \frac{\partial V_{t+1}}{\partial d}(F_t, g) \frac{\partial^2 F_t}{\partial d \partial \delta} f_\varrho(x, y) dx dy.$$

Finally, we prove strict concavity of the function  $V_t$  in the  $d$  variable. First we consider the case  $0 < \hat{\delta}_t(\bar{d}, \bar{r}) < \Delta_t$ . Then the value  $\hat{\delta}_t(\bar{d}, \bar{r})$  is determined from the first order necessary condition

$$\frac{\partial \phi}{\partial \delta}(d, r, \hat{\delta}_t(d, r)) = 0. \quad (23)$$

It follows from the analytic version of the implicit function theorem (see [19]) that the function  $\hat{\delta}_t(d, r)$  is a real analytic function in the  $d$  variable in some neighborhood of

$\bar{d}$ . Let us denote  $v = \left( 1, \frac{\partial \hat{\delta}_t}{\partial d} \right)^T$  and by  $A$  a  $2 \times 2$  matrix

$$A = \begin{pmatrix} \frac{\partial^2 \phi}{\partial d^2} & \frac{\partial^2 \phi}{\partial d \partial \delta} \\ \frac{\partial^2 \phi}{\partial d \partial \delta} & \frac{\partial^2 \phi}{\partial \delta^2} \end{pmatrix}$$

evaluated at the point  $(\bar{d}, \bar{r}, \hat{\delta}_t(\bar{d}, \bar{r}))$ . We have

$$\frac{\partial^2 V_t}{\partial d^2} = v^T A v + \frac{\partial \phi}{\partial \delta} \frac{\partial^2 \hat{\delta}}{\partial d^2} \leq v^T A v \quad (24)$$

because of concavity of  $\phi$  and the condition (23). Due to the structure of the function  $F_t$  we have  $\frac{\partial^2 F_t}{\partial d \partial \delta} = \frac{1}{d} \frac{\partial F_t}{\partial \delta}$ . Therefore  $\int_{R^2} \frac{\partial V_{t+1}}{\partial d} (F_t, g) \frac{\partial^2 F_t}{\partial d \partial \delta} f_\theta(x, y) dx dy = 0$  because of (22) and the first order necessary condition (23).

$$\frac{\partial^2 \phi}{\partial d \partial \delta} = \int_{R^2} \frac{\partial^2 V_{t+1}}{\partial d^2} (F_t, g) \frac{\partial F_t}{\partial d} \frac{\partial F_t}{\partial \delta} f_\theta(x, y) dx dy$$

at  $(\bar{d}, \bar{r}, \hat{\delta}_t)$ . Now it follows from the Cauchy-Schwartz inequality

$$\left( \int_{R^2} p(x, y) q(x, y) dx dy \right)^2 \leq \left( \int_{R^2} p(x, y)^2 dx dy \right) \left( \int_{R^2} q(x, y)^2 dx dy \right) \text{ with}$$

$$p = (-\partial_d V_{t+1} f_\theta)^{\frac{1}{2}} \partial_d F_t \text{ and } q = (-\partial_d V_{t+1} f_\theta)^{\frac{1}{2}} \partial_\delta F_t$$

$$\begin{aligned} & \left( \int_{R^2} -\frac{\partial^2 V_{t+1}}{\partial d^2} (F_t, g) \frac{\partial F_t}{\partial d} \frac{\partial F_t}{\partial \delta} f_\theta(x, y) dx dy \right)^2 \\ & < \int_{R^2} -\frac{\partial^2 V_{t+1}}{\partial d^2} (F_t, g) \left( \frac{\partial F_t}{\partial d} \right)^2 f_\theta(x, y) dx dy \int_{R^2} -\frac{\partial^2 V_{t+1}}{\partial d^2} (F_t, g) \left( \frac{\partial F_t}{\partial \delta} \right)^2 f_\theta(x, y) dx dy. \end{aligned}$$

The equality cannot occur. Indeed, in the case of equality the functions  $\frac{\partial F_t}{\partial d}$  and  $\frac{\partial F_t}{\partial \delta}$  must be a scalar multiple of each other which is clearly impossible. It means that

$$\left( \frac{\partial^2 \phi}{\partial d \partial \delta} \right)^2 < \frac{\partial^2 \phi}{\partial d^2} \frac{\partial^2 \phi}{\partial \delta^2} \quad (25)$$

and the matrix  $A$  is negative definite. Hence  $\frac{\partial^2 V_t}{\partial d^2} < 0$  and the function  $V_t$  is strictly concave at  $(\bar{d}, \bar{r})$ .

In the rest of the proof we concentrate on the case when either  $\hat{\delta}_t(\bar{d}, \bar{r})$  attains the boundary of the interval  $[0, \Delta_t]$ , i.e.  $\hat{\delta}_t(\bar{d}, \bar{r}) = \Delta_t$  or  $\hat{\delta}_t(\bar{d}, \bar{r}) = 0$ . If  $\hat{\delta}_t(d, \bar{r}) = \Delta_t$  for all  $d > 0$  or  $\hat{\delta}_t(d, \bar{r}) = 0$  for all  $d > 0$  then  $\partial_d \hat{\delta}_t \equiv 0$ . It should be obvious from (24) and (25) that  $V_t$  is smooth in  $d$ -variable and  $\partial_d^2 V_{t+1} < 0$  for every  $d > 0$ . Let us concentrate

on the case when there exists  $d > 0$  such that  $0 < \hat{\delta}_t(d, \bar{r}) < \Delta_t$ . Let us define an auxiliary function  $\tilde{\delta}_t(d, r)$  as an unconstrained argument of the maximum of the function  $\phi(d, r, \delta)$ . Then  $\tilde{\delta}_t(d, \bar{r})$  is a unique solution the first order condition (23) defined for all  $d > 0$  and, moreover,

$$\hat{\delta}_t(d, \bar{r}) = \max(\min(\tilde{\delta}_t(d, \bar{r}), \Delta_t), 0).$$

Again, by the analytic version of the implicit function theorem the function  $\tilde{\delta}_t(d, \bar{r})$  is real analytic in the  $d$ -variable and it is continuous in  $\bar{r}$ . Hence  $\hat{\delta}_t$  is continuous in both variables. Furthermore, for any finite  $0 < d_{max} < \infty$  the set  $D^0 = \{d \in (0, d_{max}), \tilde{\delta}_t(d, \bar{r}) = 0 \vee \tilde{\delta}_t(d, \bar{r}) = \Delta_t\}$  is finite, i.e.  $D^0$  consists of isolated points. Indeed, due to analyticity of  $\tilde{\delta}_t$ , it should be constant if the set  $D^0$  contains an accumulation point. It means that the function  $\hat{\delta}_t(d, \bar{r})$  is real analytic in the interval  $(0, d_{max})$  except of at most finitely many isolated points from the set  $D^0$ . It means that either  $\hat{\delta}_t(d, \bar{r}) = \tilde{\delta}_t(d, \bar{r})$  or  $\hat{\delta}_t(d, \bar{r}) = const$  on the complement of  $D^0$ . Therefore  $\partial_\delta \phi \partial_d^2 \hat{\delta}_t = \partial_\delta \phi \partial_d \hat{\delta}_t = 0$  on the set  $(0, d_{max}) \setminus D^0$ . By (24) we have  $\partial_d^2 V_t(d, \bar{r}) < 0$  for  $d \in (0, d_{max}) \setminus D^0$ . Since

$$\frac{\partial V_t}{\partial d} = \frac{\partial \phi}{\partial d} + \frac{\partial \phi}{\partial \delta} \frac{\partial \hat{\delta}_t}{\partial d}$$

we have

$$\frac{\partial V_t}{\partial d}(d, \bar{r}) = \frac{\partial \phi}{\partial d}(d, \bar{r}, \hat{\delta}_t(d, \bar{r})) \quad (26)$$

and so  $\partial_d V_t(d, r)$  is continuous and decreasing at points from the set  $D^0$ . It completes the proof of strict concavity of the function  $V_t(d, r)$  in  $d$ -variable and the proof of Proposition 1 follows. ♦

**Proof of Proposition 2.** It follows from (26) that

$$\frac{\partial^2 V_t}{\partial d^2}(d, r) = \frac{\partial^2 \phi}{\partial^2 d}(d, r, \hat{\delta}_t(d, r)) + \frac{\partial^2 \phi}{\partial d \partial \delta}(d, r, \hat{\delta}_t(d, r)) \frac{\partial \hat{\delta}_t}{\partial d}(d, r)$$

for any  $d > 0$  except of a set of isolated points. Differentiating the first order condition (23) with respect to  $d$  we obtain

$$\frac{\partial^2 V_t}{\partial d^2}(d, r) = \frac{\partial^2 \phi}{\partial^2 d}(d, r, \hat{\delta}_t(d, r)) - \frac{\partial^2 \phi}{\partial^2 \delta}(d, r, \hat{\delta}_t(d, r)) \left( \frac{\partial \hat{\delta}_t}{\partial d}(d, r) \right)^2.$$

Since  $\phi$  is concave in  $\delta$  variable we have

$$-d \frac{\partial^2 V_t}{\partial d^2}(d, r) \leq -d \frac{\partial^2 \phi}{\partial d^2}(d, r, \hat{\delta}_t(d, r)).$$

Thus

$$\begin{aligned} -d \frac{\partial^2 V_t}{\partial d^2}(d, r) &\leq -d \int_{R^2} \frac{\partial^2 V_{t+1}}{\partial d^2}(F_t, g(r, x)) \left( \frac{\partial F_t}{\partial d} \right)^2 f_\varrho(x, y) dx dy \\ &\leq C_{t+1} \int_{R^2} \frac{\partial V_{t+1}}{\partial d}(F_t, g(r, x)) \frac{d}{F_t} \left( \frac{\partial F_t}{\partial d} \right)^2 f_\varrho(x, y) dx dy. \end{aligned}$$

Since  $d = (F_t - \tau) / \partial_d F_t$  and  $\tau > 0$  we have

$$\begin{aligned} -d \frac{\partial^2 V_t}{\partial d^2}(d, r) &< C_{t+1} \int_{R^2} \frac{\partial V_{t+1}}{\partial d}(F_t, g(r, x)) \frac{\partial F_t}{\partial d} f_\varrho(x, y) dx dy \\ &= C_{t+1} \frac{\partial V_t}{\partial d}(d, r). \end{aligned}$$

Recall that  $\partial_d V_t > 0$ . Taking the supremum over the interval  $(0, \infty)$  we obtain  $C_t \leq C_{t+1}$  and the proof of Proposition 2 is complete. ♦

**Proof of Proposition 3.** Let us compute  $\partial_\delta \phi(d, r, 0)$ . Clearly, we have

$$\frac{\partial \phi}{\partial \delta}(d, r, 0) = \int_{R^2} \frac{\partial V_{t+1}}{\partial d}(F_t(d, r, 0, y), g(r, x)) \frac{\partial F_t}{\partial \delta}(d, r, 0, y) f_0(x, y) dx dy$$

Since  $F_t(d, r, 0, \cdot) = \frac{d}{1+\beta_t} e^{B(1)r - \ln A(1)} + \tau$  is independent of the  $y$  variable and  $\partial_\delta F_t(d, r, 0, y) = \frac{d}{1+\beta_t} \left( e^{\mu^s + \sigma^s y} - e^{B(1)r - \ln A(1)} \right)$  and  $\int_R e^{\sigma y} f_0(x, y) dy = \frac{1}{\sqrt{2\pi}} e^{\frac{\sigma^2}{2} \frac{x^2}{2}}$ ,

we obtain

$$\frac{\partial \phi}{\partial \delta}(d, r, 0) = \int_R \frac{\partial V_{t+1}}{\partial d}(F_t(d, r, 0, \cdot), g(r, x)) \frac{d}{1+\beta_t} \left( e^{\mu^s + \frac{\sigma^s}{2}} - e^{B(1)r - \ln A(1)} \right) dx.$$

As  $\partial_d V_{t+1} > 0$  we conclude that the condition (1) from Proposition 3 is equivalent to the statement

$$\frac{\partial \phi}{\partial \delta}(d, r, 0) \leq 0.$$

The strict concavity of  $\phi$  in the  $\delta$ -variable enables us to conclude that the condition (1) is equivalent to  $\hat{\delta}_t(d, r) = 0$  for all  $d > 0$  and the proof of Proposition 3 follows. ♦

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