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On a construction of integrally invertible graphs and their spectral properties

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Godsil (1985) defined a graph to be invertible if it has a non-singular adjacency matrix whose inverse is diagonally similar to a nonnegative integral matrix; the graph defined by the last matrix is then the inverse of the original graph. In this paper we call such graphs positively invertible and introduce a new concept of a negatively invertible graph by replacing the adjective 'nonnegative' by 'nonpositive in Godsil's definition; the graph defined by the negative of the resulting matrix is then the negative inverse of the original graph. We propose new constructions of integrally invertible graphs (those with non-singular adjacency matrix whose inverse is integral) based on an operation of 'bridging' a pair of integrally invertible graphs over subsets of their vertices, with sufficient conditions for their positive and negative invertibility. We also analyze spectral properties of graphs arising from bridging and derive lower bounds for their least positive eigenvalue. As an illustration we present a census of graphs with a unique 1-factor on $m \leq 6$ vertices and determine their positive and negative invertibility.

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1. Introduction

A number of ways of introducing inverses of graphs have been proposed, all based on inverting adjacency matrices. For a graph with a non-singular adjacency matrix a first thought might be to hope that the inverse matrix defines a graph again. It turns out, however, that this happens to be the case only for unions of isolated edges [6]. A successful approach was initiated by Godsil [4] who defined a graph to be invertible if the inverse of its (non-singular) adjacency matrix is diagonally similar (cf. [15]) to a nonnegative integral matrix representing the adjacency matrix of the inverse graph in which positive labels determine edge multiplicities. This way of introducing invertibility has the appealing property that inverting an inverse graph gives back the original graph. For a survey of results and other approaches to graph inverses we recommend [9].

Inverse graphs are of interest in estimating the least positive eigenvalue in families of graphs, a task for which there appears to be lack of suitable bounds. However, if the graphs are invertible, one can apply one of the (many) known upper bounds on largest eigenvalues of the inverse graphs instead (cf. [10–12]). Properties of spectra of inverse graphs can also be used to estimate the difference between the minimal positive and maximal negative eigenvalue (the so-called HOMO-LUMO gap) for structural models of chemical molecules, as it was done e.g. for graphene in [14].

Godsil's ideas have been further developed in several ways. Akbari and Kirkland [7] and Bapat and Ghorbani [1] studied inverses of edge-labeled graphs with labels in a ring, Ye et al. [13] considered connections of graph inverses with median eigenvalues, and Pavlíková [10,12] developed constructive methods for generating invertible graphs by edge overlapping. A large number of related results, including a unifying approach to inverting graphs, were proposed in a recent survey paper by McLeman and McNicholas [9], with emphasis on inverses of bipartite graphs and diagonal similarity to nonnegative matrices.

Less attention has been given to the study of invertibility of non-bipartite graphs and their spectral properties which is the goal of this paper. After introducing basic concepts, in Section 2 we present an example of a non-bipartite graph representing an important chemical molecule of fulvene. Its adjacency matrix has the remarkable additional property that its inverse is integral and diagonally similar to a nonpositive rather than a nonnegative matrix. This motivated us to introduce negative invertibility as a natural counterpart of Godsil [4] concept: A negatively invertible graph is one with a non-singular adjacency matrix whose inverse is diagonally similar to a nonpositive matrix. The negative of this matrix is then the adjacency matrix of the inverse graph. Positively and negatively invertible graphs are subfamilies of integrally invertible graphs, whose adjacency matrices have an integral inverse. The corresponding inverse graphs, however, would have to be interpreted as labeled graphs with both positive and negative (integral) edge labels.

Results of the paper are organized as follows. In Section 3 we develop constructions of new integrally invertible graphs from old ones by 'bridging' two such graphs over subsets of their vertices. This yields a wide range of new families of integrally invertible graphs. We derive sufficient conditions for their positive and negative invertibility. In contrast to purely graph-theoretical approach we use methods of matrix analysis and in particular results on inverting block matrices such as the Schur complement theorem and the Woodbury and Morrison–Sherman formulae. Using this approach enables us to derive useful bounds on the spectra of graphs arising from bridging construction in Section 4. We then illustrate our results in Section 5 on a recursively defined family of fulvene-like graphs. In the final Section 6 we discuss arbitrariness in the bridging construction and give a census of all invertible graphs on at most 6 vertices with a unique 1-factor.

2. Invertible graphs

In this section we recall a classical concept of an invertible graph due to Godsil [4]. Let G be an undirected graph, possibly with multiple edges, and with a (symmetric) adjacency matrix A_G. Conversely, if A is a nonnegative integral symmetric matrix, we will use the symbol G_A to denote the graph with the adjacency matrix A.

The spectrum $\sigma(G)$ of G consists of eigenvalues (i.e., including multiplicities) of A_G (cf. [3,2]). If the spectrum does not contain zero then the adjacency matrix A is invertible. We begin with a definition of an integrally invertible graph.

Definition 1. A graph $G = G_A$ is said to be integrally invertible if the inverse A^{-1} of its adjacency matrix exists and is integral.

It follows that a graph G_A is integrally invertible if and only if $det(A) = \pm 1$ (cf. [7]). Note that, in such a case the inverse matrix A^{-1} need not represent a graph as it may contain negative entries.

Following the idea due to Godsil, the concept of the inverse graph G_A^{-1} is based on the inverse matrix A^{-1} for which we require signability to a nonnegative or nonpositive matrix. We say that a symmetric matrix B is positively (negatively) signable if there exists a diagonal ± 1 matrix D such that DBD is nonnegative (nonpositive). We also say that D is a signature matrix.

Definition 2. A graph G_A is called positively (negatively) invertible if A^{-1} exists and is signable to a nonnegative (nonpositive) integral matrix. If D is the corresponding signature matrix, the positive (negative) inverse graph $H = G_A^{-1}$ is defined by the adjacency matrix $A_H = DA^{-1}D$ ($A_H = -DA^{-1}D$).

The concept of positive invertibility coincides with the original notion of invertibility introduced by Godsil [4]. Definition 2 extends Godsil's original concept to a larger class of integrally invertible graphs with inverses of adjacency matrices signable to nonpositive matrices.

Notice that for a diagonal matrix D^A containing ± 1 elements only, we have $D^A D^A = I$. It means that $(D^A)^{-1} = D^A$.

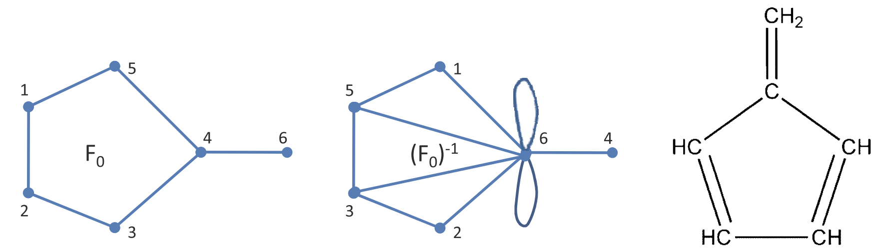

Fig. 1. An example of a negatively invertible non-bipartite graph F_0 (left) and its inverse graph $(F_0)^{-1}$ (middle) of the fulvene chemical organic molecule (right).

Remark 1. The idea behind the definition of an inverse graph consists of the following useful property. If G is a positively (negatively) invertible graph then G^{-1} is again a positively (negatively) invertible graph and $G = (G^{-1})^{-1}$. As far as the spectral properties are concerned, we have

$$\sigma(G^{-1}) = 1/\sigma(G) = \{1/\lambda,\ \lambda \in \sigma(G)\},$$

for any positively invertible graph G. On the other hand, if G is negatively invertible then

$$\sigma(G^{-1}) = -1/\sigma(G) = \{-1/\lambda,\ \lambda \in \sigma(G)\}.$$

Fig. 1 (left) shows the graph F_0 on 6 vertices representing the organic molecule of the fulvene hydrocarbon (5-methylidenecyclopenta-1,3-diene) (right). The graph F_0 is negatively (but not positively) invertible with the inverse graph $(F_0)^{-1}$ depicted in Fig. 1 (middle). The spectrum consists of the following eigenvalues:

$$\sigma(F_0) = \{-1.8608, -q, -0.2541, 1/q, 1, 2.1149\},$$

where $q = (\sqrt{5}+1)/2$ is the golden ratio with the least positive eigenvalue $\lambda_1^+(F_0) = 1/q$. The inverse adjacency matrix $A_{F_0}^{-1}$ is signable to a nonpositive integral matrix by the signature matrix $D^{A_{F_0}} = diag(-1, -1, 1, 1, 1 - 1)$.

2.1. Bipartite graphs and their invertibility

A graph G_B is called bipartite if the set of vertices can be partitioned into two disjoint subsets such that no two vertices within the same subset are adjacent. The adjacency matrix B of a bipartite graph G_B can be given in a block form:

$$B = \begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix},$$

where K is a matrix with nonnegative integer entries. Clearly, the adjacency matrix B of a bipartite graph G_B is invertible if and only if the number of its vertices is even and the matrix K is invertible. If we consider the labeled graph corresponding to the adjacency matrix B^{-1} and the product of edge labels on every cycle in this graph is positive, then B^{-1} is signable to a nonnegative matrix and so G_B is a positively invertible graph (see [8]).

Recall that a 1-factor (a perfect matching) of a graph is a spanning 1-regular subgraph with all vertices of degree 1. If G is a bipartite graph with a 1-factor M such that the graph G/M obtained from G by contracting edges of M is bipartite then G is a positively invertible graph (cf. [4,11]).

Bipartiteness and invertibility are related as follows.

Theorem 1. *Let G be an integrally invertible graph. Then G is bipartite if and only if G is simultaneously positively and negatively invertible.*

Proof. Let $G = G_B$ be an integrally invertible bipartite graph. Assume that G_B is positively invertible, i.e., there exists a signature matrix $D^+ = diag(D_1, D_2)$ such that the matrix

$$D^+B^{-1}D^+ = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} 0 & (K^{-1})^T \\ K^{-1} & 0 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} 0 & D_1(K^{-1})^T D_2 \\ D_2K^{-1}D_1 & 0 \end{pmatrix}$$

contains nonnegative integer entries only. Then for the $\{\pm 1\}$ diagonal matrix $D^- = diag(D_1, -D_2)$ the matrix

$$D^-B^{-1}D^- = \begin{pmatrix} 0 & -D_1(K^{-1})^T D_2 \\ -D_2K^{-1}D_1 & 0 \end{pmatrix}$$

contains nonpositive integers only. Hence G_B is negatively invertible, and vice versa.

On the other hand, suppose that G is simultaneously positively and negatively invertible. We will prove that G is a bipartite graph with even number of vertices. Let n be the number of vertices of the graph G. Let $D^{\pm}$ be diagonal $\{\pm 1\}$-matrices such that $D^+A^{-1}D^+$ contains nonnegative integers and $D^-A^{-1}D^-$ contains nonpositive integers only. Since $(D^{\pm}A^{-1}D^{\pm})_{ij} = D^{\pm}_{ii}(A^{-1})_{ij}D^{\pm}_{jj}$ we conclude that $(D^{\pm}A^{-1}D^{\pm})_{ij} \neq 0$ if and only if $(A^{-1})_{ij} \neq 0$. Hence

$$D^+A^{-1}D^+ = -D^-A^{-1}D^-. \tag{1}$$

As $det(A^{-1}) = det(D^+A^{-1}D^+) = (-1)^n det(D^-A^{-1}D^-) = (-1)^n det(A^{-1})$ we conclude that n is even, i.e. $n = 2m$.

Recall that for the trace operator $tr(Z) = \sum_{i=1}^n Z_{ii}$ of an $n \times n$ matrix Z we have $tr(XY) = tr(YX)$ where X, Y are $n \times n$ matrices. With respect to (1) we obtain:

$$tr(D^+D^-) = tr(AD^+D^-A^{-1}) = tr(D^-AD^+D^-A^{-1}D^-)$$
$$= -tr(D^-AD^+D^+A^{-1}D^+) = -tr(D^-D^+) = -tr(D^+D^-).$$

Thus $tr(D^+D^-) = 0$. Since $D^\pm$ are diagonal $\{\pm1\}$-matrices, there exists an $n \times n$ permutation matrix P such that

$$P^T D^+D^- P = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad i.e. \quad D^+D^- = D^-D^+ = P \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} P^T,$$

where I is the $m \times m$ identity matrix. It follows from (1) that

$$A^{-1} = -D^+D^-A^{-1}D^-D^+ = -P \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} P^T A^{-1} P \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} P^T.$$

Since $P^T P = PP^T = I$ we have

$$P^T A^{-1} P = -\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} P^T A^{-1} P \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

If we write $P^T A^{-1} P$ as a block matrix we obtain

$$P^T A^{-1} P \equiv \begin{pmatrix} V & H \\ H^T & W \end{pmatrix} = -\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} V & H \\ H^T & W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$
$$= \begin{pmatrix} -V & H \\ H^T & -W \end{pmatrix}.$$

Therefore $V = W = 0$ and

$$P^T A^{-1} P = \begin{pmatrix} 0 & H \\ H^T & 0 \end{pmatrix} \Longrightarrow P^T A P = \begin{pmatrix} 0 & (H^T)^{-1} \\ H^{-1} & 0 \end{pmatrix}.$$

This means that the adjacency matrix A represents a bipartite graph $G = G_A$ after a permutation of its vertices given by the matrix P. $\diamondsuit$

3. Integrally invertible graphs arising by bridging

Let G_A and G_B be undirected graphs on n and m vertices, respectively. By $\mathcal{B}_k(G_A, G_B)$ we shall denote the graph G_C on $n + m$ vertices which is obtained by bridging the last k vertices of the graph G_A to the first k vertices of G_B. The adjacency matrix C of the graph G_C has the form:

$$C = \begin{pmatrix} A & H \\ H^T & B \end{pmatrix},$$

where the $n \times m$ matrix H has the block structure:

$$H = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} = FE^T, \quad \text{where} \quad F = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad E = \begin{pmatrix} I \\ 0 \end{pmatrix},$$

and I is the $k \times k$ identity matrix.

Assume that A and B are symmetric $n \times n$ and $m \times m$ invertible matrices, respectively. With regard to the Schur complement theorem we obtain

$$C^{-1} = \begin{pmatrix} A & H \\ H^T & B \end{pmatrix}^{-1} = \begin{pmatrix} S^{-1} & -S^{-1}HB^{-1} \\ -B^{-1}H^TS^{-1} & B^{-1} + B^{-1}H^TS^{-1}HB^{-1} \end{pmatrix}, \tag{2}$$

where $S = A - HB^{-1}H^T$ is the Schur complement (see e.g. [5, Theorem A.6]). To facilitate further notation let us introduce the following matrices:

$$P = F^TA^{-1}F, \qquad R = E^TB^{-1}E.$$

In order to compute the inverse of the Schur complement S we follow derivation of the Woodbury and Morrison–Sherman formulae (cf. [5, Corollary A.6, A.7]). More precisely, equation $Sx = y$ can be rewritten as follows:

$$y = (A - HB^{-1}H^T)x = Ax - FE^TB^{-1}EF^Tx = Ax - FRF^Tx$$

and thus $x = A^{-1}y + A^{-1}FRF^Tx$. Hence

$$F^Tx = F^TA^{-1}y + F^TA^{-1}FRF^Tx = F^TA^{-1}y + PRF^Tx.$$

If we assume that the matrix $I - PR$ is invertible then $F^Tx = (I - PR)^{-1}F^TA^{-1}y$, and

$$S^{-1} = (A - HB^{-1}H^T)^{-1} = A^{-1} + A^{-1}FR(I - PR)^{-1}F^TA^{-1}. \tag{3}$$

Note that the matrix $I - PR$ is integrally invertible provided that either $PR = 0$ or $PR = 2I$.

Clearly, an $m \times m$ matrix with a zero principal $k \times k$ diagonal block is invertible only for $k \leq m$. Consequently, there are no connected invertible graphs with $E^TB^{-1}E = 0$ for $k > m/2$.

Theorem 2. *Let G_A and G_B be integrally invertible graphs on n and m vertices, and let R and P be the upper left and lower right $k \times k$ principal submatrices of B^{-1} and A^{-1}, respectively.*

Let $G_C = \mathcal{B}_k(G_A, G_B)$ be the graph obtained by bridging G_A and G_B over the last k vertices of G_A and the first k vertices of G_B. If $PR = 0$ or $PR = 2I$, then the graph G_C is integrally invertible.

Proof. Since $PR = 0$ or $PR = 2I$, then the inverse $(I - PR)^{-1}$ exists and is equal to $\pm I$. Hence the inverse S^{-1} of the Schur complement is an integral matrix. Therefore the block matrix C given by

$$C = \begin{pmatrix} A & H \\ H^T & B \end{pmatrix}$$

is invertible, and hence so is the bridged graph G_C. Moreover, C^{-1} is an integral matrix because A^{-1}, B^{-1}, S^{-1} are integral. ◇

Definition 3. Let G_B be a graph on m vertices with an invertible adjacency matrix B. We say that G_B is arbitrarily bridgeable over a subset of $k \le m/2$ vertices if the $k \times k$ upper principal submatrix $R \equiv E^T B^{-1} E$ of the inverse matrix B^{-1} is a null matrix, that is $R = 0$.

In view of Definition 3, the bridged graph $G_C = \mathcal{B}_k(G_A, G_B)$ is integrally invertible provided that G_B is arbitrarily bridgeable over the set of its 'first' k vertices.

In the next theorem we address the question of invertibility of the bridged graph $G_C = \mathcal{B}_k(G_A, G_B)$ under the assumption that G_A and G_B are positive (negative) invertible graphs.

Theorem 3. *Let G_A and G_B be graphs on n and m vertices, respectively. Assume that they are either both positively invertible or both negatively invertible graphs with signature matrices D^A and D^B. Then the graph $G_C = \mathcal{B}_k(G_A, G_B)$ is positively (negatively) invertible if we have $PR = 0$ and either the matrix $D^A H D^B$ or $-D^A H D^B$ contains nonnegative integers only.*

Proof. Let C be the adjacency matrix to the graph $G_C = \mathcal{B}_k(G_A, G_B)$. If $PR = 0$ then for the inverse of the Schur complement (see (3)) we have

$$S^{-1} = A^{-1} + A^{-1} F R F^T A^{-1} = A^{-1} + A^{-1} H B^{-1} H^T A^{-1}$$

because $FRF^T = FE^T B^{-1} E F^T = H B^{-1} H^T$. Therefore

$$\begin{aligned} D^A S^{-1} D^A &= D^A A^{-1} D^A + D^A A^{-1} H B^{-1} H^T A^{-1} D^A \\ &= D^A A^{-1} D^A \\ &\quad + (D^A A^{-1} D^A)(D^A H D^B)(D^B B^{-1} D^B)(D^B H^T D^A)(D^A A^{-1} D^A) \end{aligned}$$

and so $D^A S^{-1} D^A$ is a nonnegative (nonpositive) integer matrix because the matrices $D^A A^{-1} D^A$ and $D^B B^{-1} D^B$ are simultaneously nonnegative (nonpositive) and $D^A H D^B$ or $-D^A H D^B$ contains nonnegative integers only.

In the case when $D^A H D^B$ is nonnegative we will prove that C^{-1} is diagonally similar to a nonnegative (nonpositive) integer matrix with $D^C = diag(D^A, -D^B)$ ($D^C = diag(D^A, D^B)$). With regard to (2) we have

$$D^C C^{-1} D^C = \begin{pmatrix} D^A & 0 \\ 0 & -D^B \end{pmatrix} \begin{pmatrix} S^{-1} & -S^{-1}HB^{-1} \\ -B^{-1}H^T S^{-1} & B^{-1} + B^{-1}H^T S^{-1} H B^{-1} \end{pmatrix} \times \begin{pmatrix} D^A & 0 \\ 0 & -D^B \end{pmatrix}$$

$$= \begin{pmatrix} D^A S^{-1} D^A & D^A S^{-1} D^A D^A H D^B D^B B^{-1} D^B \\ D^B B^{-1} D^B D^B H^T D^A D^A S^{-1} D^A & D^B B^{-1} D^B + W, \end{pmatrix}$$

where

$$W = (D^B B^{-1} D^B)(D^B H^T D^A)(D^A S^{-1} D^A)(D^A H D^B)(D^B B^{-1} D^B).$$

In the expression for the matrix W we have intentionally used the matrices $D^A D^A = D^B D^B = I$ instead of the identity matrix I. Since the matrices $D^A A^{-1} D^A$, $D^B B^{-1} D^B$, and $D^A S^{-1} D^A$ contain nonnegative (nonpositive) integers only and $D^A H D^B$ is nonnegative, we conclude that C^{-1} is diagonally similar to a nonnegative (nonpositive) integral matrix.

In the case when $-D^A H D^B$ is nonnegative we can proceed similarly as before and conclude that C^{-1} is diagonally similar to a nonnegative (nonpositive) integral matrix.

Hence the graph G_C is positively (negatively) invertible, as claimed. $\diamondsuit$

As a consequence we obtain the following:

Corollary 1. *Let G_A, G_B be two positively (negatively) invertible graphs such that $(B^{-1})_{11} = 0$. Then the graph $G_C = \mathcal{B}_1(G_A, G_B)$ bridged over the first vertex is again positively (negatively) invertible.*

Proof. For $k = 1$ the condition $(B^{-1})_{11} = 0$ implies $R \equiv E^T B^{-1} E = 0$, i.e. G_B is arbitrarily bridgeable. The matrix $D^A H D^B$ contains only one nonzero element, equal to $\pm H$. Hence the assumptions of Theorem 3 are fulfilled and so G_C is positively (negatively) invertible. $\diamondsuit$

With regard to Theorem 1 and Theorem 3 we obtain the following result:

Corollary 2. *Let G_A, G_B be two bipartite positively and negatively invertible graphs such that G_B is arbitrarily bridgeable over the first k vertices. Let D^A_+ and D^B_+ (D^A_- and*

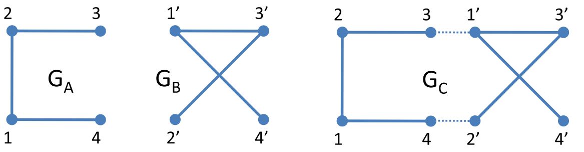

Fig. 2. An example of bridging of two bipartite positively and negatively invertible graphs G_A and G_B through vertices $3 \leftrightarrow 1', 4 \leftrightarrow 2'$. The resulting graph $G_C = \mathcal{B}_2(G_A, G_B)$.

D_-^B) be diagonal $\{\pm 1\}$-matrices such that $D_+^A A^{-1} D_+^A$ and $D_+^B B^{-1} D_+^B$ ($D_-^A A^{-1} D_-^A$ and $D_-^B B^{-1} D_-^B$) are nonnegative (nonpositive) matrices. If $D_\pm^A H D_\pm^B$ are either both nonnegative or both nonpositive then the bridged graph $G_C = \mathcal{B}_k(G_A, G_B)$ is again bipartite positively and negatively invertible.

Example 1. In what follows, we present an example showing that the assumption made on nonnegativity or nonpositivity of matrices $D_+^A H D_+^B$ and $D_-^A H D_-^B$ cannot be relaxed. To do so, we will construct a bridged graph G_C from two integrally invertible bipartite graphs such that G_C is only positively but not negatively invertible graph and, as a consequence of Theorem 1, the graph G_C is not bipartite.

Let G_A and G_B be two simultaneously positively and negatively invertible bipartite graphs shown in Fig. 2. We will bridge them over a set of $k = 2$ vertices to obtain the graph G_C with inverses given by

$$A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

$$C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 2 \end{pmatrix}.$$

The graphs G_A, G_B are isomorphic with eigenvalues: $\{\pm 1.6180, \pm 0.6180\}$. The upper left 2×2 principal submatrix R of B^{-1} is zero, so that the graph G_B can be arbitrarily bridged to an integrally invertible graph G_A.

It is easy to verify that the inverse matrices A^{-1} and B^{-1} can be signed to nonnegative matrices by signature matrices $D_+^A = D_+^B = diag(-1, 1, 1, -1)$. At the same time they can be signed to a nonpositive matrix by $D_-^A = diag(-1, -1, 1, 1)$ and $D_-^B = diag(-1, 1, -1, 1)$. Furthermore, $D_+^A H D_+^B = -H$ is a nonpositive matrix. By Theorem 3, the graph G_C is positively invertible. On the other hand,

$$D_-^A H D_-^B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is neither nonnegative nor nonpositive. Indeed, the graph G_C is not bipartite and it is just positively (and not negatively) invertible, with spectrum

$$\sigma(G_C) = \{-1.9738, -1.8019, -0.7764, -0.445, 0.2163, 1.247, 1.4427, 2.0912\}.$$

Remark 2. If the graph G_B is arbitrarily bridgeable over the first k vertices then $R = 0$, and, consequently the assumption $PR = 0$ appearing in Theorem 3 is satisfied. On the other hand, if we consider the graph G_C with the vertex set $\{1, 2, 3, 4, 1', 2', 3', 4'\}$ shown in Fig. 2 then the inverse matrix C^{-1} contains the principal submatrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

corresponding to vertices $4, 1', 2', 3'$. Consider the same graph $\tilde{G}_{\tilde{C}}$ on the vertex set $\{\tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{1}', \tilde{2}', \tilde{3}', \tilde{4}'\}$. Then, after permuting vertices, the inverse matrix $\tilde{C}^{-1}$ has the upper left principal 4×4 submatrix

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $PR = 0$ but neither P nor R is an all-zero 4×4 matrix. By Theorem 3 the bridged graph $\mathcal{B}_4(G_C, \tilde{G}_{\tilde{C}})$ over the set of vertices $4 \leftrightarrow \tilde{2}', 1' \leftrightarrow \tilde{3}', 2' \leftrightarrow \tilde{4}, 3' \leftrightarrow \tilde{1}'$ is integrally invertible.

4. Spectral bounds for graphs arising by bridging

In this section we derive a lower bound for the least positive eigenvalue of bridged graphs $\mathcal{B}_k(G_A, G_B)$ in terms of the least positive eigenvalues of graphs G_A and G_B. Throughout this section we assume that the adjacency matrices A, B are invertible but we do not require their integral invertibility.

Before stating and proving our spectral estimate we need the following auxiliary Lemma.

Lemma 1. *Assume that D is an $n \times m$ matrix and $\alpha, \beta > 0$ are positive constants. Then, for the optimal value λ^* of the following constrained optimization problem:*

$$\begin{aligned} \lambda^* = \max \quad & \alpha\|x - Dy\|^2 + \beta\|y\|^2 \\ \text{s.t.} \quad & \|x\|^2 + \|y\|^2 = 1, \quad x \in \mathbb{R}^n, y \in \mathbb{R}^m, \end{aligned} \tag{4}$$

we have an explicit expression of the form:

$$\lambda^* = \max\left\{\lambda, \ \frac{(\lambda - \alpha)(\lambda - \beta)}{\alpha\lambda} \in \sigma(D^T D)\right\}$$

$$= \frac{\alpha\mu^* + \alpha + \beta + \sqrt{(\alpha\mu^* + \alpha + \beta)^2 - 4\alpha\beta}}{2},$$

where $\mu^* = \max\{\sigma(D^T D)\}$ *is the maximal eigenvalue of the matrix* $D^T D$*.*

Proof. The proof is straightforward and is based on standard application of the Lagrange multiplier method (see e.g. [5]). We give details for the reader's convenience.

Let us introduce the Lagrange function:

$$\begin{aligned} L(x, y, \lambda) &= \alpha\|x - Dy\|^2 + \beta\|y\|^2 - \lambda(\|x\|^2 + \|y\|^2) \\ &= \alpha x^T x - 2\alpha x^T Dy + \alpha y^T D^T Dy + \beta y^T y - \lambda x^T x - \lambda y^T y. \end{aligned}$$

Now, it follows from the first order necessary conditions for constrained maximum (x, y) (see e.g. [5]) that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$0 = L'_x \equiv 2\alpha x^T - 2\alpha y^T D^T - 2\lambda x^T, \tag{5}$$

$$0 = L'_y \equiv -2\alpha x^T D + 2\alpha y^T D^T D + 2\beta y^T - 2\lambda y^T. \tag{6}$$

In the case $\lambda \neq \alpha$ we obtain

$$x = \frac{\alpha}{\alpha - \lambda} Dy, \qquad \left[\alpha D^T D - (\lambda - \beta)\right] y = \alpha D^T x = \frac{\alpha^2}{\alpha - \lambda} D^T Dy.$$

Therefore,

$$D^T Dy = \frac{(\lambda - \alpha)(\lambda - \beta)}{\alpha\lambda} y \;\Rightarrow\; \frac{(\lambda - \alpha)(\lambda - \beta)}{\alpha\lambda} \in \sigma(D^T D).$$

Now, from the constraint $x^T x + y^T y = 1$ we deduce that

$$1 = x^T x + y^T y = \frac{\alpha^2}{(\alpha - \lambda)^2} y^T D^T Dy + y^T y = \left(\frac{\alpha^2}{(\alpha - \lambda)^2} \frac{(\lambda - \alpha)(\lambda - \beta)}{\alpha\lambda} + 1\right) \|y\|^2.$$

Hence

$$\|y\|^2 = \frac{(\lambda - \alpha)\lambda}{\lambda^2 - \alpha\beta}, \qquad \|x\|^2 = 1 - \|y\|^2 = \frac{(\lambda - \beta)\alpha}{\lambda^2 - \alpha\beta}.$$

Finally, for the value function $f(x, y) = \alpha\|x - Dy\|^2 + \beta\|y\|^2$ of the constrained optimization problem (4) we obtain

$$\begin{aligned} f(x, y) &= \alpha x^T x - 2\alpha x^T Dy + \alpha y^T D^T Dy + \beta y^T y \\ &= \alpha x^T x - 2\frac{\alpha^2}{\alpha - \lambda} y^T D^T Dy + \alpha y^T D^T Dy + \beta y^T y \end{aligned}$$

$$= \frac{\alpha^2(\lambda-\beta)}{\lambda^2-\alpha\beta} + \left(\alpha - \frac{2\alpha^2}{\alpha-\lambda}\right) \frac{(\lambda-\alpha)(\lambda-\beta)}{\alpha\lambda} \frac{(\lambda-\alpha)\lambda}{\lambda^2-\alpha\beta} + \beta\frac{(\lambda-\alpha)\lambda}{\lambda^2-\alpha\beta}$$
$$= \lambda.$$

In the case $\lambda = \alpha$, one sees from (5) that $Dy = 0$ and so $f(x, y) = \alpha\|x\|^2 + \beta\|y\|^2 \leq \max\{\alpha, \beta\} \leq \lambda^*$. In summary,

$$\lambda^* = \max\left\{\lambda,\ \frac{(\lambda-\alpha)(\lambda-\beta)}{\alpha\lambda} \in \sigma(D^T D)\right\},$$

as claimed. ♢

We are in a position to present our spectral bound.

Theorem 4. *Let G_A and G_B be graphs on n and m vertices with invertible adjacency matrices. Assume G_B is arbitrarily bridgeable over the first k vertices. Then the least positive eigenvalue $\lambda_1^+(G_C)$ of its adjacency matrix C of the bridged graph $G_C = \mathcal{B}_k(G_A, G_B)$ satisfies*

$$\lambda_1^+(G_C) \geq \lambda_{lb}(G_A, G_B, k) := \frac{2}{\alpha\mu^* + \alpha + \beta + \sqrt{(\alpha\mu^* + \alpha + \beta)^2 - 4\alpha\beta}},$$

where $\mu^* = \max\{\sigma(B^{-1}H^T HB^{-1})\}$ is the maximal eigenvalue of the positive semidefinite $m \times m$ matrix $B^{-1}H^T HB^{-1}, \alpha = 1/\lambda_1^+(G_A)$ and $\beta = 1/\lambda_1^+(G_B)$.

Proof. The idea of the proof is based on estimation of the numerical range of the matrix C^{-1}. Since $\lambda_1^+(G_C) = \lambda_1^+(C) = 1/\lambda_{max}(C^{-1})$ where $\lambda_{max}(C^{-1})$ is the maximal eigenvalue of the inverse matrix C^{-1}, the lower bound for $\lambda_1^+(C)$ can be derived from the upper bound for $\lambda_{max}(C^{-1})$. As stated in Definition 3, the assumption that G_B is an arbitrarily bridgeable graph implies $S^{-1} = A^{-1}$. Thus formula (2) for the inverse matrix C^{-1} becomes:

$$C^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}HB^{-1} \\ -B^{-1}H^T A^{-1} & B^{-1} + B^{-1}H^T A^{-1}HB^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} I & 0 \\ -B^{-1}H^T & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} I & -HB^{-1} \\ 0 & I \end{pmatrix}.$$

Let $z = (x, y)^T \in \mathbb{R}^{n+m}$ where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. For the Euclidean inner product $\langle C^{-1}z, z\rangle$ in $\mathbb{R}^{n+m}$ we obtain

$$\langle C^{-1}z, z\rangle = \langle A^{-1}(x - HB^{-1}y), x - HB^{-1}y\rangle + \langle B^{-1}y, y\rangle$$
$$\leq \lambda_{max}(A^{-1})\|x - HB^{-1}y\|^2 + \lambda_{max}(B^{-1})\|y\|^2.$$

Letting $\alpha = \lambda_{max}(A^{-1}), \beta = \lambda_{max}(B^{-1})$ and $D = HB^{-1}$, by Lemma 1 we obtain

$$\langle C^{-1}z, z\rangle \le \frac{1}{\lambda_{lb}(G_A, G_B, k)}\|z\|^2,$$

for any $z \in \mathbb{R}^{n+m}$. Since

$$\lambda_{max}(C^{-1}) = \max_{z\neq 0}\frac{\langle C^{-1}z, z\rangle}{\|z\|^2} \le \frac{1}{\lambda_{lb}(G_A, G_B, k)}$$

our Theorem follows because $\alpha = \lambda_{max}(A^{-1}) = 1/\lambda_1^+(A) = 1/\lambda_1^+(G_A)$ and $\beta = \lambda_{max}(B^{-1}) = 1/\lambda_1^+(B) = 1/\lambda_1^+(G_B)$. $\diamondsuit$

To illustrate this on an example, for the graph $G_C = \mathcal{B}_k(G_A, G_B)$ shown in Fig. 2 we have $\lambda_1^+(G_C) = 0.2163$ and the lower bound derived above gives $\lambda_{lb}(G_A, G_B, k) = 0.1408$.

5. A "fulvene" family of integrally invertible graphs

The aim of this section is to present construction of a family of integrally invertible graphs grown from the "fulvene" graph of Fig. 1 (left), which is the same as the H_{10} in Fig. 5. With regard to Table 2 (see Section 6), the graph $F_0 \equiv H_{10}$ can be arbitrarily bridged over the pair of vertices labeled by $1, 2$ (see the left part of Fig. 1) to any integrally invertible graph.

Our construction begins with the fulvene graph F_0. The next iteration F_1 is obtained by bridging F_0 to another copy of F_0 over the vertex set $\{1, 2\}$ in both copies (see Fig. 3).

We now describe a recursive construction of graphs F_n from F_{n-1}. For $n \ge 2$, the graph F_n will be obtained from F_{n-1} by bridging a certain number f_n (to be described below) copies of the graph F_0 over the vertex set $\{1, 2\}$ to vertices of F_{n-1} of degree 1 or 2. By definition, we set $f_1 = f_2 := 2$. Let $|V^{(i)}(F_n)|,\ i = 1, 2, 3$, denote the number of vertices of F_n with degree i.

The order of bridging is as follows:

- two copies of F_0 are bridged to every vertex of degree 1 which belonged to F_{n-2} and remained in F_{n-1} with degree 1. The other vertex of F_0 is bridged to the shortest path distance vertex of degree 2 belonging to F_{n-1}. The number $f_n^{(1)}$ of graphs F_0 added to F_{n-1} is given by: $f_n^{(1)} = 2f_{n-2}$. This way one uses $|V^{(2)}(F_{n-1})| - 2f_{n-2}$ of vertices of degree 2 from F_{n-1}.
- The remaining $f_n^{(2)} = f_n - f_n^{(1)}$ copies of F_0 are bridged to F_{n-1} through $|V^{(2)}(F_{n-1})| - 2f_{n-2}$ vertices of degree 2 in such a way that the graph is bridged to the pair vertices of degree 2 with the shortest distance. By construction we have $|V^{(2)}(F_n)| = 2f_n$. Hence, the number $|V^{(2)}(F_{n-1})| - 2f_{n-2} = 2(f_{n-1} - f_{n-2})$ is even and so $f_n^{(2)} = f_{n-1} - f_{n-2}$. Moreover, the number $|V^{(1)}(F_n)|$ of vertices of degree 1 is given by: $|V^{(1)}(F_n)| = f_n + f_{n-1}$ as the vertices of degree 1 from $F_{n-1} \setminus F_{n-2}$ have not been bridged.

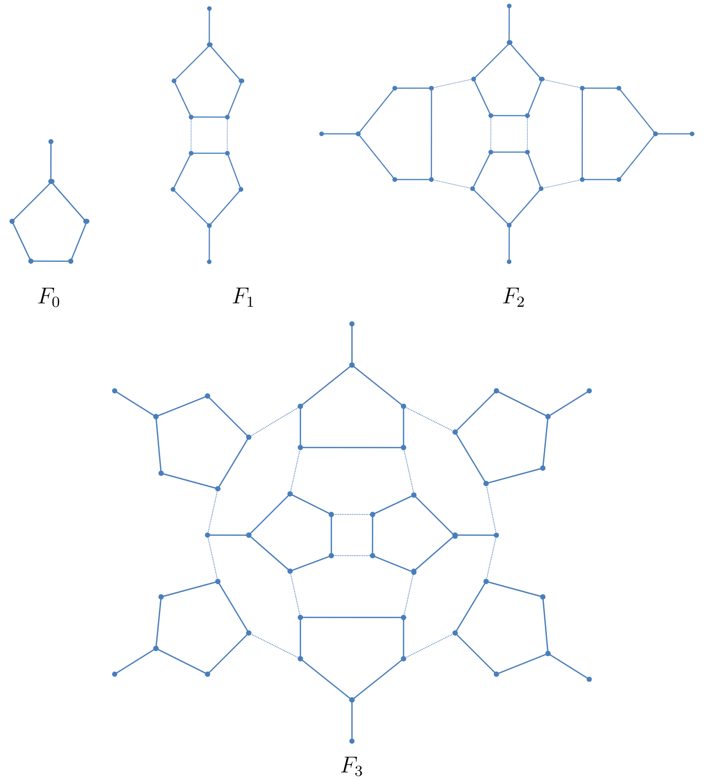

Fig. 3. The graphs F_0, F_1, F_2, F_3 of the "fulvene" family of integrally invertible graphs.

Since $f_n^{(1)} = 2f_{n-2}$ and $f_n^{(2)} = f_{n-1} - f_{n-2}$ the total number $f_n = f_n^{(1)} + f_n^{(2)}$ of newly added graphs F_0 satisfies the Fibonacci recurrence

$$f_n = f_{n-1} + f_{n-2}, \; f_1 = f_2 = 2.$$

It can be explicitly expressed as:

$$f_n = \frac{2}{\sqrt{5}} \left(q^n - q^{-n} \right),$$

where $q = (1 + \sqrt{5})/2$ is the golden ratio.

Properties of the fulvene family of graphs F_n, $n \geq 0$, (for F_0, F_1, F_2, F_3 see Fig. 3) can be summarized as follows.

Theorem 5. *Let $F_n, n \geq 0$, be a graph from the fulvene family of graphs. Then*

1. *F_n is integrally invertible;*
2. *F_n is a planar graph of maximum degree 3, with*

$$|V^{(1)}(F_n)| = f_n + f_{n-1},$$
$$|V^{(2)}(F_n)| = 2f_n,$$
$$|V^{(3)}(F_n)| = |V(F_n)| - |V^{(1)}(F_n)| - |V^{(2)}(F_n)| = 6\sum_{k=1}^{n} f_k - 3f_n - f_{n-1},$$

where $|V(F_n)| = 6\sum_{k=1}^{n} f_k$ *is the number of vertices of* F_n*;*

3. F_n *is asymptotically cubic in the sense that*

$$\lim_{n\to\infty} \frac{|V^{(3)}(F_n)|}{|V(F_n)|} = 1;$$

4. *the least positive eigenvalue* $\lambda_1^+(F_n)$ *satisfies the estimate:*

$$\lambda_1^+(F_n) \ge \frac{1}{q}\frac{5}{6^{n+1}-1}.$$

Proof. The number of vertices and integral invertibility of F_n have been derived during construction of F_n.

To prove the lower bound for the least positive eigenvalue $\lambda_1^+(F_n)$ of the integrally invertible graph F_n constructed in Section 5. Recall that the next generation F_n is constructed from F_{n-1} by bridging f_n basic fulvene graphs F_0 to vertices of degree 1 and 2 of F_{n-1}, which can be described as

$$F_n = \mathcal{B}_{2f_n}(F_{n-1}, G_{B_n}),$$

where the graph G_{B_n} has an $M \times M$ adjacency matrix B_n of the block diagonal form:

$$B_n = diag(\underbrace{B, \cdots, B}_{f_n\ times}).$$

Here $M = 6f_n$ and $B = A_{F_0}$ is the adjacency matrix to the graph F_0. Therefore

$$A_{F_n} = \begin{pmatrix} A_{F_{n-1}} & H_n \\ H_n^T & B_n \end{pmatrix}$$

where $H_n = (H_n^1, \cdots, H_n^{f_n})$ is an $N \times M$ block matrix with $N = |V(F_{n-1})|$. Each H_n^r is an $N \times 6$ $\{0,1\}$-matrix of the form $H_n^r = (u^r, v^r, 0, 0, 0, 0)$ where $u_i^r = 1$ ($v_i^r = 1$) if and only if the vertex 1 (2) of the r-th fulvene graph F_0 is bridged to the i-th vertex of F_{n-1}.

In order to apply the spectral estimate from Theorem 4 we will derive an upper bound on the optimum value of $\mu^* = \max\sigma(B_n^{-1}H_n^T H_n B_n^{-1})$. Clearly, the matrix $B_n^{-1}H_n^T H_n B_n^{-1}$ satisfies

$$(B_n^{-1} H_n^T H_n B_n^{-1})_{rs} = B^{-1} (H_n^r)^T H_n^s B^{-1}.$$

Now,

$$(H_n^r)^T H_n^s = \begin{cases} diag(1,1,0,0,0,0), & \text{if } r = s, \\ diag(1,0,0,0,0,0), & \text{if } r \neq s \text{ and the } r\text{-th and } s\text{-th graph } F_0 \\ & \text{are bridged to the same vertex of } F_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 & -2 \end{pmatrix}$$

it can be verified by an easy calculation that

$$\max \sigma(B^{-1} (H_n^r)^T H_n^s B^{-1}) = \begin{cases} 3, & \text{if } r = s, \\ 2, & \text{if } r \neq s \text{ and the } r\text{-th and } s\text{-th graph } F_0 \\ & \text{are bridged to the same vertex of } F_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for any vector $z = (z^1, \cdots, z^{f_n}) \in \mathbb{R}^M$, $z^i \in \mathbb{R}^6$, we have

$$\begin{aligned} z^T B_n^{-1} H_n^T H_n B_n^{-1} z &= \sum_{r,s=1}^{f_n} (z^r)^T B^{-1} (H_n^r)^T H_n^s B^{-1} z^s \\ &\leq 3\|z\|^2 + \sum_{r \neq s} (z^r)^T B^{-1} (H_n^r)^T H_n^s B^{-1} z^s \\ &\leq 3\|z\|^2 + 2 \sum_{r \neq s} \frac{1}{2} (\|z^r\|^2 + \|z^s\|^2) \leq 5\|z\|^2, \end{aligned}$$

because for the symmetric matrix $W = B^{-1} (H_n^r)^T H_n^s B^{-1}$ it holds: $|u^T W v| \leq \max |\sigma(W)| \frac{1}{2} (\|u\|^2 + \|v\|^2)$. Hence,

$$\mu^* = \max \sigma(B_n^{-1} H_n^T H_n B_n^{-1}) \leq 5.$$

Finally, we establish the lower bound for the least positive eigenvalue $\lambda_1^+(F_n)$. With regard to Theorem 4 we have

Fig. 4. The family of graphs on 4 vertices with a unique 1-factor.

$$\lambda_1^+(F_n) \geq \frac{2}{\alpha(\mu^*+1)+\beta+\sqrt{(\alpha(\mu^*+1)+\beta)^2-4\alpha\beta}},$$

where $\alpha = 1/\lambda_1^+(F_{n-1}), \beta = 1/\lambda_1^+(G_{B_n}) = 1/\lambda_1^+(F_0) = q$. If we denote $y_n = 1/\lambda_1^+(F_n)$ we obtain

$$\begin{aligned} y_n &\leq \frac{1}{2}\left((\mu^*+1)y_{n-1}+q+\sqrt{((\mu^*+1)y_{n-1}+q)^2-4qy_{n-1}}\right) \\ &\leq (\mu^*+1)y_{n-1}+q \leq 6y_{n-1}+q. \end{aligned}$$

Solving the above difference inequality yields $y_n \leq \frac{q}{5}(6^{n+1}-1)$ and so

$$\lambda_1^+(F_n) \geq \frac{1}{q}\frac{5}{6^{n+1}-1},$$

as claimed. $\diamondsuit$

Remark 3. The asymptotic behavior of $\lambda_1^+(F_n) \to 0$ as $n \to \infty$ is not surprising. For example, if we consider a cycle C_N on N vertices then, we have $\lambda_1^+(C_N) = 2\cos\frac{\pi}{2}\frac{N-1}{N}$ and so $\lambda_1^+(C_N) = O(N^{-1}) = O(|V(C_N)|^{-a})$ with the polynomial decay rate $a = 1$. In the case of the graph F_n the number of its vertices growths exponentially $N = O(q^{n+1})$, and so the lower bound $\lambda_1^+(F_n) \geq O(6^{-n-1}) = O(|V(F_n)|^{-a})$ with the polynomial decay rate $a = \ln 6/\ln q \doteq 3.7234$ as $|V(F_n)| \to \infty$ can be expected.

6. Arbitrarily bridgeable connected graphs with a unique 1-factor

In this section we present a census of invertible graphs on $m \leq 6$ vertices with a unique 1-factor, such that they can be arbitrarily bridged to an invertible graph through a set of $k \leq m/2$ vertices. Recall that a graph G has a unique 1-factor if G contains a unique 1-regular spanning subgraph (i.e., a perfect matching). Note that any graph having a 1-factor should have even number of vertices.

For $m = 2$ the graph K_2 is the unique connected graph with a unique 1-factor. It is a positively invertible bipartite graph with the spectrum $\sigma(K_2) = \{-1, 1\}$.

For $m = 4$ there are two connected graphs Q_1, Q_2 with a unique 1-factor shown in Fig. 4. Both graphs are positively invertible with the spectra

$$\sigma(Q_1) = \{\pm 1.6180, \pm 0.6180\}, \qquad \sigma(Q_2) = \{-1.4812, -1, 0.3111, 2.1701\}.$$

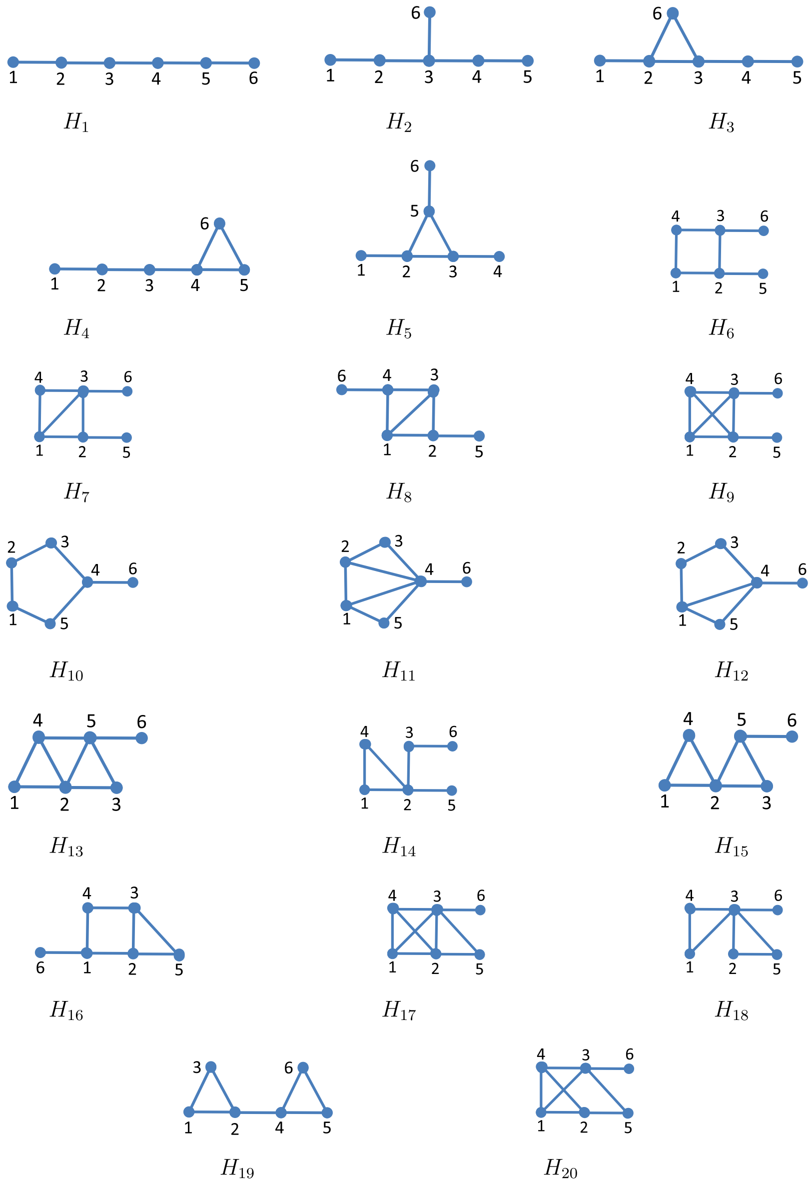

Fig. 5. The family of graphs on 6 vertices with a unique 1-factor.

The graph Q_1 can be arbitrarily bridged over the singleton sets $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and over pairs of vertices: $\{2,3\}, \{1,3\}, \{2,4\}$. The graph Q_1 can be arbitrarily bridged over the singletons $\{2\}, \{3\}, \{4\}$ and over the pair $\{2,3\}$.

The situation is more interesting and, at the same time, more complicated, for connected graphs on $m = 6$ vertices with a unique 1-factor. To this end, we recall the well-known Kotzig's theorem stating that a graph with a unique 1-factor has a bridge

Table 1
The family of graphs on 6 vertices with a unique 1-factor, their signability and spectrum. Graphs H_8 and H_{18} are iso-spectral but not isomorphic.

Graph	invertibility	spectrum
H_1	pos, neg	$\{-1.8019, -1.2470, -0.4450, 0.4450, 1.2470, 1.8019\}$
H_2	pos, neg	$\{-1.9319, -1.0000, -0.5176, 0.5176, 1.0000, 1.9319\}$
H_3	pos	$\{-1.7397, -1.3738, -0.5945, 0.2742, 1.0996, 2.3342\}$
H_4	pos	$\{-1.7746, -1.0000, -1.0000, 0.1859, 1.3604, 2.2283\}$
H_5	neg	$\{-1.6180, -1.6180, -0.4142, 0.6180, 0.6180, 2.4142\}$
H_6	pos, neg	$\{-2.2470, -0.8019, -0.5550, 0.5550, 0.8019, 2.2470\}$
H_7	pos	$\{-1.8942, -1.3293, -0.6093, 0.3064, 0.7727, 2.7537\}$
H_8	pos	$\{-1.9032, -1.0000, -1.0000, 0.1939, 1.0000, 2.7093\}$
H_9	pos	$\{-1.6180, -1.3914, -1.0000, 0.2271, 0.6180, 3.1642\}$
H_{10}	neg	$\{-1.8608, -1.6180, -0.2541, 0.6180, 1.0000, 2.1149\}$
H_{11}	int inv	$\{-1.8241, -1.6180, -0.5482, 0.3285, 0.6180, 3.0437\}$
H_{12}	neg	$\{-2.1420, -1.3053, -0.3848, 0.4669, 0.7661, 2.5991\}$
H_{13}	pos	$\{-1.8563, -1.4780, -0.7248, 0.1967, 0.8481, 3.0143\}$
H_{14}	pos	$\{-1.9202, -1.0000, -0.7510, 0.2914, 1.0000, 2.3799\}$
H_{15}	pos	$\{-1.6783, -1.3198, -1.0000, 0.1397, 1.2297, 2.6287\}$
H_{16}	pos	$\{-2.1364, -1.2061, -0.5406, 0.2611, 1.0825, 2.5395\}$
H_{17}	pos	$\{-1.8619, -1.2827, -1.0000, 0.2512, 0.4897, 3.4037\}$
H_{18}	pos	$\{-1.9032, -1.0000, -1.0000, 0.1939, 1.0000, 2.7093\}$
H_{19}	nonint inv	$\{-1.7321, -1.0000, -1.0000, -0.4142, 1.7321, 2.4142\}$
H_{20}	pos	$\{-2.3117, -1.0000, -0.6570, 0.3088, 0.7272, 2.9327\}$

'pos'/'neg' stands for a positively/negatively invertible graph, 'int inv' means an integrally invertible graph which is neither positively nor negatively invertible, 'nonint inv' stands for a graph with an adjacency matrix which is invertible but it is not integral.

that belongs to the 1-factor sub-graph. Splitting of 6 vertices into two subsets of 3 vertices connected by a bridge leads to graphs H_1, H_4, H_{19} shown in Fig. 5. Splitting into subsets of 2 and 4 vertices is impossible because the bridge should belong to the 1-factor and so the hanging leaf vertex of a 2-vertices sub-graph is not contained in the 1-factor. Splitting into a 1 vertex graph and 5-vertices graph lead to the remaining 17 graphs shown in Fig. 5. One can construct these 17 graphs from the set of all 10 graphs on four vertices (including disconnected graphs) by bridging to K_2 using up to 4 edges.

In summary, there exist 20 undirected connected graphs on $m = 6$ vertices with a unique 1-factor shown in Fig. 5. (See Table 1.) All of them have invertible adjacency matrix. Except of the graph H_{19} they are integrally invertible.

In this census, there are three bipartite graphs H_1, H_2, H_6 which are simultaneously positively and negatively invertible. There are twelve graphs

$$H_3, H_4, H_7, H_8, H_9, H_{13}, H_{14}, H_{15}, H_{16}, H_{17}, H_{18}, H_{20},$$

which are positively invertible. The three graphs H_5, H_{10}, H_{12} are negatively invertible. The integrally invertible graph H_{11} is neither positively nor negatively invertible. The graphs H_8 and H_{18} are iso-spectral but not isomorphic.

Acknowledgements

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Table 2
The family of graphs on 6 vertices with a unique 1-factor which can be arbitrarily bridged through $k = 1, 2, 3$ vertices.

Graph	$k = 1$	$k = 2$	$k = 3$
H_1	$\{6\}, \{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{4,6\}, \{2,6\}, \{4,5\}, \{3,5\}, \{2,5\}, \{1,5\}, \{2,4\}, \{2,3\}, \{1,3\}$	$\{2,4,6\}, \{2,4,5\}, \{2,3,5\}, \{1,3,5\}$
H_2	$\{6\}, \{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{4,6\}, \{2,6\}, \{3,5\}, \{2,5\}, \{1,5\}, \{3,4\}, \{2,4\}, \{1,4\}, \{2,3\}, \{1,3\}$	$\{2,4,6\}, \{2,3,5\}, \{1,3,5\}, \{2,3,4\}, \{1,3,4\}$
H_3	$\{6\}, \{5\}, \{4\}, \{3\}, \{2\}$	$\{4,6\}, \{2,6\}, \{3,5\}, \{2,5\}, \{3,4\}, \{2,4\}, \{2,3\}$	$\{2,4,6\}, \{2,3,5\}, \{2,3,4\}$
H_4	$\{6\}, \{5\}, \{4\}, \{2\}$	$\{4,6\}, \{2,6\}, \{4,5\}, \{2,5\}, \{2,4\}$	$\{2,4,6\}, \{2,4,5\}$
H_5	$\{6\}, \{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{3,6\}, \{2,6\}, \{4,5\}, \{3,5\}, \{2,5\}, \{1,5\}, \{2,4\}, \{2,3\}, \{1,3\}$	$\{2,3,6\}, \{2,4,5\}, \{2,3,5\}, \{1,3,5\}$
H_6	$\{6\}, \{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{5,6\}, \{4,6\}, \{2,6\}, \{3,5\}, \{1,5\}, \{3,4\}, \{2,4\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{2,4,6\}, \{1,3,5\}, \{2,3,4\}, \{1,2,3\},$
H_7	$\{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{3,5\}, \{1,5\}, \{3,4\}, \{2,4\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{1,3,5\}, \{2,3,4\}, \{1,2,3\}$
H_8	$\{4\}, \{3\}, \{2\}, \{1\}$	$\{3,4\}, \{2,4\}, \{1,4\}, \{2,3\}, \{1,2\}$	$\{2,3,4\}, \{1,2,4\}$
H_9	$\{4\}, \{3\}, \{2\}, \{1\}$	$\{3,4\}, \{2,4\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{2,3,4\}, \{1,2,3\}$
H_{10}	$\{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{4,5\}, \{2,5\}, \{3,4\}, \{2,4\}, \{1,4\}, \{1,3\}, \{1,2\}$	$\{2,4,5\}, \{1,3,4\}, \{1,2,4\}$
H_{11}	$\{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{4,5\}, \{2,5\}, \{3,4\}, \{2,4\}, \{1,4\}, \{1,3\}, \{1,2\}$	$\{2,4,5\}, \{1,3,4\}, \{1,2,4\}$
H_{12}	$\{6\}, \{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{5,6\}, \{4,5\}, \{2,5\}, \{3,4\}, \{2,4\}, \{1,4\}, \{1,3\}, \{1,2\}$	$\{2,4,5\}, \{1,3,4\}, \{1,2,4\}$
H_{13}	$\{5\}, \{4\}, \{2\}, \{1\}$	$\{4,5\}, \{2,5\}, \{1,5\}, \{2,4\}, \{1,2\}$	$\{2,4,5\}, \{1,2,5\}$
H_{14}	$\{6\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{4,6\}, \{2,6\}, \{1,6\}, \{3,4\}, \{2,4\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{2,4,6\}, \{1,2,6\}, \{2,3,4\}, \{1,2,3\}$
H_{15}	$\{5\}, \{4\}, \{2\}, \{1\}$	$\{4,5\}, \{2,5\}, \{1,5\}, \{2,4\}, \{1,2\}$	$\{2,4,5\}, \{1,2,5\}$
H_{16}	$\{6\}, \{5\}, \{3\}, \{2\}, \{1\}$	$\{5,6\}, \{3,5\}, \{1,5\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{1,3,5\}, \{1,2,3\}$
H_{17}	$\{4\}, \{3\}, \{2\}, \{1\}$	$\{3,4\}, \{2,4\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{2,3,4\}, \{1,2,3\}$
H_{18}	$\{5\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{4,5\}, \{3,5\}, \{1,5\}, \{3,4\}, \{2,4\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{3,4,5\}, \{1,3,5\}, \{2,3,4\}, \{1,2,3\}$
H_{19}	–	–	–
H_{20}	$\{6\}, \{4\}, \{3\}, \{2\}, \{1\}$	$\{4,6\}, \{1,6\}, \{3,4\}, \{2,4\}, \{2,3\}, \{1,3\}, \{1,2\}$	$\{1,2,3\}, \{2,3,4\}$

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