INVERTIBILITY OF GRAPHS WITH A UNIQUE PERFECT MATCHING

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Abstract.   In this paper we investigate invertibility of graphs with a unique perfect matching, i.e. graphs having a unique 1-factor. We recall the new notion of the so-called negatively invertible graphs investigated by the authors in the recent paper. It is an extension of the classical definition of an inverse graph due to Godsil. We characterize all graphs with a unique perfect matching on $m \leq 6$ vertices with respect to their positive and negative invertibility. We show that negatively invertible graphs exhibit properties like selfinvertibility which cannot be observed within the class of positively invertible non-bipartite graphs with a unique perfect matching.

Keywords: Integrally invertible graph, positively an negatively invertible graphs, 1-factor, unique perfect matching

Mathematics subject classification: Primary 05C50; Secondary 15A09

1 Introduction and preliminaries

The main purpose of this paper is to investigate invertibility of graphs with a unique perfect matching (graphs having a unique 1-factor) (c.f. Pavlíková *et al.* [10, 9]). Following our recent paper [12], we recall the new notion of the so-called negatively invertible graphs. It is an extension of the classical definition of an inverse graph due to Godsil [4]. Our goal is to characterize all graphs with a unique perfect matching on $m \leq 6$ vertices with respect to their positive and negative invertibility. To our best knowledge, the concept of negative invertibility of graphs is a novel approach in the theory of graph inverses when compared to the classical concept of (positive) invertibility of graphs due to Godsil.

According to the result due to Sachs (c.f. [2, p. 32]), it is known that, for a bipartite graph G having a unique perfect matching, the adjacency matrix A is integrally invertible. The aim of this paper is to investigate a larger class of graphs which are not necessarily bipartite but still have a unique 1-factor and an integrally invertible adjacency matrix.
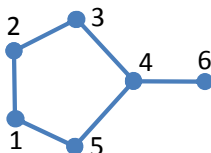
Fig. 1. A graph representing the structural graph of a chemical molecule of fulvene

Let $G = (V, E)$ be an undirected graph with the set of vertices V and the set of edges E. By A_G we denote its adjacency matrix. Conversely, if A is a $\{0, 1\}$-symmetric matrix then G_A denotes the graph with the adjacency matrix A. The spectrum $\sigma(G)$ of an undirected graph G consists of eigenvalues of its adjacency symmetric matrix A_G, i.e. $\sigma(G) = \sigma(A_G) = \{\lambda, \lambda$ is an eigenvalue of $A_G\}$ (c.f. [3, 2]). If the spectrum does not contain zero then the adjacency matrix A is invertible.

Clearly, a graph G_A has the integral inverse matrix A^{-1} if and only if the determinant $\det(A) = \pm 1$ (c.f. Kirkland and Akbari [6]). The inverse matrix A^{-1} however may contain positive as well as negative entries. In fact, it is well known that the inverse matrix of the adjacency matrix of a graph is nonnegative if and only if the graph is a union of isolated edges (c.f. Harary [5]). Nevertheless, the concept of an inverse graph is indeed based on the inverse of its adjacency matrix. In [4] Godsil defined a graph to be invertible if the inverse of its (non-singular) adjacency matrix is diagonally similar (c.f. [13]) to a non-negative matrix.

Definition 1 (Godsil [4]) *An undirected vertex-labeled graph G_A is called positively invertible (or invertible in Godsil's sense) if inverse of its adjacency matrix is signable to a non-negative integer matrix, i.e. there exists a diagonal matrix D^A containing ± 1 elements and such that all elements of the matrix $D^A A^{-1} D^A$ are nonnegative integers only. The inverse graph $H = G_A^{-1}$ is defined by the adjacency matrix $A_H = D^A A^{-1} D^A$.*

Unfortunately, the class of positively invertible graphs does not contain many important examples of graphs having integral inverse matrix. Among them there is a graph representing the chemical organic molecule of fulvene (see Fig. 1). The inverse of its adjacency matrix is integral but it is not signable to a nonnegative matrix.

The structural graph of fulvene however plays an important role in computational chemistry as it represents a chemical molecule. For instance, spectral properties of inverse graphs are important in order to compute the so-called binding energies of molecular orbitals.

Motivated by this shortcoming of positive invertibility of a graph, in the recent paper [12], Pavlíková and Ševčovič introduced the new concept of the so-called negative invertibility of a graph with integral inverse of its adjacency matrix.

Definition 2 (Pavlíková and Ševčovič [12]) *An undirected vertex-labeled graph G_A is called negative invertible if the inverse matrix of its adjacency matrix is signable to a nonpositive integer matrix, i.e. there exists a diagonal matrix D^A containing ± 1 elements and such that all elements of the matrix $D^A A^{-1} D^A$ are nonpositive integers only. The inverse graph $H = G_A^{-1}$ is defined by the adjacency matrix $A_H = -D^A A^{-1} D^A$.*

The adjacency matrix A of the fulvene graph and its inverse matrix A^{-1} are as follows:

$$
A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad
A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 & -2 \end{pmatrix}.
$$

The matrix A^{-1} is signable to a nonpositive matrix by the diagonal matrix $D^A = diag(1, 1, -1, -1, -1, 1)$.

Both positive as well as negatively invertible graphs have the following useful property. If $G = G_A$ is positively invertible then $A_H = D^A A^{-1} D^A$ is the adjacency matrix of the inverse graph $H = G^{-1}$. As $A = D^A A_H^{-1} D^A$ we have the identity $G = (G^{-1})^{-1}$. This property also remains true in the case of negatively invertible graphs. Indeed, if $G = G_A$ is negatively invertible then $A_H = -D^A A^{-1} D^A$ is the adjacency matrix of the inverse graph $H = G^{-1}$. As $A = -D^A A_H^{-1} D^A$ we again obtain $G = (G^{-1})^{-1}$.

In the next section we recall the complete list of graphs with a unique perfect matching on $m \leq 6$ vertices discovered by Pavlíková and Ševčovič [12]. In addition, we present their inverse graphs and maximal subgraphs with a unique perfect matching. Interestingly enough, there are no positively invertible non-bipartite graphs with a unique perfect matching that are selfinvertible. On the other hand, there exist two negatively invertible graphs which are selfinvertible.

2 Connected graphs with a unique 1-factor

In this section we present the complete list of invertible graphs on $m \leq 6$ vertices with a unique 1-factor. Recall that a graph G has a unique 1-factor if there exists a unique subgraph spanning G and such that each vertex has the degree 1. Notice that any graph having a 1-factor should have even number of vertices.

For $m = 2$ the graph K_2 is the unique connected graph and it has a unique 1-factor. It is a positively and negatively invertible bipartite graph.
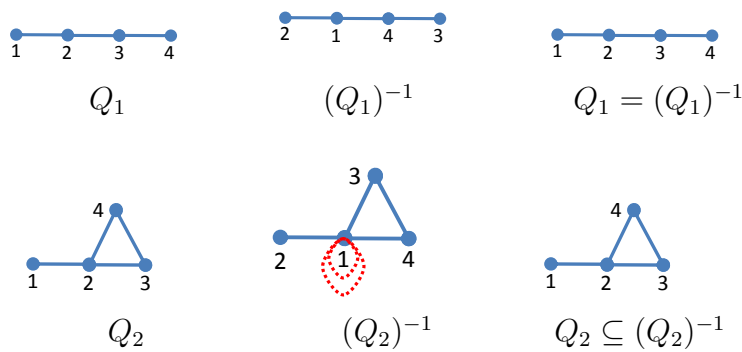
Fig. 2. Invertible graphs on four vertices with a unique perfect matching, their inverse graphs and maximal subgraphs with a unique perfect matching

For $m = 4$ there are two connected graphs Q_1, Q_2 with a unique 1-factor shown in Fig. 2. The graph Q_1 is bipartite and selfinvertible. The graph Q_2 is positively invertible but it is not selfinvertible.

Next we analyse the connected graphs on $m = 6$ vertices with a unique 1-factor. We have the following theorem.

Theorem 1 *[12] There exist 20 undirected connected graphs on $m = 6$ vertices with a unique 1-factor shown in Figures 3,4,5,6,7. All of them have invertible adjacency matrices. Except of the graph H_{19} they are integrally invertible.*

There are three bipartite graphs H_1, H_2, and H_6 which are simultaneously positively and negatively invertible. There are twelve graphs

$$H_3, H_4, H_7, H_8, H_9, H_{13}, H_{14}, H_{15}, H_{16}, H_{17}, H_{18}, H_{20},$$

which are positively invertible. The three graphs H_5, H_{10}, and H_{12} are negatively invertible. The integrally invertible graph H_{11} is neither positively nor negatively invertible. The graphs H_8, and H_{18} are iso-spectral but not isomorphic.

Recall that the proof of this result is based on the well-known Kotzig's theorem. It states that a graph with a unique 1-factor should contain a bridge belonging to the 1-factor sub-graph. Using this characterization one can construct all 20 connected graphs on 6 vertices having a unique 1-factor. We refer the reader to [12] for details.

Tab. 1. The family of graphs on 6 vertices with a unique 1-factor and their signability. Source: [12]

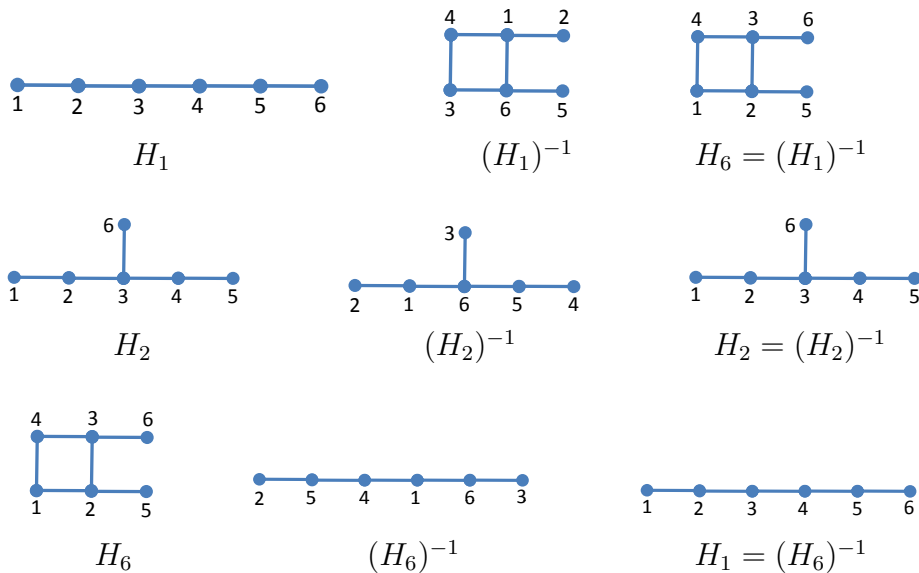| Graph	invertibility
H_1	positively and negatively invertible (bipartite)
H_2	positively and negatively invertible (bipartite)
H_3	positively invertible
H_4	positively invertible
H_5	negatively invertible
H_6	positively and negatively invertible (bipartite)
H_7	positively invertible
H_8	positively invertible
H_9	positively invertible
H_{10}	negatively invertible
H_{11}	noninvertible with integral inverse of the adjacency matrix
H_{12}	negatively invertible
H_{13}	positively invertible
H_{14}	positively invertible
H_{15}	positively invertible
H_{16}	positively invertible
H_{17}	positively invertible
H_{18}	positively invertible
H_{19}	noninvertible with nonintegral inverse of the adjacency matrix
H_{20}	positively invertible

Fig. 3. Positively and negatively invertible (bipartite) graphs with a unique perfect matching, their inverses and maximal subgraphs with a unique perfect matching

2.1 Positively and negatively invertible (bipartite) graphs with a unique perfect matching on $m = 6$ vertices

Recall that a graph G_B is called bipartite if the set of vertices can be divided into two disjoint subsets such that every edge connects a vertex from the first subset into the one from the second subset.

Theorem 2 (Pavlíková and Ševčovič [12]) *Let G be an integrally invertible graph. Then G is a bipartite graph if and only if G is simultaneously positively and negatively invertible.*

In [8] McLeman and McNicholas considered graphs which can be obtained from a given connected graph by attaching pendant edges to each of vertices. They proved that such graphs are selfinvertible, i.e. $H^{-1} = H$. An example illustrating their result is the graph H_2 which is obtained from a path with three vertices by adding pendant edges to each of them.

In Fig. 3 we show all three bipartite graphs on $m = 6$ vertices with a unique perfect matching. According to the previous theorem they are simultaneously positive as well as negative invertible.

2.2 Positively invertible graphs with a unique perfect matching on $m = 6$ vertices

In this section we present the complete list of twelve non-bipartite graphs with a unique perfect matching on $m = 6$ vertices which are positively invariant (see Fig. 4,5). Interestingly, none of them is selfinvertible and their inverse graphs contain vertices with loops with even multiplicity. The inverse graphs $(H_4)^{-1}, (H_8)^{-1}, (H_{13})^{-1}, (H_{15})^{-1}$ contain multiple edges. There are four graphs denoted by $H_9, H_{13}, H_{14},$ and H_{18} such that they are maximal subgraphs of their inverse graphs, i.e. $H_i \subseteq (H_i)^{-1}, i = 9, 13, 14, 18$. There are two pairs of graphs (H_i, H_j) such that $H_i \subseteq (H_j)^{-1}$ and $H_j \subseteq (H_i)^{-1}$ for pairs $(i, j) \in \{(3, 7), (15, 17)\}$.

2.3 Negatively invertible graphs with a unique perfect matching on $m = 6$ vertices

In this section we present the complete list of all three non-bipartite graphs with a unique perfect matching on $m = 6$ vertices which are negatively invariant. The graphs H_5, and H_{12} are selfinvertible. Recall that according to the results due to McLeman and McNicholas [8] a graph which is obtained from a given connected graph by attaching pendant edges to each of vertices is selfinvertible. Among negatively invertible graphs with unique perfect matching on $m = 6$ vertices as an example illustrating this property one can present the graph H_5. In contrast to positively invertible bipartite graphs there exists the graph H_{12} which is selfinvertible but it cannot be obtained by means of the construction due to McLeman and McNicholas.

2.4 Noninvertible graphs with a unique perfect matching on $m = 6$ vertices

In this section we present two remaining graphs on $m = 6$ vertices which are not invertible. The first graph is denoted by H_{11} and it has the adjacency matrix which is integrally invertible. Nevertheless, A^{-1} is neither positively nor negatively signable, so H_{11} is not invertible. The graph H_{19} has the adjacency matrix with $\det(A) = 3$ and, as a consequence, A^{-1} is not integer matrix. Hence H_{19} is noninvertible as well.
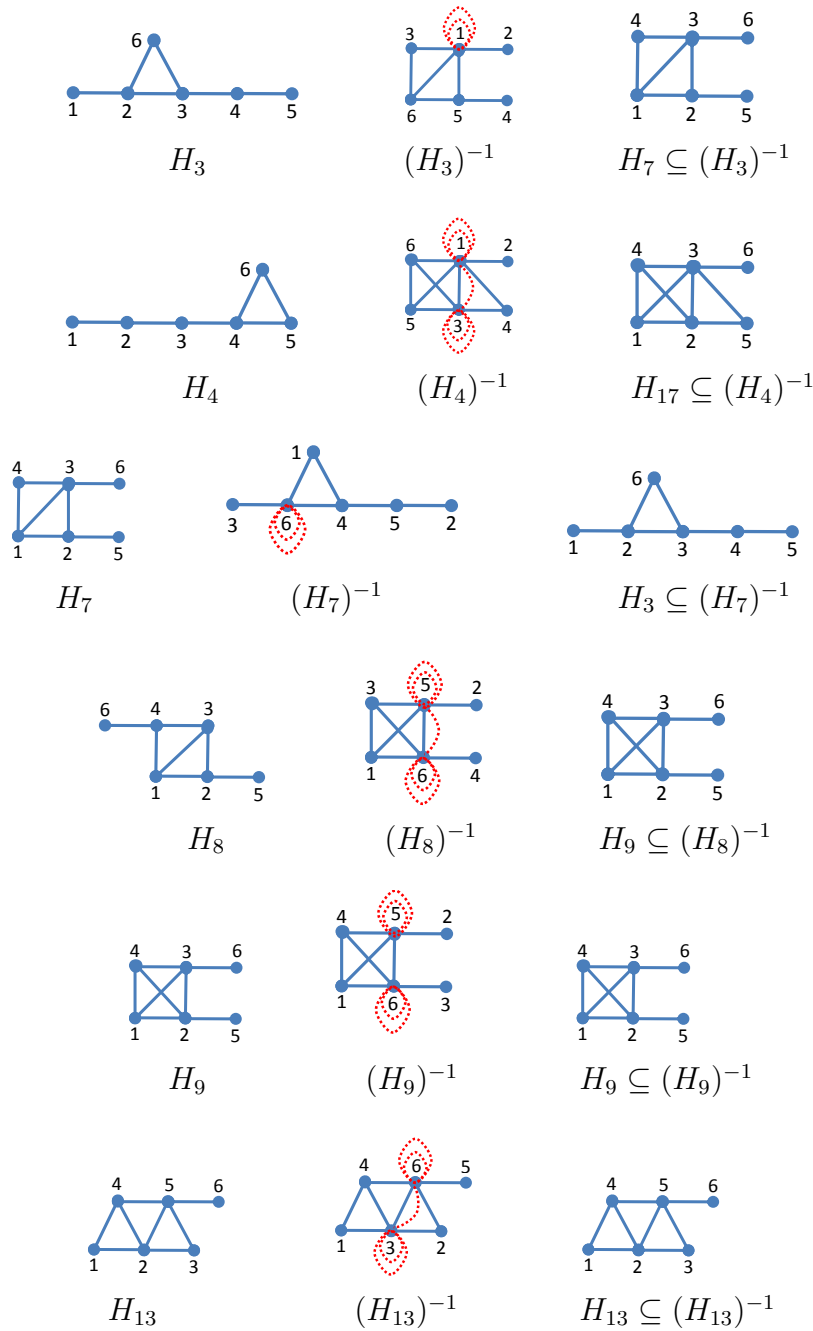
Fig. 4. Positively invertible graphs with a unique perfect matching, their inverses and maximal subgraphs with a unique perfect matching (1. part)

3 Conclusions

In this note we investigated invertible graphs with a unique perfect matching on $m \leq 6$ vertices. We recalled classical concept of graph inversion due to Godsil which we referred to as positive invertibility of a graph. We also recalled the new concept of negative invertibility proposed by the authors in [12]. By inspecting properties of the inverse graphs we showed that negatively invertible graphs exhibits properties like selfinvertibility which cannot be observed within the class of positively invertible non-bipartite graphs with a unique perfect matching on $m \leq 6$ vertices.
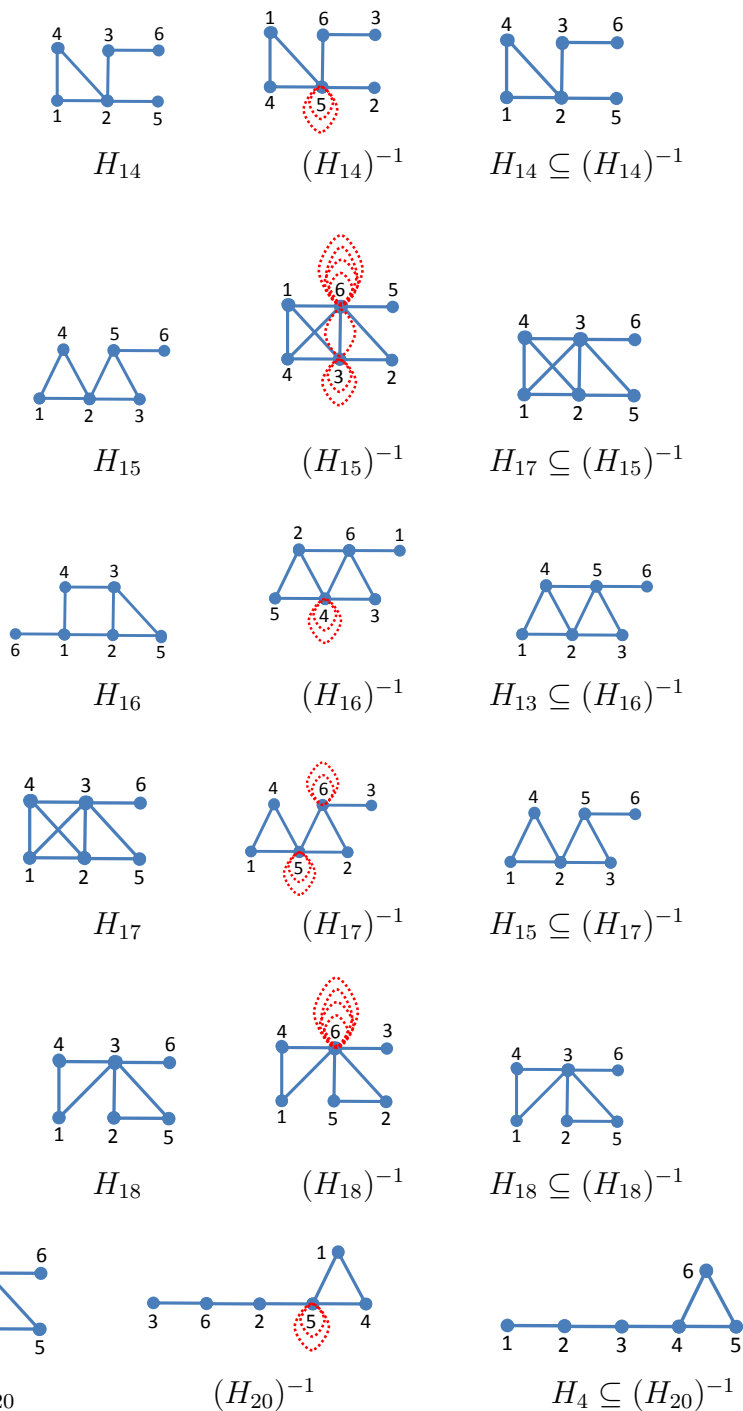
Fig. 5. Positively invertible graphs with a unique perfect matching, their inverses and maximal subgraphs with a unique perfect matching (2. part)
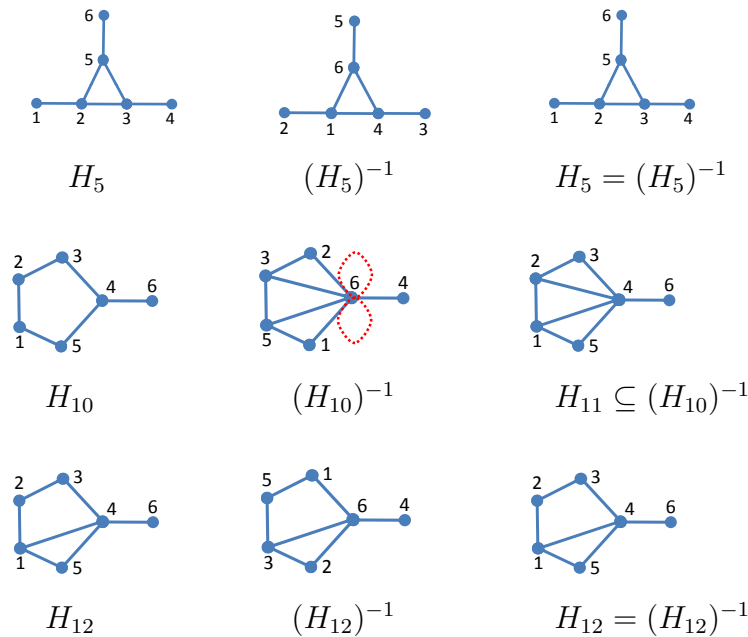
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Fig. 6. Negatively invertible graphs with a unique perfect matching, their inverses and maximal subgraphs with a unique perfect matching
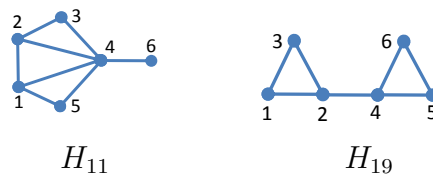

Fig. 7. Noninvertible graphs with a unique perfect matching: H_{11} has integral inverse of its adjacency matrix but it is neither positively nor negatively invertible, H_{19} is not integrally invertible

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