Dynamic model of pension savings management with stochastic interest rates and stock returns, In: Mathematical and Statistical Methods for Actuarial Sciences and Finance Perna, C., Sibillo, M. (Eds.) 1st Edition., Springer Verlag, Berlin, Heidelberg, 2012, pp. 295-304, ISBN 978-88-470-2341-3

# Dynamic Model of Pension Savings Management with Stochastic Interest Rates and Stock Returns

Igor Melicherčík and Daniel Ševčovič

**Abstract** In this paper we recall and summarize results on a dynamic stochastic accumulation model for determining optimal decision between stock and bond investments during accumulation of pension savings. The model has been proposed and analyzed by the authors in [8]. We assume stock prices to be driven by a geometric Brownian motion whereas interest rates are modeled by means of a one factor interest rate model. It turns out that the optimal decision representing stock to bond proportion is a function of the duration of saving, the level of savings and the short rate. We furthermore summarize the results of testing the model on the fully funded second pillar of the Slovak pension system.

**Key words:** dynamic stochastic programming, utility function, Bellman equation

## 1 Introduction

The ongoing demographic crisis has motivated pension reforms across the world. One can observe a shift from public pay-as-you-go systems towards funded defined-contribution (DC) ones. The DC system is considered to be more resistant to the demographic change. On the other hand, the risk of asset returns during the accumulation phase is charged to members. A natural question is whether a future pensioner should invest savings to assets with low risk and low returns (bonds with low duration and money market instruments) or to assets with higher risk associated with higher expected returns (stocks). Conventional wisdom is that stock returns should outperform bond ones in the long term run. Consequently, young people should invest their savings to stocks. On the other hand, being close to the retirement age, it is too risky to invest the savings to stocks because of the high risk

Igor Melicherčík and Daniel Ševčovič
Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia, e-mail: {igor.melichercik, daniel.sevcovic}@fmph.uniba.sk

of fall in the asset value without a sufficient time to recovery. In [9, 11], Merton and Samuelson showed that when one considers a model with one-shot investment with maximizing the expected CRRA utility function of the final wealth, the stocks to bonds proportion is independent of the time to maturity and depends only on the savers risk aversion. However, if one considers a series of defined contributions throughout a lifespan a fall in the asset value early in life does not affect value of accumulated future contributions, while if it occurs close to retirement it affects all past accumulated contributions and returns on them, i.e. most of one's pension wealth. Therefore, in the case of successive contributions the investment decision should depend on the time to maturity of saving. A similar argument was used in [2] by Bodie et al. They concluded that pension saving becomes more conservative as retirement approaches. In [3] the authors investigated the stochastic dynamic accumulation model with stochastic wages and its application to optimal asset allocation for defined contribution pension plans. The dynamic accumulation with stochastic interest rates (following CIR process) with no contributions has been studied by Deelstra et al. [4] in which the authors were able to derive explicit formulae for optimal porfolio decisions. A model for a defined-contribution pension fund in continuous time with exponential utility was investigated in [1, 7]. In [5] Kilianová and the authors developed a simple dynamic stochastic model of pension fund management with regular yearly contributions. Future pensioner can choose from finitely many funds with different risk profiles. The bond investments were supposed to have independent in time and normally distributed returns. In the present paper we improve the simplified model proposed in [5]. We describe bond returns by means of one factor short rate model. Furthermore, instead of choosing from a finite number of funds, the decision variable is the weight of the portfolio invested to stocks.

## 2 The two factor dynamic stochastic accumulation model

Suppose that a future pensioner deposits once a year a $\tau$-part of his/her yearly salary $w_t$ to a pension fund with a $\delta$-part of assets in stocks and a $(1-\delta)$-part of assets in bonds where $\delta \in [0,1]$. Denote by $\gamma_t$, $t = 1,2,\dots T$, the accumulated sum at time $t$ where $T$ is the expected retirement time. Then the budget-constraint equations read as follows:

$$\gamma_{t+1} = \delta\gamma_t \exp(R^s(t,t+1)) + (1-\delta)\gamma_t \exp(R^b(t,t+1)) + w_{t+1}\tau \qquad (1)$$

for $t = 1,2,\dots,T-1$, where $\gamma_1 = w_1\tau$. $R^s(t,t+1)$ and $R^b(t,t+1)$ are the annual returns on stocks and bonds in the time interval $[t,t+1)$. When retiring, a pensioner will strive to maintain his/her living standards in the level of the last salary. ¿From this point of view, the saved sum $\gamma_T$ at the time of retirement $T$ is not precisely what a future pensioner cares about. For a given life expectancy, the ratio of the cumulative sum $\gamma_T$ and the yearly salary $w_T$ is of a practical importance. Using the quantity $d_t = \gamma_t/w_t$ one can reformulate the budget-constraint equation (1) as follows:

$$d_{t+1} = d_t \frac{\delta \exp(R^s(t,t+1)) + (1-\delta)\exp(R^b(t,t+1))}{1+\beta_t} + \tau \tag{2}$$

for $t = 1,2,\dots,T-1$, where $d_1 = \tau$ and $\beta_t$ denotes the wage growth: $w_{t+1} = w_t(1+\beta_t)$. We shall assume that the term structure of the wage growth $\beta_t, t = 1,...,T$, is known and can be externally estimated from a macroeconometric model. Notice that for a future pensioner it might be also reasonable to express her post retirement income as a percentage of the yearly salary $\gamma_T$. For this purpose assumptions concerning the annuization rate should be introduced. Moreover, in many countries (including Slovakia, where the model is tested) the annuitization is not compulsory immediately after reaching the retirement age. Therefore, the problem of optimal annuization time arises. This problem, however, can be treated separately and this is why we do not discuss this issue in the present paper. These problems are investigated by many authors. We only refer to the paper [10] among others.

The term structure development is driven by one factor short rate rate model:

$$\mathrm{d}r_t = \mu(r_t,t)\,\mathrm{d}t + \omega(r_t,t)\mathrm{d}Z_t, \tag{3}$$

where $r_t$ stands for a short rate and $Z_t$ is the Wiener process. Suppose that the bond part of the fund consists of 1-year zero coupon bonds. If $R^b(t,t+1)$ is the return on a one year maturing zero coupon bond at time $t$ then it can be expressed as a function of the short rate $r_t$, $R^b(t,t+1) = R_1(r_t,t)$. Using a discretization of the short rate process (3) we obtain $r_{t+1} = g(r_t,\Phi)$ where $\Phi \sim N(0,1)$ is a normally distributed random variable. We shall assume the stock prices $S_t$ are driven by the geometric Brownian motion. The annual stock return $R^s(t,t+1) = \ln(S_{t+1}/S_t)$ can be therefore expressed as: $R^s(t,t+1) = \mu^s + \sigma^s\Psi$ where $\mu^s$ and $\sigma^s$ are the mean value and volatility of annual stock returns in the time interval $[t,t+1)$, $\Psi \sim N(0,1)$ is a normally distributed random variable. The random variables $\Phi,\Psi$ are assumed to be correlated with correlation $\rho = \mathbb{E}(\Phi\Psi) \in (-1,1)$. Based on historical data, the correlation coefficient $\rho$ has typically negative values.

Suppose that each year the saver has the possibility to choose a level of stocks included in the portfolio $\delta_t(I_t)$, where $I_t$ denotes the information set consisting of the history of bond and stock returns $R^b(t',t'+1)$, $R^s(t',t'+1)$, and wage growths $\beta_{t'}$, $t' = 1,2,\dots,t-1$. We suppose that the forecasts of the wage growths $\beta_t$, $t = 1,2,\dots,T-1$ are deterministic, the stock returns $R^s(t,t+1)$ are assumed to be random, independent for different times $t = 1,2,\dots,T-1$, and the interest rates are driven by the Markov process (3). Then the only relevant information are the quantities $d_t$ and the short rate $r_t$. Hence $\delta_t(I_t) \equiv \delta_t(d_t,r_t)$. One can formulate a problem of dynamic stochastic programming:

$$\max_{\delta} \mathbb{E}(U(d_T)) \tag{4}$$

subject to the following recurrent budget constraints:

$$d_{t+1} = F_t(d_t,r_t,\delta_t(d_t,r_t),\Psi)\,,\ t = 1,2,\dots,T-1, \quad \text{where } d_1 = \tau, \tag{5}$$

$$F_t(d,r,\delta,y) = d\,\frac{\delta\exp[\mu_t^s + \sigma_t^s y] + (1-\delta)\exp[R_1(r,t)]}{1+\beta_t} + \tau \tag{6}$$

and the short rate process is driven by a discretization of (3):

$$r_{t+1} = g(r_t, \Phi)\,,\ t = 1,2,\dots,T-1\,, \tag{7}$$

with $r_1 = r_{init}$. In particular, a general form of the AR(1) process (7) includes various one-factor interest rate models like e.g. the Vasicek model or Cox-Ingersoll-Ross model (CIR). In our calculations the term structure is driven by the one factor CIR model, where equation (3) has the form

$$\mathrm{d}r_t = \kappa(\theta - r_t)\,\mathrm{d}t + \sigma^b |r_t|^{\frac{1}{2}}\,\mathrm{d}Z_t. \tag{8}$$

Here $Z_t$ stands for the Wiener process, $\theta > 0$ is the long term interest rate, $\kappa > 0$ is the rate of reversion and $\sigma^b > 0$ is the volatility of the process. In this case

$$g(r,x) = \theta + e^{-\kappa}(r-\theta) + \sigma^b |r|^{\frac{1}{2}} e^{-\kappa}\left((e^{2\kappa}-1)/2\kappa\right)^{\frac{1}{2}} x \tag{9}$$

and $R_1(r,t)$ is an affine function of the short rate $r$. In the dynamic stochastic optimization problem (4) the maximum is taken over all non-anticipative strategies $\delta = \delta_t(d_t, r_t)$. We assume the stock part of the portfolio is bounded by a given upper barrier function $\Delta_t : 0 \le \delta_t(d_t, r_t) \le \Delta_t$. The function $\Delta_t : \{1,...,T-1\} \mapsto [0,1]$ is subject to governmental regulations. In our modeling we shall use the constant relative risk aversion (CRRA) utility function $U(d) = -d^{1-a},\ d > 0$ where $a > 1$ is the constant coefficient of relative risk aversion. Let us denote by $V_t(d,r)$ saver's intermediate utility function at time $t$ defined as:

$$V_t(d,r) = \max_{0\le\delta\le\Delta_t} \mathbb{E}(U(d_T)|d_t = d, r_t = r)\,. \tag{10}$$

Then, by using the law of iterated expectations we obtain the Bellman equation

$$V_t(d,r) = \max_{0\le\delta\le\Delta_t} \mathbb{E}[V_{t+1}(F_t(d,r,\delta,\Psi), g(r,\Phi))] \tag{11}$$

for every $d,r > 0$ and $t = 1,2,\dots,T-1$. Using $V_T(d,r) = U(d)$ the optimal strategy can be calculated backwards. One can prove (see [8]) that there exists the unique argument of the maximum in (11) $\hat{\delta}_t = \hat{\delta}_t(d_t, r_t)$. An efficient numerical procedure how to solve the recurrent Bellman equation (11) and determine the value $\hat{\delta}_t(d,r)$ has been also discussed in [8].

*Remark 1.* At the end of this section, we shall discuss the dependence of the level of savings $d_t$ and the optimal stock to bond ratio $\hat{\delta}_t$ with respect to the contribution rate $\tau > 0$. Let us denote by $V_t(d,r;\tau)$ and $\hat{\delta}_t(d,r;\tau)$ the value function and the optimal stock to bond ratio corresponding to the contribution rate $\tau > 0$. One can prove the identity: $V_t(\lambda d, r; \lambda\tau) = \lambda^{1-a} V_t(d,r;\tau)$, for any constant $\lambda > 0$, provided that $U(d) = -d^{1-a}$. The statement is obvious for $t = T$ where $V_T(\lambda d, r; \lambda\tau) = U(\lambda d) =$

$\lambda^{1-a}V_T(d,r;\tau)$. As $F_t(\lambda d,r,\delta,y;\lambda\tau) = \lambda F_t(d,r,\delta,y;\tau)$ the statement easily follows from the backward mathematical induction for $t = T, T-1, ..., 2, 1$ with the optimal stock to bond ratio satisfying the relationship: $\hat{\delta}_t(\lambda d,r;\lambda\tau) = \hat{\delta}_t(d,r;\tau)$. As a consequence, by a forward mathematical induction, one can prove that the stochastic variable $d_t$ defined recursively $d_{t+1} = F_t(d_t,r_t,\hat{\delta}_t(d_t,r_t;\tau),\Psi;\tau)$ depends linearly on the contribution rate $\tau$ and corresponding decision $\hat{\delta}_t$ is invariant with respect to $\tau$.

## 3 Computational results

Since January 2005, pensions in Slovakia are operated by a three-pillar system: the mandatory non-funded $1^{st}$ pay-as-you-go pillar, the mandatory funded $2^{nd}$ pillar and the voluntary funded $3^{rd}$ pillar. The old-age contribution rates were set at 9% for $1^{st}$ and $2^{nd}$ pillars, i.e. $\tau = 0.09$. The savings in the second pillar are managed by pension asset administrators. Each pension administrator manages three funds: Growth Fund, Balanced Fund and Conservative fund, each of them with different limits for investment (see Tab. 1). At the same time instant savers may hold assets in one fund only. In the last 15 years preceding retirement, a saver may not hold assets in the Growth Fund and in the last 7 years all assets must be deposited in the Conservative Fund.

**Table 1** Governmental limits for investment for the pension funds

| Fund type | Stocks | Bonds and money market instruments |
|---|---|---|
| Growth Fund | up to 80% | at least 20% |
| Balanced Fund | up to 50% | at least 50% |
| Conservative Fund | no stocks | 100% |

**Table 2** Expected wage growths from 2007 ($t = 1$) to 2048 (t=40) in Slovakia. Source: [6]

| | $1 \le t \le 4$ | $5 \le t \le 9$ | $10 \le t \le 14$ | $15 \le t \le 19$ | $20 \le t \le 24$ | $25 \le t \le 29$ | $30 \le t \le 34$ | $35 \le t \le 40$ |
|---|---|---|---|---|---|---|---|---|
| $\beta_t$ | 7% | 7.1% | 6.4% | 5.9% | 5.6% | 5.2% | 4.9% | 4.5% |

Our model is applied to the $2^{nd}$ pillar. According to Slovak legislature the percentage of salary transferred each year to a pension fund is 9% ($\tau = 0.09$). We have assumed the period $T = 40$ of saving. The forecast for the expected wage growth $\beta_t$ in Slovakia has been taken from [6]. The term structure $\{\beta_t, t = 1, \dots, T\}$ from 2007 to 2048 is shown in Tab. 2. Stocks have been represented by the S&P500 Index. The stock returns were assumed to be normally distributed. As for the calibration, we chose the same time period (Jan 1996-June 2002) as in Kilianová et al. [5] with average return $\mu^s = 0.1028$ and standard deviation $\sigma^s = 0.169$. The model parameters describing the Slovakian term structure of the zero coupon bonds have been adopted from the paper by Ševčovič and Urbánová Csajková [12]. We assumed the long term interest rate $\theta = 0.029$, $\sigma^b = 0.15$, $\kappa = 1$ and $\lambda = 0$. The correlation between stock and bond returns was set to $\rho = -0.1151$ (the same as in [5]).

In Fig. 1 we present a typical result of our analysis with the risk aversion coefficient $a = 9$ and the time $T = 40$ years of the pension savings. It contains optimal decisions (without governmental regulations) $\hat{\delta}_t(d,r)$ with fixed short rate $r = 4\%$. One can see that pension saving becomes more conservative as the retirement approaches. The reason for such a behavior is that more contributions are accumulated and higher part of the future pension is affected by asset returns. The dependence of the decision on the level of savings gradually decreases. This is due to the fact that less amount of forthcoming contributions is expected. In the case of no future contributions, a decision based on a CRRA utility function is independent of the level of savings (see e.g. Samuelson [11]).

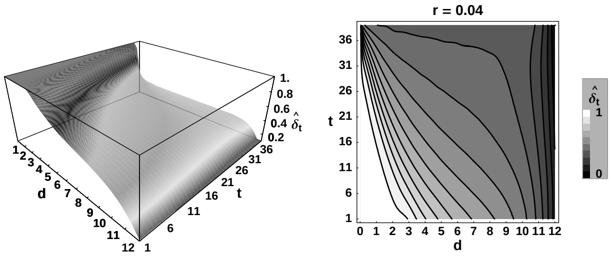


**Fig. 1** 3D and contour plots of the function $\hat{\delta}_t(d,r)$ for $r = 4\%$ with no limitations. Source: [8]

One can see the impact of governmental regulations in Fig. 2 and Tab. 3. The mean wealth $\mathbb{E}(d_t)$ and standard deviations were calculated using 10 000 simulations with the risk aversion coefficient $a = 9$. It is clear that the average wealth achieved is higher without governmental regulations. The regulations reduce standard deviations of the wealth achieved. The values of the average final wealth and standard deviations for various risk aversion parameters $a$ can be found in Tab. 3. One can observe that the higher the risk aversion, the lower the expected wealth associated with lower risk (standard deviation).

Even before the financial crisis, pension asset managers used very conservative investment strategies. In March 2007 growth funds contained only up to 20% of stock investments. In this case the difference between the pension funds was insignificant. In our calculations we have supposed that this proportion will be linearly increased up to 50% in the next 3 years. After that the proportion of the stock investment in the balanced fund will be 30%. The development of the average level of savings and average proportion of the stock investment with standard deviations for such a cautious investment strategies can be found in Fig. 3 and Tab. 3. In order to demonstrate that these strategies are still too conservative, we have considered very high risk aversion coefficient $a = 12$. One can observe that even in this case, it

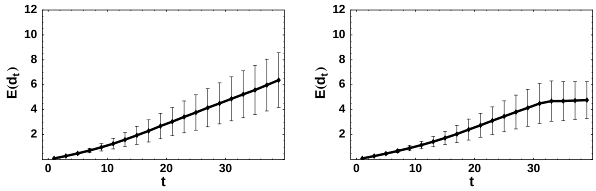


**Fig. 2** The average value $\mathbb{E}(d_t)$ for the risk aversion parameter $a = 9$. No governmental limitations on the optimal choice of $\hat{\delta}_t$ (left); governmental limitations imposed (right). The error bars show the standard deviation of $d_t$. Source: [8]

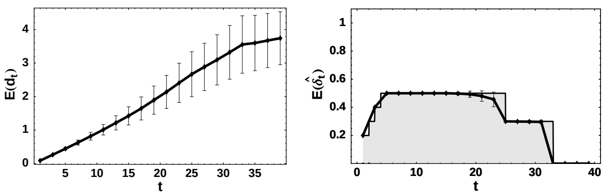


**Fig. 3** The average values $\mathbb{E}(d_t)$ (left) and $\mathbb{E}(\hat{\delta}_t)$ (right). Error bars depicts standard deviations for the cautious investment strategy. The risk aversion coefficient $a = 12$. Source: [8]

**Table 3** The average value $\mathbb{E}(d_T)$ of $d_T$ and its standard deviation $\sigma(d_T)$ for various risk aversion parameters $a$. Source: [8]

| $a$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| | Governmental limitations | | | | | | | | | |
| $\mathbb{E}(d_T)$ | 5.264 | 5.261 | 5.247 | 5.203 | 5.109 | 4.966 | 4.791 | 4.6 | 4.427 | 4.275 |
| $\sigma(d_T)$ | 2.033 | 2.026 | 1.997 | 1.928 | 1.809 | 1.644 | 1.462 | 1.288 | 1.143 | 1.023 |
| | No limits | | | | | | | | | |
| $\mathbb{E}(d_T)$ | 9.871 | 9.574 | 9.04 | 8.402 | 7.738 | 7.112 | 6.561 | 6.089 | 5.697 | 5.375 |
| $\sigma(d_T)$ | 3.075 | 3.024 | 3.002 | 2.912 | 2.736 | 2.496 | 2.233 | 1.968 | 1.718 | 1.505 |
| | Cautious investment | | | | | | | | | |
| $\mathbb{E}(d_T)$ | 3.818 | 3.818 | 3.818 | 3.818 | 3.818 | 3.817 | 3.814 | 3.806 | 3.793 | 3.774 |
| $\sigma(d_T)$ | 0.848 | 0.848 | 0.848 | 0.848 | 0.848 | 0.846 | 0.839 | 0.825 | 0.805 | 0.78 |

is optimal to stay in the growth and balanced funds as long as possible (according to governmental regulations). If we compare the cautious strategies with the ones that undergo just governmental regulations, the level of savings is significantly lower (see Fig. 2 (right) and Tab. 3). Therefore, the stock investments should be soon increased to higher levels.

## 4 Conclusions

We have applied a dynamic model of saving with incremental contributions to the funded pillar of the Slovak pension system. Stock prices were assumed to be driven by the geometric Brownian motion. Interest rates were modeled by one factor short rate model. The optimal decision strategy is dynamic and depends on the duration of saving $t$, the level of savings $d_t$ and the short rate $r_t$. In accord with [2] the results confirmed that saving becomes more conservative close to the retirement time. This is a consequence of gradual saving. As the retirement approaches, the model resembles the one with one-shot investment ([9, 11]) and therefore the decision becomes less sensitive to the level of savings. We have used a family of CRRA utility functions with a parameter representing individual risk preferences. In accord with intuition, the higher the risk aversion, the lower the expected level of savings associated with lower standard deviations. Not surprisingly, the strategies respecting the governmental regulations have lower expected level of savings associated with lower risk (standard deviation). Cautious strategies of pension asset managers in Slovakia imply that savers stay in the most risky funds as long as possible (respecting the governmental regulations). Such strategies could lead to insufficient pensions.

**Acknowledgements** This work has been supported by ERDF-017/2009/4.1/OPVaV-CESIUK and VEGA 1/0381/09 projects.

## References

1. Battochio, P. & Menoncin, F. Optimal pension management in a stochastic framework. Insurance: Mathematics & Economics **34**, 79–95 (2004).
2. Bodie, Z., Merton, R. & Samuelson, W.F. Labor supply flexibility and portfolio choice in a life-cycle model. Journal of Economic Dynamics & Control **16**, 427–449 (1992).
3. Cairns, A., Blake, D. & Dowd, K. Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. Journal of Economic Dynamics & Control **30**, 843-877 (2006).
4. Deelstra, G., Grasselli, M., Koehl, P. Optimal investment strategies in a CIR framework. Journal of Applied Probability **37**, 936-946 (2000).
5. Kilianová, S., Melicherčík, I. & Ševčovič, D. Dynamic accumulation model for the second pillar of the Slovak pension system. Czech Journal for Economics and Finance **11-12**, 506–521 (2006).
6. Kvetan, V., Mlýnek, M., Páleník, V. & Radvanský, M. Starnutie, zdravotný stav a determinanty výdavkov na zdravie v podmienkach Slovenska. Research studies of Institute of Economics SAV, Bratislava, ISBN: 978-80-7144-160-1, (2007).
7. Ma, Q.-P. On ”optimal pension management in a stochastic framework” with exponential utility. Insurance: Mathematics & Economics **49**, 61–69 (2011).
8. Melicherčík, I. & Ševčovič D. Dynamic stochastic accumulation model with application to pension savings management. Yugoslav Journal of Operations Research **20**, 1–24 (2010).
9. Merton, R.C. & Samuelson, P.A. Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods. Journal of Financial Economics **1**, 67–94 (1974).
10. Milevsky, M. Optimal asset allocation towards the end of the life cycle: to annuitize or not to annuitize? Journal of Risk and Insurance **65**, 401–426 (1998).
11. Samuelson, P.A. Lifetime portfolio selection by dynamic stochastic programming. The Review of Economics and Statistics **51**, 239–246 (1969).
12. Ševčovič, D. & Urbánová Csajková A. On a two-phase minmax method for parameter estimation of the Cox, Ingersoll, and Ross interest rate model. Central European Journal of Operational Research **13**, 169–188 (2005).