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Riccati Transformation Method for Solving Constrained Dynamic Stochastic Optimal Allocation Problem

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Abstract

In this paper we present our recent results on application of the Riccati transformation for solving the evolutionary Hamilton-Jacobi-Bellman equation arising from the stochastic dynamic optimal allocation problem. It turns out that the fully nonlinear Hamilton-Jacobi-Bellman equation governing evolution of the value function can be transformed into a quasi-linear parabolic equation. Its diffusion function is obtained as a value function of certain parametric convex optimization problem. A solution is then constructed by means of an implicit iterative finite volume numerical approximation scheme. As an application we present results of computing optimal strategies for a portfolio investment problem.

Key words: Hamilton–Jacobi–Bellman equation, Riccati transformation, quasi-linear parabolic equation, finite volume approximation scheme
MSC 2000: 34E05 70H20 91B70 90C15 91B16

1 Introduction

The goal of this paper is to investigate a novel method based on the Riccati transformation for solving a time dependent Hamilton-Jacobi-Bellman equation arising from a stochastic dynamic optimal allocation problem on a finite time horizon. Our motivation arises from a dynamic stochastic optimization problem in which the purpose is to maximize the conditional expected value

$$\max_{\boldsymbol{\theta}|_{[0,T)}} \mathbb{E}\left[U(X_T^{\boldsymbol{\theta}}) \,\middle|\, X_0^{\boldsymbol{\theta}} = x_0 \right], \tag{1}$$

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of the terminal utility $U(X_T^{\boldsymbol{\theta}})$ of a portfolio. Here $\{X_t^{\boldsymbol{\theta}}\}$ is an Itō's stochastic process on the finite time horizon $[0,T]$, $U:\mathbb{R}\to\mathbb{R}$ is a given terminal utility function and x_0 a given initial state condition of $\{X_t^{\boldsymbol{\theta}}\}$ at $t=0$. The function $\boldsymbol{\theta}:\mathbb{R}\times[0,T)\to\mathbb{R}^n$ mapping $(x,t)\mapsto\boldsymbol{\theta}(x,t)$ represents an unknown control function governing the underlying stochastic process $\{X_t^{\boldsymbol{\theta}}\}_{t\geq 0}$. Here $\boldsymbol{\theta}|_{[t,T)}$ for $0\leq t<T$ denotes the restriction of the control function $\boldsymbol{\theta}$ to the time interval $[t,T)$. We assume that $X_t^{\boldsymbol{\theta}}$ is driven by the stochastic differential equation

$$\mathrm{d}X_t^{\boldsymbol{\theta}} = \left\{\varepsilon e^{-X_t} + r + \mu(\boldsymbol{\theta}) - \sigma(\boldsymbol{\theta})^2/2\right\}\mathrm{d}t + \sigma(\boldsymbol{\theta})\mathrm{d}W_t, \tag{2}$$

where W_t denotes the standard Brownian motion and the functions $\mu(\boldsymbol{\theta})$ and $\sigma(\boldsymbol{\theta})$ are the drift and volatility functions depending on the control function $\boldsymbol{\theta}$. The parameter $\varepsilon\geq 0$ represents a constant inflow rate of property to the system whereas $r\geq 0$ is the interest rate. Throughout the paper we shall assume that the control parameter $\boldsymbol{\theta}\in\mathcal{S}^n$ belongs to the compact simplex

$$\mathcal{S}^n = \{\boldsymbol{\theta}\in\mathbb{R}^n \mid \boldsymbol{\theta}\geq\mathbf{0}, \mathbf{1}^T\boldsymbol{\theta}=1\}\subset\mathbb{R}^n, \tag{3}$$

where $\mathbf{1}=(1,\cdots,1)^T\in\mathbb{R}^n$. It should be noted that the process $\{X_t^{\boldsymbol{\theta}}\}$ is a logarithmic transformation of a stochastic process $\{Y_t^{\tilde{\boldsymbol{\theta}}}\}_{t\geq 0}$ driven by the SDE:

$$\mathrm{d}Y_t^{\tilde{\boldsymbol{\theta}}} = \left\{\varepsilon + [r+\mu(\tilde{\boldsymbol{\theta}})]Y_t^{\tilde{\boldsymbol{\theta}}})\right\}\mathrm{d}t + \sigma(\tilde{\boldsymbol{\theta}})Y_t^{\tilde{\boldsymbol{\theta}}}\mathrm{d}W_t, \tag{4}$$

where $\tilde{\boldsymbol{\theta}}(y,t)=\boldsymbol{\theta}(x,t)$ with $x=\ln y$.

As a typical example leading to the stochastic dynamic optimization problem (1) in which the underlying stochastic process satisfies SDE (2) one can consider a problem of dynamic portfolio optimization in which the assets are labeled as $i=1,\cdots,n$, and associated with the price processes $\{Y_t^i\}_{t\geq 0}$, each of them following a geometric Brownian motion

$$\frac{\mathrm{d}Y_t^i}{Y_t^i} = \mu_i\mathrm{d}t + \sum_{j=1}^{n}\bar{\sigma}_{ij}\mathrm{d}W_t^j,$$

(cf. Merton [12, 13], Browne [4], Bielecki and Pliska [3]. The value of a portfolio with weights $\tilde{\boldsymbol{\theta}}=\tilde{\boldsymbol{\theta}}(y,t)$ is denoted by $Y_t^{\tilde{\boldsymbol{\theta}}}$. We have $\mu(\boldsymbol{\theta})=\boldsymbol{\mu}^T\boldsymbol{\theta}$ and $\sigma(\boldsymbol{\theta})^2=\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta}$ with $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_n)^T$ and $\boldsymbol{\Sigma}$ is a positive definite symmetric covariance matrix, $\boldsymbol{\Sigma}=\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{\Sigma}}^T$ where $\bar{\boldsymbol{\Sigma}}=(\bar{\sigma}_{ij})$. It can be shown that $\{Y_t^{\tilde{\boldsymbol{\theta}}}\}_{t\geq 0}$ satisfies (4) with $\varepsilon=r=0$. The assumption $\boldsymbol{\theta}\in\mathcal{S}^n$ corresponds to the situation in which borrowing of assets is not allowed, i.e. $\theta_i\geq 0$ and $\sum_{i=1}^n\theta_i=1$. A function $U(x)$ represents a given terminal utility function representing investor's risk preferences.

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2 Hamilton-Jacobi-Bellman Equation and Method of Riccati transformation

It is known from the theory of stochastic dynamic programming that the so-called value function

$$V(x,t) := \sup_{\boldsymbol{\theta}|_{[t,T)}} \mathbb{E}\left[U(X_T^{\boldsymbol{\theta}}) | X_t^{\boldsymbol{\theta}} = x\right] \tag{5}$$

subject to the terminal condition $V(x,T) := U(x)$ can be used for solving the stochastic dynamic optimization problem (1) (cf. Bertsekas [2]). If the process $X_t^{\boldsymbol{\theta}}$ is driven by (2), then the value function $V = V(x,t)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\partial_t V + \max_{\boldsymbol{\theta}\in\mathcal{S}^n} \left\{ \left(\varepsilon e^{-x} + r + \mu(\boldsymbol{\theta}) - \frac{1}{2}\sigma(\boldsymbol{\theta})^2\right) \partial_x V + \frac{1}{2}\sigma(\boldsymbol{\theta})^2 \partial_x^2 V \right\} = 0\,, \tag{6}$$

for all $x \in \mathbb{R},\ t \in [0,T)$ subject to the terminal condition $V(x,T) := U(x)$ (see e.g. Macová and Ševčovič [11] or Ishimura and Ševčovič [6]).

Following the methodology of the Riccati transformation studied by Ishimura *et al.* [1, 5, 7] and further analyzed by Ishimura and Ševčovič [6], we introduce the following Riccati like transformation:

$$\varphi(x,t) = 1 - \frac{\partial_x^2 V(x,t)}{\partial_x V(x,t)}. \tag{7}$$

According to [8, Theorem 3.2], the transformed function φ is a solution to a Cauchy problem for the following quasi-linear parabolic equation

$$\begin{aligned} &\partial_t\varphi + \partial_x^2\alpha(\varphi) + \partial_x[(\varepsilon e^{-x} + r)\varphi + (1-\varphi)\alpha(\varphi)] = 0, \quad x\in\mathbb{R}, t\in[0,T), \\ &\varphi(x,T) = 1 - U''(x)/U'(x), \quad x \in \mathbb{R}, \end{aligned} \tag{8}$$

where the diffusion function $\alpha(\varphi)$ is obtained as the value function of the parametric non-linear constrained optimization problem.

$$\alpha(\varphi) = \min_{\boldsymbol{\theta}\in\mathcal{S}^n} \{-\mu(\boldsymbol{\theta}) + \frac{\varphi}{2}\sigma(\boldsymbol{\theta})^2\}\,. \tag{9}$$

In our application the problem (9) is a convex quadratic programming problem with $\mu(\boldsymbol{\theta}) := \boldsymbol{\mu}^T\boldsymbol{\theta}$ and $\sigma(\boldsymbol{\theta})^2 := \boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta}$ where $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\boldsymbol{\Sigma}$ is a positive definite $n\times n$ matrix.

Unfortunately, the value function $\alpha(\varphi)$ need not be sufficiently smooth. Indeed, according to [8, Theorem 4.1] $\alpha \in C^{1,1}(\mathbb{R}^+)$, i.e. its derivative is Lipschitz continuous only. Moreover, with regard to [8] there are concrete market data examples of German DAX 30 stock index for which the value function can have a finite number of discontinuities in the second derivative of α.

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Applying the methodology of Schauder estimates we were able to prove the following result on existence and smoothness of classical solutions to (8) belonging to the parabolic Hölder spaces $H^{2+\lambda,1+\lambda/2}(\mathbb{R}\times[0,T])$ for some $0<\lambda<1$. The detailed proof can be found in the recent paper [8] by the authors.

Theorem 2.1 *Suppose that* $\mathbf{\Sigma}$ *is positive definite,* $\boldsymbol{\mu}\in\mathbb{R}^n,\varepsilon,r\geq 0$*, and the optimal value function* $\alpha(\varphi)$ *is given by (9). Assume that the terminal condition* $\varphi(x,T)=1-U''(x)/U'(x)$*,* $x\in\mathbb{R}$*, is positive and uniformly bounded for* $x\in\mathbb{R}$ *and belongs to the Hölder space* $H^{2+\lambda}(\mathbb{R})$ *for some* $0<\lambda<1/2$*. Then there exists a unique classical solution* $\varphi(x,t)$ *to the backward quasi-linear parabolic equation (8) satisfying the terminal condition* $\varphi(x,T)$*. The function* $t\mapsto\partial_t\varphi(x,t)$ *is* $\lambda/2$*-Hölder continuous for all* $x\in\mathbb{R}$ *whereas* $x\mapsto\partial_x\varphi(x,t)$ *is Lipschitz continuous for all* $t\in[0,T]$*. Moreover,* $\alpha(\varphi(.,.))\in H^{2+\lambda,1+\lambda/2}(\mathbb{R}\times[0,T])$*.*

3 Application to portfolio optimization

In [8], the authors proposed an iterative numerical approximation scheme for solving the Cauchy problem for the quasi-linear parabolic equation (8). We followed the method of a finite volume approximation scheme (cf. LeVeque [10]) combined with a nonlinear equation iterative solver proposed by Mikula and Kútik in [9]. The scheme has been tested with semi-explicit traveling wave solutions (see [8, Sections 6,7]) and it turned out that the scheme is of the second experimental order of convergence. We furthermore applied the scheme to a practical example in which our goal was to optimize a portfolio consisting of $n=30$ assets forming the German DAX 30 Index. The regular contribution to the portfolio was set to $\varepsilon=1$ and $r=0$. As far as the utility function is concerned, we considered the constant absolute risk aversion (CARA) utility function of the form $U(x)=-\frac{1}{a-1}\exp(-(a-1)x)$ with a coefficient of the absolute risk aversion $a=9$. In terms of the transformed variable $x=\ln y$ the CARA utility function corresponds to the constant relative risk aversion (CRRA) function $\widetilde{U}(y)=-\frac{1}{a-1}y^{-a+1}$. We considered the finite time horizon $T=10$.

Using the finite volume approximation scheme we constructed a numerical solution $\varphi(x,t)$ to the quasilinear parabolic equation (8). Then, by solving the parametric quadratic programming problem (9) for $\varphi=\varphi(x,t)$ we found optimal response strategies $\boldsymbol{\theta}$ as a function of the logarithmic level of property x and time t. Results of numerical calculation are shown at Fig. 1.

It turned out shows that there are only a few relevant assets out of the set of thirty assets entering the DAX 30 Index. The figure reveals the highest portion of Merck stocks for the early period of saving and for low account values y. It is indeed reasonable to invest in an asset with the highest expected return, although with the highest volatility, when the account value is low, in early times of saving. Evident fast decrement of the Merck company weight can be observed for increasing account value. It should be noted that Fresenius Medical company has the lowest volatility out of the considered five assets (and

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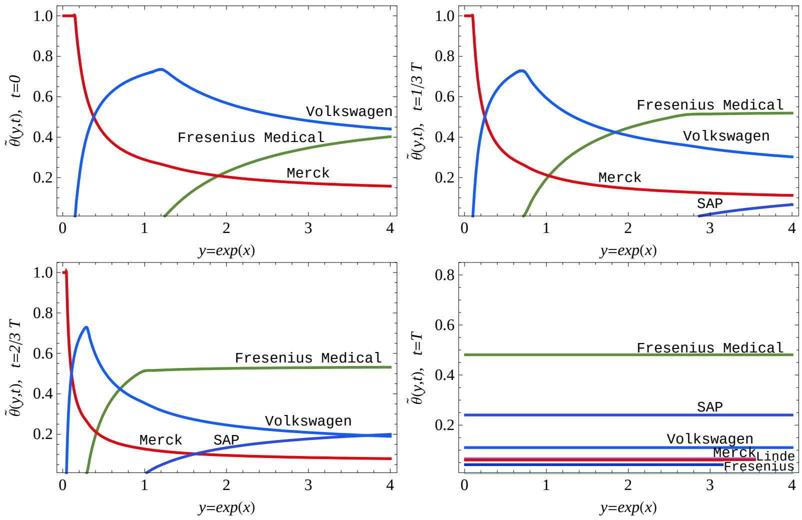

Figure 1: Nonzero components $\tilde{\boldsymbol{\theta}}_i, i \in \{1, \cdots, n\}$ of tptimal response strategy vector $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}(y,t) = \boldsymbol{\theta}(\ln y, t)$ for the DAX 30 index portfolio optimization, for time instances $t = 0$, $t = T/3$, $t = 2T/3$ and $t = T$ where $T = 10$ Source: [8].

third lowest out of all thirty assets) and third best mean return, which is reflected in its major representation in the portfolio.

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