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Enhanced semi-definite relaxation method with application to optimal anisotropy function construction

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Abstract

Non-convex quadratic programs find applications in many areas and they are known to be NP hard in general. The standard approach is to relax the non-convex quadratic optimization problem via a convex conic problem, for which effective algorithms are already known. The fundamental question in the theory of semi-definite relaxation is to give conditions under which the relaxation is exact. In this case it is often said that the problem possesses a hidden convexity property and this problem is closely related to finding strong duality conditions for non-convex quadratic problems. The conditions are already known for quadratic problems with one or two (in the complex case) quadratic constraints and the results are based on the so-called S-lemma. We propose an exact semi-definite relaxation method for quadratic problems with non-convex objective and subject to convex quadratic, linear and eventually semi-definite constraints, based on enhancing the linear equality constraints by quadratic ones. Sufficient conditions that guarantee the exact relaxation are given and the technique is applied to the problem of optimal anisotropy function construction as a minimizer for the anisoperimetric ratio for a given Jordan curve.

KEYWORDS

Enhanced semi-definite relaxation method, semi-definite programming, inverse Wulff problem.

1. INTRODUCTION

In this paper we present of survey of our recent results on the enhanced semi-definite relaxation method for solving the following wide class of nonlinear optimization problems:

$$\begin{aligned} \min\ & x^T P_0 x + 2q_0^T x + r_0 \\ \text{s.t.}\ & x^T P_l x + 2q_l^T x + r_l \le 0, \quad l = 1,\cdots,d, \\ & Ax = b, \\ & H_0 + \sum_{j=1}^{n} x_j H_j \succeq 0, \end{aligned} \tag{1}$$

where $x \in R^n$ is the variable. The input data of the problem: P_0, P_l are $n \times n$ real symmetric matrices, $q_0, q_l \in R^n$, $r_0, r_l \in R$, A is an $m \times n$ real matrix of the full rank, $b \in R^m$ and $H_0, H_1, \cdots, H_n$ are $k \times k$ complex Hermitian matrices. The last constraint $H_0 + \sum_{j=1}^{n} x_j H_j \succeq 0$ is the so-called linear matrix inequality (LMI) and it means that the matrix on the left-hand side is positive semi-definite. As the matrix P_0 need not be positive semi-definite the objective function in (1) is need not be necessarily convex, in general. The aim of this paper is to review recent advances on how to solve general non-convex optimization problems of the form (1) by means of the so-called enhanced semi-definite relaxation method recently proposed and investigated by Ševčovič and Trnovská (2015a, 2015b).

The paper is organized as follows: in the next section, we recall the so-called enhanced semi-definite relaxation method. We also provide sufficient conditions made on matrices P_0 and A guaranteeing zero duality gap and, as a consequence, tightness of relaxation. Furthermore, we present a simple example illustrating the capability of the enhanced semi-definite relaxation in comparison to usual relaxations. Finally, in section 3 we apply this method to the problem of construction of the optimal anisotropy function. The anisotropy function σ describing the so-called Finsler metric in the plane occurs in various models arising from applications. It particular, it enters the anisotropic Ginzburg-Landau free energy and the nonlinear parabolic Allen-Cahn equation with a diffusion coefficient depending on σ (cf. Belletini and Paolini (1996) and references therein).

2. ENHANCED SEMI-DEFINITE RELAXATION METHOD

Notice that for the case there are neither linear constraints $A=0$ nor LMI constraints and just one quadratic constraint ($d=1$) the method of semi-definite relaxation of (1) is discussed in a detail in Boyd and Vanderberghe (2004). In such a particular case, the proof of tightness of relaxation yielding the zero duality gap is based on the S-lemma. We refer the reader to Boyd and Vanderberghe (2004), Bao *et al.* (2011), Shor (1987) for an overview of semi-definite relaxation techniques for solving various classes of non-convex quadratic optimization problems. In what follows, we present a method for solving non-convex optimization problems of the form (1) recently studied by Ševčovič and Trnovská (2015a, 2015b). The method is based on enhancing the usual semi-definite relaxation methodology by enhancing linear equality constraints $Ax=b$ the quadratic-linear constraint $Axx^T=bx^T$. Since this is a dependent constraint, it should be obvious that problem (1) is equivalent to the following augmented problem:

$$\begin{aligned} \hat{p}_1 = \min\ & x^T P_0 x + 2q_0^T x + r_0 \\ \text{s. t.}\ & x^T P_l x + 2q_l^T x + r_l \le 0, \quad l=1,\cdots,d, \\ & Ax=b, \quad Axx^T = bx^T, \\ & H_0 + \sum_{j=1}^n x_j H_j \succeq 0. \end{aligned} \tag{2}$$

A formal way of construction semi-definite relaxation to (2) is simple and it is based on the identity $x^T P_l x = \mathrm{tr}(x^T P_l x) = \mathrm{tr}(P_l xx^T)$ and, on relaxation of the non-convex quadratic constraint $X=xx^T$ by the convex linear matrix inequality $X \succeq xx^T$. The enhanced semi-definite relaxation of problem (1) is, in fact, semi-definite relaxation of the augmented problem (2) and it has the form:

$$\begin{aligned} \hat{p}_2 = \min\ & \mathrm{tr}(P_0 X) + 2q_0^T x + r_0 \\ \text{s. t.}\ & \mathrm{tr}(P_l X) + 2q_l^T x + r_l \le 0, \quad l=1,\cdots,d, \\ & Ax=b, \quad AX = bx^T, \ X \succeq xx^T, \\ & H_0 + \sum_{j=1}^n x_j H_j \succeq 0. \end{aligned} \tag{3}$$

A systematic way of deriving the semi-definite relaxation of (2) is based on construction of the second Lagrangian dual problem to (3). Notice that the LMI constraint

$$H_0 + \sum_{j=1}^n x_j H_j \succeq 0$$

is preserved in (3) and it has no impact on enhanced relaxation of the problem. In the next Theorem we summarize results obtained in Proposition 1 and Theorem 1 from Ševčovič and Trnovská (2015a).

Theorem 1. *Suppose that problem (2) is feasible. Let* $\hat{p}_1$ *be the optimal value of (2) and* $\hat{p}_2$ *be the optimal value of problem (3) obtained by the enhanced semi-definite relaxation method. Then* $\hat{p}_1 \ge \hat{p}_2$. *If we furthermore assume* $P_l \succeq 0$ *for* $l=1,\cdots,d$, *and there exists a real* $m \times n$ *matrix* V *satisfying such that* $P_0 + \frac{1}{2}(V^T A + A^T V) \succeq 0$ *then we have* $\hat{p}_1 = \hat{p}_2$. *Hence there is no duality gap between (2) and (3) and the enhanced semi-definite relaxation is tight. Moreover, if* $(\tilde{x}, \tilde{X})$ *is an optimal solution to (3) then* $\tilde{x}$ *is the optimal solution to (2).*

2.1 A simple example

The aforementioned sufficient condition $P_0 + \frac{1}{2}(V^T A + A^T V) \succeq 0$ for some matrix V is more general then the classical Finsler condition $P_0 + \rho A^T A \succeq 0$ for some $\rho \ge 0$ which is sufficient (but not necessary) for tightness of enhanced semi-definite relaxation. An example of a matrix V which is not proportional to A was presented in Ševčovič and Trnovská (2015a). In this section we generalize this example and show importance of the condition for tightness of the enhanced semi-definite relaxation. Moreover, we shall demonstrate the importance of the enhanced semi-definite relaxation method in comparison to a usual semi-definite relaxation based on the second Lagrangian dual of (1) without consideration of the augmented condition $Axx^T = bx^T$ in (2). Let us consider the following simple example:

$$\begin{array}{ll} \hat{p}_1 = \min & x^T P_0 x \\ \text{s.t.} & Ax = 0 \end{array} \qquad \text{where } P_0 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}. \tag{3}$$

Clearly, the matrix P_0 is not positive semi-definite and $P_0 + \frac{1}{2}(V^T A + A^T V) = diag(1,0,1) \succeq 0$ for the matrix $V = \begin{pmatrix} 0 & 2 & 0 \end{pmatrix}$. This solution can be obtained by solving the following LMI constrained quadratic problem:

$$\begin{array}{ll} \min & \frac{1}{2} tr(V^T V) \\ \text{s.t.} & P_0 + \frac{1}{2}(V^T A + A^T V) \succeq 0. \end{array}$$

It should be noted that there is no $\rho \geq 0$ satisfying the Finsler condition: $P_0 + \rho A^T A \succeq 0$. We have $x^T P_0 x = x_1^2 - 2x_1 x_2 - 2x_2 x_3 + x_3^2 = 2x_1^2$ for any x such that $Ax = 0$. So the optimal value of (2) is $\hat{p}_1 = 0$.

Now, let us examine the enhanced semi-definite relaxation of problem (3), i.e. the second Lagrangian dual of (3) augmented by the condition $Axx^T = 0$. It has in the following form:

$$\begin{array}{l} \hat{p}_2 = \min \ tr(P_0 X) \\ \quad \text{s.t.} \ \ Ax = 0, \ AX = 0, \ X \succeq xx^T. \end{array}$$

As $tr(P_0 X) = X_{11} - 2X_{12} - 2X_{23} + X_{33} = X_{11} + X_{33} \geq 0$ for any matrix X satisfying $AX = 0$, and, in particular, $X_{12} + X_{23} = 0$ and $X \succeq xx^T \succeq 0$ we conclude that $\hat{p}_1 = \hat{p}_2 = 0$. Hence the enhanced semi-definite relaxation of (1) is tight, i.e. there is no duality gap. This is, of course, in accordance with our Theorem 1.

On the other hand, if we consider just the second Lagrangian dual to (3) without augmenting (3) with constraint $AX = 0$, we obtain the following problem:

$$\begin{array}{l} p_2 = \min \ tr(P_0 X) \\ \quad \text{s.t} \ \ Ax = 0, \ X \succeq xx^T, \end{array}$$

yielding unbounded solution because

$$X = \begin{pmatrix} 1 & a & 0 \\ a & a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $x = 0$ is its feasible solution for any a, and $tr(P_0 X) = X_{11} - 2X_{12} - 2X_{23} + X_{33} = 1 - 2a \to -\infty$ for $a \to \infty$ and so $p_2 = -\infty < \hat{p}_1 = 0$. It means that, without application of the enhanced semi-definite relaxation method, there exists a nontrivial duality gap.

3. APPLICATION OF ENHANCED SEMI-DEFINITE RELAXATION METHOD TO OPTIMAL ANISOTROPY FUNCTION CONSTRUCTION

In various applications, e.g. material science, Finsler geometry, image processing, etc., an important role is played by the so-called Finsler anisotropy function $\sigma(v) \geq 0$ which is a 2π - periodic real valued non-negative function depending on the tangent angle $v \in R$ of a curvilinear boundary Γ enclosing a two dimensional connected area. For instance, in the Finsler relative anisotropic geometry the lengths of curves depend on their tangent angle $v \in R$. The total anisotropic interface energy $L_\sigma(\Gamma) = \int_\Gamma \sigma(v) ds$ of a closed curve $\Gamma \subset R^2$ depends on the anisotropy function σ and, in material science it measures surface tension energy of the phase interface Γ. Furthermore, the anisotropy function is closely related to the fundamental notion describing the Finsler geometry in the plane and it is characterized by the so-called Wulff shape

$$W_\sigma = \bigcap_{v \in [0,2\pi]} \left\{ x \,|\, -x^T n \leq \sigma(v) \right\},$$

where $n = (-\sin v, \cos v)^T$ is the unit inward vector. Since the Wulff shape is the intersection of hyper-planes it must be a convex set. Furthermore, the curvature of the boundary ∂W_σ is given by $\kappa = [\sigma(v) + \sigma''(v)]^{-1}$ and so we have that the anisotropy function must satisfy the condition $\sigma(v) + \sigma''(v) \geq 0$ for any $v \in \mathbb{R}$.

The classical isoperimetric inequality $L(\Gamma)^2/(4\pi A(\Gamma)) \geq 1$ is a relation between the length $L(\Gamma)$ and the enclosed area $A(\Gamma)$ by a given Jordan curve Γ. The equality is attained for circles only. It is related to the Dido problem and it was first proved by Pappus of Alexandria (cf. Ver Eecke 1933). Later, it has been generalized by German crystallographer G.Wulff (1901) to the case of the relative geometry. Wulff's anisoperimetric inequality reads as follows:

$$\Pi_\sigma(\Gamma) \equiv \frac{L_\sigma(\Gamma)^2}{4|W_\sigma|A(\Gamma)} \geq 1$$

and the equality is attained by a curve Γ similar to the boundary of the Wulff shape. Our approach in construction of the optimal anisotropy function σ is based on minimization of the anisoperimetric ratio $\Pi_\sigma(\Gamma)$ provided that the Jordan curve Γ is given. The curve Γ may represent a boundary of an important planar object for which we want to reconstruct the underlying anisotropy function σ.

A 2π - periodic real valued anisotropy function $\sigma(v) \geq 0$ can be represented by its complex Fourier series expansion:

$$\sigma(v) = \sum_{k=-\infty}^{\infty} \sigma_k e^{ikv}, \text{ where } \sigma_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikv}\sigma(v)\,\mathrm{d}v, \quad \sigma_{-k} = \overline{\sigma}_k .$$

The area $|W_\sigma|$ of the Wulff shape can be expressed in terms of the Fourier coefficients as follows:

$$|W_\sigma| = \frac{1}{2}\int_0^{2\pi} |\sigma|^2 - |\sigma'|^2\,\mathrm{d}v = \pi\sigma_0^2 + 2\pi\sum_{k=1}^{\infty}(1-k^2)|\sigma_k|^2 .$$

Similarly, we can express the interface energy

$$L_\sigma(\Gamma) = \int_\Gamma \sigma(v)\,\mathrm{d}s = \sum_{k=-\infty}^{\infty} \sigma_k \int_\Gamma e^{ikv}\mathrm{d}s = c_0\sigma_0 + 2\Re\sum_{k=1}^{\infty}\overline{c}_k\sigma_k,$$

where the complex coefficients

$$c_k = \int_\Gamma e^{-ikv}\mathrm{d}s,\ k \in Z,$$

form the so-called Fourier length spectrum of the curve Γ . For further details on properties of the Fourier length spectrum we refer the reader to Section 4 of Ševčovič and Trnovská (2015a). Minimization of the anisoperimetric ratio with respect to σ can be equivalently reformulated as the problem of maximization of the area $|W_\sigma|$ (or minimization of $-|W_\sigma|$) assuming that the interface energy $L_\sigma(\Gamma) = const$ is fixed leading to linear constraints of the form $Ax = b$ for the vector $x = (\Re\sigma, \Im\sigma)$ representing real and imaginary parts of the vector of complex Fourier coefficients σ_k . For computational purposes, we can restrict infinite Fourier series to its fine finite Fourier modes representation. According to characterization due McLean and Woerdeman (2001) a finite Fourier series expansion

$$\sigma(v) = \sum_{k=-N+1}^{N-1}\sigma_k e^{ikv} \ ,$$

where $\sigma_0 \in R,\ \sigma_k = \overline{\sigma}_{-k} \in C$, is a nonnegative function $\sigma(v) \geq 0$ for $v \in \mathbb{R}$, if and only if there exists a Hermitian matrix $F \in H^N$, $F \succeq 0$, such that

$$\sum_{p=k+1}^{N} F_{p,p-k} = \sigma_k \text{ for each } k = 0,1,\cdots,N-1.$$

Similarly, the condition $\sigma(v) + \sigma''(v) \geq 0$ can be expressed in terms LMI constraints. In summary, the optimization problem for minimizing the anisoperimetric ratio (the isoperimetric ratio in the Finsler geometry described by the anisotropy function σ) can be stated as follows:

$$\begin{aligned} &\min\ x^T P_0 x \\ &s.t.\quad Ax = b, \\ &\qquad \sum_{p=k+1}^{N} F_{p,p-k} = \sigma_k,\ \sum_{p=k+1}^{N} G_{p,p-k} = (1-k^2)\sigma_k,\ F,G \succeq 0,\ F,G \in H^N,\ x \in R^{2N},\ k = 1,\cdots,N-1, \end{aligned} \tag{4}$$

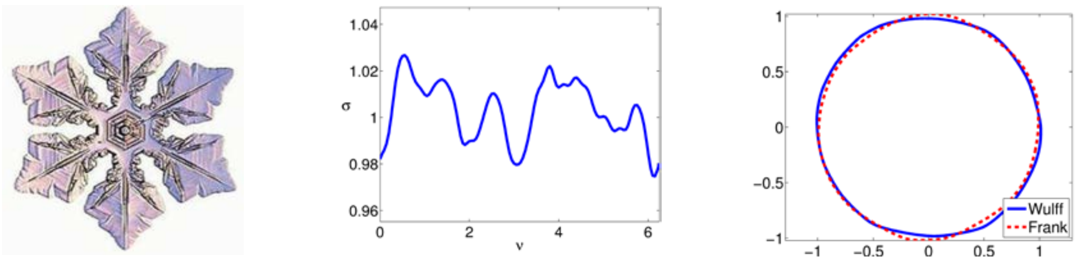
which is a quadratic optimization problem with linear equalities and linear matrix inequality constraints and non-convex objective function fitting into a general class of problems of the form (1). According to Theorem 1, its enhanced semi-definite relaxation has the following form:

$$
\begin{aligned}
&\min \; x^T P_0 x \\
&s.t. \quad Ax = b, \; AX = bx^T, \; X \succeq xx^T, \\
&\qquad \sum_{p=k+1}^{N} F_{p,p-k} = \sigma_k, \; \sum_{p=k+1}^{N} G_{p,p-k} = (1-k^2)\sigma_k, \; F, G \succeq 0, \; F, G \in H^N, \; x \in R^{2N}, \; k = 1, \cdots, N-1.
\end{aligned}
\tag{5}
$$

3.2 Computational results

Finally, we present a numerical example of construction of the optimal anisotropy function based on minimization of the anisoperimetric ratio. We considered a curve Γ which is a boundary of a snowflake. We employed the powerful Matlab computational toolbox SeDuMi due to J. Sturm (2001) for computing the enhanced semi-definite relaxation problem (5). The results of computation are shown in the following Figure 1. We can observe six local maxima in the optimal anisotropy function σ (Figure 1 b) corresponding to the hexagonal anisotropy forming the snowflake a).

Figure 10 A boundary of a snowflake a). The optimal anisotropy function σ computed by means of a solution to (5) (the semi-definite relaxation of (4)) is shown in b). The Wulff shape and its dual Frank diagram are depicted in c). Source: Ševčovič and Trnovská (2015b).



4. CONCLUSIONS

We presented a brief survey of a novel method of enhanced semi-definite relaxation for solving a class of non-convex quadratic optimization problems. The method was applied to the inverse Wulff problem of construction of the optimal anisotropy function minimizing the anisoperimetric ratio.

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