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Pricing American call options using the Black–Scholes equation with a nonlinear volatility function

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ABSTRACT

In this paper, we investigate a nonlinear generalization of the Black–Scholes equation for pricing American-style call options, where the volatility term may depend on both the underlying asset price and the Gamma of the option. We propose a numerical method for pricing American-style call options that involves transforming the free boundary problem for a nonlinear Black–Scholes equation into the so-called Gamma variational inequality with a new variable depending on the Gamma of the option. We apply a modified projected successive over-relaxation method in order to construct an effective numerical scheme for discretization of the Gamma variational inequality. Finally, we present several computational examples of the

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nonlinear Black–Scholes equation for pricing American-style call options in the presence of variable transaction costs.

Keywords: variational inequality; finite-difference scheme; American option pricing; nonlinear Black–Scholes equation; variable transaction costs; projected successive over-relaxation (PSOR) method.

1 INTRODUCTION

In a stylized financial market, the price of a European-style option can be computed by solving the well-known Black–Scholes linear parabolic equation derived by Black and Scholes (1973). Recall that a European call option gives its owner the right, but no obligation, to purchase an underlying asset at the expiration price E at the expiration time T. In this paper, we consider American-style options that can be exercised anytime in the time interval $[0, T]$.

The classical linear Black–Scholes model was derived under several restrictive assumptions, namely the presence of no transaction costs; frictionless, liquid and complete markets, etc. However, more realistic models are required for market data analysis to overcome the drawbacks due to these restrictions of the classical Black–Scholes theory. One of the first nonlinear models to take transaction costs into account is the jumping volatility model of Avellaneda and Paras (1994). A nonlinear modification of the original Black–Scholes model can also arise from feedback and illiquid market effects due to the influence of large traders choosing given stock-trading strategies (Frey and Patie 2002; Frey and Stremme 1997; Schönbucher and Wilmott 2000), imperfect replication and investors' preferences (Barles and Soner 1998), and the risk from unprotected portfolios (Jandačka and Ševčovič 2005; Kratka 1998; Ševčovič 2009). In this paper, we focus on a new nonlinear model that was derived recently by Ševčovič and Žitňanská (2016) for pricing call or put options in the presence of variable transaction costs. This model generalizes the well-known Leland model with a constant transaction costs function (see Hoggard *et al* 1994; Leland 1985) and the Amster *et al* (2005) model with a linearly decreasing transaction costs function. It leads to the following generalized Black–Scholes equation with a nonlinear volatility function $\hat{\sigma}$ depending on the product $H = S\partial_S^2 V$ of the underlying asset price S and the second derivative (Gamma) of the option price V:

$$\partial_t V + \tfrac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2 \partial_S^2 V + (r-q)S\partial_S V - rV = 0, \quad V(T, S) = (S-E)^+, \tag{1.1}$$

where $r, q \geqslant 0$ are the interest rate and the dividend yield, respectively. The price $V(t, S)$ of such a call option, in the presence of variable transaction costs, is given by a solution to the nonlinear parabolic equation (1.1) depending on the underlying stock price $S > 0$ at time $t \in [0, T]$, where $T > 0$ is the time of maturity and $E > 0$ is the exercise price.

For European-style call options, various numerical methods for solving the fully nonlinear parabolic equation (1.1) were proposed and analyzed by Ďuriš *et al* (2016). Meanwhile, Ševčovič (2007) and Ševčovič and Žitňanská (2016) investigated a new transformation technique (referred to as the Gamma transformation). They showed that the fully nonlinear parabolic equation (1.1) can be transformed into a quasilinear parabolic equation:

$$\partial_\tau H - \partial_u^2 \beta(H) - \partial_u \beta(H) - (r-q)\partial_u H + qH = 0, \quad \text{where } \beta(H) = \hat{\sigma}(H)^2 H/2, \tag{1.2}$$

of a porous-media type for the transformed quantity $H(\tau, u) = S\partial_S^2 V(t, S)$, where $\tau = T - t$, $u = \ln(S/E)$.

The advantage of solving the quasilinear parabolic equation in the divergent form (1.2) as opposed to the fully nonlinear equation (1.1) is twofold. First, from an analytical point of view, the theory of existence and uniqueness of solutions to quasilinear parabolic equations of the form (1.2) is well developed and understood. Using the general theory of quasilinear parabolic equations due to Ladyženskaya *et al* (1968), the existence of Hölder smooth solutions to (1.2) has been shown in Ševčovič and Žitňanská (2016). Second, the quasilinear parabolic equations in the divergent form can be numerically approximated by means of the finite-volume method (see LeVeque 1985). Further, the semi-implicit approximation scheme proposed in Section 4 fits into a class of numerical methods that have been shown to be of the second-order of convergence (see, for example, Kilianová and Ševčovič 2013). In a series of papers (Koleva 2011; Koleva and Vulkov 2013, 2016, 2017), Koleva investigated the transformed Gamma equation (1.2) for pricing European-style call and put options. They also derived the second-order positivity preserving numerical scheme for solving (1.1) and (1.2).

Our goal is to study American-style call options that can be described using the solution to a free boundary problem for a parabolic equation. Their prices can be computed by means of the generalized Black–Scholes equation with a nonlinear volatility function of the form (1.1). If the volatility function is constant, then it is well known that American options can be priced by means of a solution to a linear complementarity problem (see Kwok 1998). Similarly, for the nonlinear volatility model, one can construct a nonlinear complementarity problem involving the variational inequality from the left-hand side of (1.1) and the inequality $V(t, S) \geqslant (S - E)^+$. However, due to the fully nonlinear character of the differential operator in (1.1), directly computing the nonlinear complementarity problem becomes harder and numerically unstable. Therefore, we propose an alternative approach and reformulate the nonlinear complementarity problem in terms of the new transformed variable H for which the differential operator has the form of a quasilinear parabolic operator (see the left-hand side of (1.2)).

In order to apply the Gamma transformation technique to American-style options, we derive a nonlinear complementarity problem for the transformed variable H, and we solve the variational problem by means of a modified projected successive over-relaxation (PSOR) method (see Kwok 1998). Using this method, we compute American-style call option prices for the Black–Scholes nonlinear model to price call options in the presence of variable transaction costs.

This paper is organized as follows. In Section 2, we present a nonlinear option pricing model under variable transaction costs. Section 3 is devoted to the transformation of the free boundary problem into the so-called Gamma variational inequality. In Section 4, we present a finite-volume discretization of the complementarity problem and its solution, obtained using the PSOR method. Finally, in Section 5, we present the results of various numerical experiments for pricing American-style call options, the early exercise boundary position and a comparison with models with constant volatility terms.

2 NONLINEAR BLACK–SCHOLES EQUATION FOR PRICING OPTIONS IN THE PRESENCE OF VARIABLE TRANSACTION COSTS

In the original Black–Scholes theory, continuous hedging of the portfolio including underlying stocks and options is allowed. In the presence of transaction costs for purchasing and selling the underlying stock, this continuous feature may lead to an infinite number of transaction costs, yielding unbounded total transaction costs.

One of the basic nonlinear models that includes transaction costs is the Leland model for option pricing (Leland 1985), where the possibility of rearranging a portfolio at discrete time can be relaxed. Recall that, in the derivation of the Leland model (Hoggard *et al* 1994; Hull 1989; Leland 1985), it is assumed that an investor follows the delta hedging strategy in which the number δ of bought/sold underlying assets depends on the delta of the option, ie, $\delta = \partial_S V$. Then, applying self-financing portfolio arguments, one can derive the extended version of the Black–Scholes equation:

$$\partial_t V + (r - q)S\partial_S V + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 V - rV = r_{\mathrm{TC}} S. \tag{2.1}$$

Here, the transaction cost measure r_{TC} is given by

$$r_{\mathrm{TC}} = \frac{\mathbb{E}[\Delta \mathrm{TC}]}{S\Delta t}, \tag{2.2}$$

where $\Delta\mathrm{TC}$ is the change in transaction costs during a time interval of length $\Delta t > 0$. If $C \geqslant 0$ represents a percentage of the cost of the sale and purchase of a share relative to the price S, then $\Delta\text{TC} = \frac{1}{2}CS|\Delta\delta|$, where $\Delta\delta$ is the number of bought ($\Delta\delta > 0$) or sold ($\Delta\delta < 0$) underlying assets during the time interval Δt. The parameter $C > 0$ measuring transaction costs per unit of the underlying asset may either be constant or depend on the number of transacted underlying assets, ie, $C = C(|\Delta\delta|)$.

Further, assuming the underlying asset follows the geometric Brownian motion $\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}W$, it can be shown that $\Delta\delta = \Delta\partial_S V \approx \sigma S\partial_S^2 V\Phi\sqrt{\Delta t}$, where $\Phi \sim N(0,1)$ is a normally distributed random variable. Hence,

$$r_{\text{TC}} = \frac{1}{2}\frac{\mathbb{E}[C(\alpha|\Phi|)\alpha|\Phi|]}{\Delta t}, \tag{2.3}$$

where $\alpha := \sigma S|\partial_S^2 V|\sqrt{\Delta t}$ (see Jandačka and Ševčovič 2005; Ševčovič *et al* 2011). In order to rewrite (2.1), we recall the mean value modification of the transaction costs function introduced in Ševčovič and Žitňanská (2016).

DEFINITION 2.1 (Ševčovič and Žitňanská 2016, Definition 1) Let $C = C(\xi)$, $C\colon \mathbb{R}_0^+ \to \mathbb{R}$, be a transaction costs function. The integral transformation $\tilde{C}\colon \mathbb{R}_0^+ \to \mathbb{R}$ of the function C,

$$\tilde{C}(\xi) = \sqrt{\frac{\pi}{2}}\mathbb{E}[C(\xi|\Phi|)|\Phi|] = \int_0^\infty C(\xi x)x\,\mathrm{e}^{-x^2/2}\,\mathrm{d}x, \tag{2.4}$$

is called the mean value modification of the transaction costs function. Here, Φ is a random variable with a standardized normal distribution, ie, $\Phi \sim N(0,1)$.

If we assume that $C\colon \mathbb{R}_0^+ \to \mathbb{R}$ is a measurable and bounded transaction costs function, then the price of the option based on these variable transaction costs is given by the solution of the following nonlinear Black–Scholes partial differential equation (PDE) (see Ševčovič and Žitňanská 2016, Proposition 2.1):

$$\partial_t V + (r-q)S\partial_S V + \tfrac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2\partial_S^2 V - rV = 0, \tag{2.5}$$

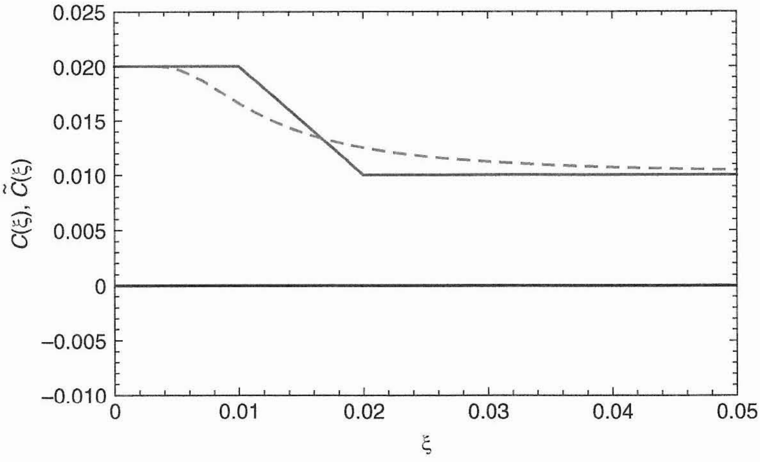
where the nonlinear diffusion coefficient $\hat{\sigma}^2$ is given by

$$\hat{\sigma}(S\partial_S^2 V)^2 = \sigma^2\left(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma S|\partial_S^2 V|\sqrt{\Delta t})\frac{\operatorname{sgn}(S\partial_S^2 V)}{\sigma\sqrt{\Delta t}}\right). \tag{2.6}$$

A realistic example of a piecewise linear decreasing transaction costs function was proposed and analyzed by Ševčovič and Žitňanská (2016). It is written as follows:

$$C(\xi) = \begin{cases} C_0 & \text{if } 0 \leqslant \xi < \xi_-, \\ C_0 - \kappa(\xi - \xi_-) & \text{if } \xi_- \leqslant \xi \leqslant \xi_+, \\ \underline{C_0} \equiv C_0 - \kappa(\xi_+ - \xi_-) & \text{if } \xi \geqslant \xi_+, \end{cases} \tag{2.7}$$

FIGURE 1 A piecewise linear transaction costs function with parameters $C_0 = 0.02$, $\kappa = 1$, $\xi_- = 0.01$, $\xi_+ = 0.02$, and its mean value modification $\tilde{C}(\xi)$ (dashed line).



where $0 < \xi_- \leqslant \xi_+$, $\kappa > 0$, $C_0 > 0$ are model parameters. Such a transaction costs function corresponds to a stylized market in which the investor pays a higher amount C_0 for a small volume of traded assets; however, if the traded volume of stocks is higher, the investor pays a smaller amount $\underline{C_0}$. The modified mean value transaction costs function can be analytically expressed using the following formula:

$$\tilde{C}(\xi) = C_0 - \kappa\xi \int_{\xi_-/\xi}^{\xi_+/\xi} \mathrm{e}^{-u^2/2}\,\mathrm{d}u, \quad \text{for } \xi \geqslant 0 \tag{2.8}$$

(see Ševčovič and Žitňanská 2016, Equation (24)). According to Ševčovič and Žitňanská (2016, Proposition 2.2), lower/upper bounds and limiting behavior exist for the mean value modification of the piecewise linear transaction costs function $\tilde{C}(\xi)$, ie,

$$\underline{C_0} \leqslant \tilde{C}(\xi) \leqslant C_0 \quad \text{and} \quad \lim_{\xi\to\infty} \tilde{C}(\xi) = \lim_{\xi\to\infty} C(\xi) = \underline{C_0}. \tag{2.9}$$

A graph of a piecewise linear transaction costs function C and its mean value modification is depicted in Figure 1.

If the transaction costs function $C \equiv C_0 > 0$ is constant (ie, $\xi_\pm = 0$), we obtain the well-known Leland model (see Hoggard *et al* 1994; Hull 1989; Leland 1985) in which the diffusion term has the form

$$\hat{\sigma}(S\partial_S^2 V)^2 = \sigma^2(1-\mathrm{Le}\,\mathrm{sgn}(\partial_S^2 V)) = \sigma^2(1-\mathrm{Le}\,\mathrm{sgn}(S\partial_S^2 V)), \quad \mathrm{Le} = \sqrt{\frac{2}{\pi}}\frac{C_0}{\sigma\sqrt{\Delta t}},$$

where Le is the Leland number.

Amster *et al* (2005) investigated a linear nonincreasing transaction costs function of the form

$$C(\xi) = C_0 - \kappa\xi, \quad \text{where } \xi \geqslant 0,$$

ie, $\xi_- = 0$, $\xi_+ = \infty$. The mean value modification function has the form

$$\tilde{C}(\xi) = C_0 - \sqrt{\pi/2}\kappa\xi, \quad \text{where } \xi \geqslant 0.$$

Clearly, such a transaction costs function can attain negative values; this can be considered a drawback of this model.

3 TRANSFORMATION OF THE FREE BOUNDARY PROBLEM INTO THE GAMMA VARIATIONAL INEQUALITY

In the context of European-style options, the transformation method of the Gamma equation was proposed and analyzed by Jandačka and Ševčovič (2005). If we consider the generalized nonlinear Black–Scholes equation (1.1) for our European-style option, then, making the change of variables $u = \ln(S/E)$ and $\tau = T - t$, and computing the second derivative of (1.1) with respect to u, we derive the so-called Gamma equation (1.2), ie,

$$\partial_\tau H - \partial_u\beta(H) - \partial_u^2\beta(H) - (r-q)\partial_u H + qH = 0, \quad \text{where } \beta(H) = \tfrac{1}{2}\hat{\sigma}(H)^2 H. \tag{3.1}$$

More details on the derivation of the Gamma equation as well as the existence and uniqueness of classical Hölder smooth solutions can be found in Ševčovič and Žitňanská (2016).

LEMMA 3.1 (Ševčovič and Žitňanská 2016, Proposition 3.1, Remark 3.1) *Let us consider a call option with the payoff diagram* $V(T,S) = (S-E)^+$. *Then, the function* $H(\tau,u) = S\partial_S^2 V(t,S)$, *where* $u = \ln(S/E)$ *and* $\tau = T-t$ *is a solution of (3.1) subject to the Dirac initial condition* $H(0,x) = \delta(x)$ *if and only if the function*

$$V(t,S) = \int_{-\infty}^{+\infty} (S - E\mathrm{e}^u)^+ H(\tau,u)\,\mathrm{d}u$$

is a solution of (1.1).

3.1 American-style options

In this subsection, we investigate the transformation method of the free boundary problem for pricing American-style options by means of a solution to the so-called Gamma variational inequality.

The principal advantage of American-style option contracts is the flexibility they offer holders, as these contracts can be exercised any time before the expiration date T. The majority of derivative contracts traded in the financial markets are of the American style. When modeling American options, unlike European-style options, there is the possibility of exercising the contract early, at some time $t^* \in [0, T)$ prior to the maturity time T.

It is well known that pricing an American call option on an underlying stock paying a continuous dividend yield $q > 0$ leads to the free boundary problem. In addition to the unknown function $V(t, S)$, we have to find the early exercise boundary function $S_f(t)$, $t \in [0, T]$. The function $S_f(t)$ has the following properties:

- if $S_f(t) > S$ for $t \in [0, T]$, then $V(t, S) > (S - E)^+$;
- if $S_f(t) \leqslant S$ for $t \in [0, T]$, then $V(t, S) = (S - E)^+$.

Over the past few decades, many authors have analyzed the free boundary position function S_f. Stamicar *et al* (1999) derived an accurate approximation to the early exercise position for times t close to expiry T for the Black–Scholes model with constant volatility (see also Evans *et al* 2002; Lauko and Ševčovič 2011; Zhu 2006). Their method was generalized for the nonlinear Black–Scholes model by Ševčovič (2007).

Following Kwok (1998) (see also Ševčovič *et al* 2011), the free boundary problem for pricing American-style call options consists of finding a function $V(t, S)$ and the early exercise boundary function S_f such that V solves the Black–Scholes PDE (1.1) on a time-dependent domain: $\{(t, S),\ 0 < S < S_f(t)\}$ and $V(t, S_f(t)) = S_f(t) - E$, and $\partial_S V(t, S_f(t)) = 1$.

Alternatively, a C^1 smooth function V is a solution to the free boundary problem for (1.1) if and only if it is a solution to the nonlinear variational inequality

$$
\begin{aligned}
\partial_t V + (r - q) S \partial_S V + S\beta(S \partial_S^2 V) - rV \leqslant 0, \qquad V(t, S) \geqslant g(S), \\
(\partial_t V + (r - q) S \partial_S V + S\beta(S \partial_S^2 V) - rV) \times (V - g) = 0
\end{aligned}
\tag{3.2}
$$

for any $S > 0$ and $t \in [0, T]$, where $g(S) \equiv (S - E)^+$.

3.2 Gamma transformation of the variational inequality

In this subsection, we present a novel transformation method to transform the nonlinear complementarity problem (3.2) for the function $V(t, S)$ into the so-called

Gamma variational inequality involving the transformed function $H(\tau, x)$. We need two auxiliary lemmas.

LEMMA 3.2 *Let $V(t, S)$ be a function that is C^1 smooth in the t variable and C^4 smooth in the S variable. Let $u = \ln(S/E)$, $\tau = T - t$, and define the function $Y(\tau, u) := \partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV$. Then,*

$$-\partial_\tau H + \partial_u \beta(H) + \partial_u^2 \beta(H) + (r-q)\partial_u H - qH = \frac{1}{E}\mathrm{e}^{-u}[\partial_u^2 Y - \partial_u Y],$$

where $H(\tau, u) = S\partial_S^2 V(t, S)$.

PROOF By differentiating the function Y with respect to the u variable and using the fact that $\partial_u = S\partial_S$, we obtain

$$\left.\begin{aligned} \partial_u Y &= \partial_t(S\partial_S V) + S(\beta + \partial_u\beta) + (r-q)SH - qS\partial_S V, \\ \partial_u^2 Y &= \partial_t(S\partial_S V + S^2\partial_S^2 V) + (r-q)S(H + \partial_u H) \\ &\quad + S(\beta + \partial_u\beta) + S(\partial_x^2\beta + \partial_u\beta) - qS\partial_S V - qH, \end{aligned}\right\} \tag{3.3}$$

where $S = E\mathrm{e}^u$. Then, $\partial_u^2 Y - \partial_u Y = E\mathrm{e}^u \Psi[H]$, where

$$\Psi[H] := -\partial_\tau H + \partial_u\beta(H) + \partial_u^2\beta(H) + (r-q)\partial_u H - qH, \tag{3.4}$$

as claimed. □

In the particular case where $Y \equiv 0$, the function $V(t, S)$ represents the price of a European-style call option. It is a solution to the nonlinear Black–Scholes equation (1.1) if and only if the function H is a solution to the so-called Gamma equation (3.1) subject to the initial condition $H(x, 0) = \delta(x)$, where δ is the Dirac function (see Ševčovič 2007; Ševčovič and Žitňanská 2016).

LEMMA 3.3 *If the function Y fulfills the asymptotic behavior*

$$\lim_{u\to-\infty} Y(\tau, u) = 0 \quad \textit{and} \quad \lim_{u\to-\infty} \mathrm{e}^{-u}\partial_u Y(\tau, u) = 0,$$

then

$$\begin{aligned} \int_{-\infty}^{+\infty} (S - E\mathrm{e}^u)^+ \Psi[H](\tau, u)\,\mathrm{d}u &= Y(\tau, u)|_{u=\ln(S/E)} \\ &\equiv \partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV. \end{aligned}$$

PROOF Using Lemma 3.2 and (3.3), we can express the term

$$\int_{-\infty}^{+\infty} (S - E\mathrm{e}^{u})^{+} \Psi[H](\tau, u)\, \mathrm{d}u$$

as follows:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (S - E\mathrm{e}^{u})^{+} \frac{1}{E} \mathrm{e}^{-u} [\partial_u^2 Y - \partial_u Y]\, \mathrm{d}u \\
&= \frac{1}{E} \int_{-\infty}^{\ln(S/E)} (S\mathrm{e}^{-u} - E)[\partial_u^2 Y - \partial_u Y]\, \mathrm{d}u \\
&= \frac{1}{E} \int_{-\infty}^{\ln(S/E)} [S\mathrm{e}^{-u} \partial_u Y - (S\mathrm{e}^{-u} - E)\partial_u Y\, \mathrm{d}u] + \underbrace{[(S\mathrm{e}^{-u} - E)\partial_u Y]_{-\infty}^{\ln(S/E)}}_{0} \\
&= \frac{1}{E} \int_{-\infty}^{+\infty} E\partial_u Y\, \mathrm{d}u \\
&= Y(\tau, u)|_{u=\ln(S/E)} \\
&= \partial_t V + (r - q) S \partial_S V + S\beta(S \partial_S^2 V) - rV,
\end{aligned}$$

and the proof of the lemma is as follows. □

THEOREM 3.4 *The function $V(t, S)$ is a solution to the nonlinear complementarity problem (3.2) if and only if the transformed function H is a solution of the following Gamma variational inequality and complementarity constraint:*

$$-\int_{-\infty}^{+\infty} (S - E\mathrm{e}^{u})^{+} \Psi[H](\tau, u)\, \mathrm{d}u \geqslant 0, \quad \int_{-\infty}^{+\infty} (S - E\mathrm{e}^{u})^{+} H(\tau, u)\, \mathrm{d}u \geqslant g(S), \tag{3.5}$$

$$\int_{-\infty}^{+\infty} (S - E\mathrm{e}^{u})^{+} \Psi[H](\tau, u)\, \mathrm{d}u \times \bigg(\int_{-\infty}^{+\infty} (S - E\mathrm{e}^{u})^{+} H(\tau, u)\, \mathrm{d}u - g(S) \bigg) = 0 \tag{3.6}$$

for any $S \geqslant 0$ and $\tau \in [0, T]$.

PROOF This directly follows from Lemma 3.2 and Lemma 3.3. □

REMARK 3.5 To calculate $V(T, S)$ in Theorem 3.4, we use the fact that $H(0, u) = \bar{H}(u)$, $u \in \mathbb{R}$, where $\bar{H}(u) := \delta(u)$ is the Dirac delta function such that $\int_{-\infty}^{+\infty} \delta(u)\, \mathrm{d}u = 1$, and

$$\int_{-\infty}^{+\infty} \delta(u - u_0)\phi(u)\, \mathrm{d}u = \phi(u_0)$$

for any continuous function ϕ.

We approximate the initial Dirac delta function as follows:

$$H(x,0) \approx f(d)/(\hat{\sigma}\sqrt{\tau^*}),$$

where $0 < \tau^* \ll 1$ is a sufficiently small parameter, and $f(d)$ is the probability density function of the normal distribution, that is, $f(d) = \mathrm{e}^{-d^2/2}/\sqrt{2\pi}$ and $d = (x+(r-q-\sigma^2/2)\tau^*)/\sigma\sqrt{\tau^*}$. This approximation follows from the observation that for a solution of the linear Black–Scholes equation with a constant volatility $\sigma > 0$ at time $T - \tau^*$ close to expiry T, the value $H^{\text{lin}}(x,\tau^*) = S\partial_S^2 V^{\text{lin}}(S, T-\tau^*)$ is given by $H^{\text{lin}}(x,\tau^*) = f(d)/(\hat{\sigma}\sqrt{\tau^*})$. Moreover, $H^{\text{lin}}(\cdot,\tau^*) \to \delta(\cdot)$ as $\tau^* \to 0$ in the sense of distributions.

REMARK 3.6 Zakamouline (2008, 2009) generalized the Leland option pricing model for pricing options on multiasset portfolios under constant transaction costs. This approach has since been generalized by Amster and Mogni (2017) for the case of variable transaction costs. Since the Gamma transformation method is proposed for single underlying asset nonlinear models, it is unclear how to generalize it for multidimensional problems.

4 SOLVING THE GAMMA VARIATIONAL INEQUALITY USING THE PROJECTED SUCCESSIVE OVER-RELAXATION METHOD

According to Theorem 3.4, the American call option pricing problem can be rewritten in terms of the function $H(\tau,u)$ satisfying the Gamma variational inequality (3.5–3.6) with the complementarity constraint (3.6). We follow Ševčovič and Žitňanská (2016) in order to derive an efficient numerical scheme for solving the Gamma variational inequality for a general form of the function $\beta(H)$ including the case of a variable transaction costs model. In order to apply the PSOR method (see Kwok 1998) to the variational inequality (3.5–3.6), we have to discretize the nonlinear operator Ψ defined in (3.4).

The proposed numerical discretization is based on the finite-volume method. Assume that the spatial variable u belongs to a bounded interval $(-L, L)$ for a sufficiently large $L > 0$. We divide the spatial interval $[-L, L]$ into a uniform mesh of discrete points $u_i = ih$, where $i = -n, \dots, n$ with a spatial step $h = L/n$. The time interval $[0, T]$ is uniformly divided with a time step $k = T/m$ into discrete points $\tau_j = jk$ for $j = 1, \dots, m$. The finite-volume discretization of the operator $\Psi[H]$ leads to a tridiagonal matrix multiplied by the vector

$$H^j = (H^j_{-n+1}, \dots, H^j_{n-1})^{\mathrm{T}} \in \mathbb{R}^{2n-1}.$$

More precisely, the vector $\Psi[H]^j$ at time level τ_j is given by $\Psi[H]^j = -(A^j H^j - d^j)$, where the $(2n-1) \times (2n-1)$ matrix A^j has the form

$$A^j = \begin{pmatrix} b^j_{-n+1} & c^j_{-n+1} & 0 & \cdots & 0 \\ a^j_{-n+2} & b^j_{-n+2} & c^j_{-n+2} & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \cdots & a^j_{n-2} & b^j_{n-2} & c^j_{n-2} \\ 0 & \cdots & 0 & a^j_{n-1} & b^j_{n-1} \end{pmatrix}, \tag{4.1}$$

with coefficients

$$\begin{aligned} a_i^j &= -\frac{k}{h^2}\beta'(H_{i-1}^{j-1}) + \frac{k}{2h}(r-q), \\ c_i^j &= -\frac{k}{h^2}\beta'(H_i^{j-1}) - \frac{k}{2h}(r-q), \\ b_i^j &= (1+kq) - (a_i^j + c_i^j), \\ d_i^j &= H_i^{j-1} + \frac{k}{h}(\beta(H_i^{j-1}) - \beta(H_{i-1}^{j-1})). \end{aligned}$$

Finally, using a simple numerical integration rule, the variational inequality (3.5–3.6) can be discretized as follows:

$$V(S, T-\tau_j) = h \sum_{i=-n}^{n} (S - E\mathrm{e}^{u_i})^+ H_i^j, \quad j = 1, 2, \ldots, m. \tag{4.2}$$

Then, the full space–time discretized version of inequalities occurring in (3.5–3.6) is given by

$$h \sum_{i=-n}^{n} (S - E\mathrm{e}^{u_i})^+ [(A^j H^j)_i - d_i^j] \geqslant 0, \tag{4.3}$$

$$h \sum_{i=-n}^{n} (S - E\mathrm{e}^{u_i})^+ H_i^j \geqslant g(S) \equiv (S-E)^+. \tag{4.4}$$

Let us define the auxiliary invertible matrix $P = (P_{li})$ as follows:

$$P_{li} = h \max(S_l - E\mathrm{e}^{u_i}, 0) = hE \max(\mathrm{e}^{v_l} - \mathrm{e}^{u_i}, 0), \tag{4.5}$$

where $v_l = (u_{l+1} + u_{l-1})/2$ for $l = -n, \ldots, n$.

Next, our purpose is to solve the problem (4.3)–(4.4) by means of the PSOR method. Using the matrix P, we can rewrite the system (4.3)–(4.4) as follows:

$$(PAH)_i \geqslant (Pd)_i, \qquad (PH)_i \geqslant g_i, \\ (PAH - Pd)_i \times (PH - g)_i = 0, \quad \text{for all } i, \tag{4.6}$$

where $A = A^j$, $g_i = (S_i - E)^+$ and $H = H^j$. The complementarity problem (4.6) can be solved by means of the PSOR algorithm, given by the following iterative scheme:

(1) for $k = 0$, set $v^{j,k} = v^{j-1}$;

(2) until $k \leqslant k_{\max}$, repeat:

$$w_i^{j,k+1} = \frac{1}{\tilde{A}_{ii}}\bigg(-\sum_{l<i} \tilde{A}_{il} v_l^{j,k+1} - \sum_{l>i} \tilde{A}_{il} v_l^{j,k} + \tilde{d}_i^j\bigg),$$
$$v_i^{j,k+1} = \max\{v_i^{j,k} + \omega(w_i^{j,k+1} - v_i^{j,k}), g_i\};$$

(3) set $v^j = v^{j,k+1}$

for $i = -n, \dots, n$ and $j = 1, \dots, m$, where $v^j = PH^j$, $\tilde{d}^j = Pd^j$ and $\tilde{A} = PA^j P^{-1}$. Here, $\omega \in [1, 2]$ is a relaxation parameter that can be tuned in order to speed up the convergence process. Finally, using the value $H^j = P^{-1} v^j$ and (4.2), we can evaluate the option price V.

5 NUMERICAL EXPERIMENTS

In this section, we focus our attention on numerical experiments for computing American-style call option prices based on the nonlinear Black–Scholes equation involving a piecewise linear decreasing transaction costs function C. In Figure 2, we show the corresponding function $\beta(H)$ given by

$$\beta(H) = \frac{\sigma^2}{2}\bigg(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\operatorname{sgn}(H)}{\sigma\sqrt{\Delta t}}\bigg)H,$$

where $\tilde{C}$ is the mean value modification of the transaction costs function C.

The parameters C_0, κ, $\xi_\pm$, Δt characterizing the nonlinear piecewise linear variable transaction costs function C and other model parameters are summarized in Table 1. Here, Δt is the time interval between two consecutive portfolio rearrangements, T is the maturity time, σ is the historical volatility, q is the dividend yield, E is the exercise price and r denotes the risk-free interest rate. A small parameter

FIGURE 2 A graph of the function $\beta(H)$ related to the piecewise linear decreasing transaction costs function (see Jandačka and Ševčovič 2005).

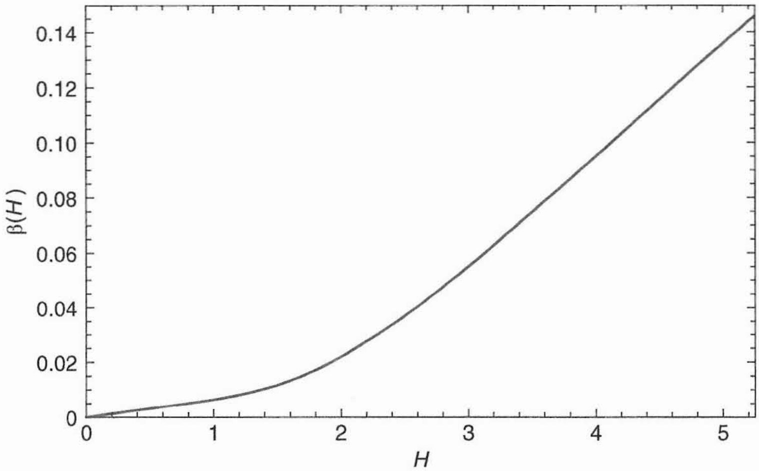


TABLE 1 Model and numerical parameters used in numerical experiments.

Model parameters	Numerical parameters
$C_0 = 0.02$	$m = 200, 800$
$\kappa = 0.3$, $\xi_- = 0.05$, $\xi_+ = 0.1$	$n = 250, 500$
$\Delta t = 1/261$	$h = 0.01$
$\sigma = 0.3$	$\tau^* = 0.005$
$r = 0.011$, $q = 0.008$	$k = T/m$
$T = 1$, $E = 50$	$L = 2.5$

$0 < \tau^* \ll 1$ represents a smoothing parameter for approximating the Dirac delta function (see Remark 3.5).

For the numerical parameters from Table 1, we computed option values V_{vtc} for several underlying asset prices S. The prices were calculated by means of numerical solutions for both bid and ask option prices. These are shown in Table 2. The bid price $V_{\mathrm{Bid}_{\mathrm{vtc}}}$ is compared with the price V_{BinMin}, which is computed by means of the binomial tree method (see Kwok 1998) with constant lower volatility,

$$\hat{\sigma}^2_{\min} = \sigma^2\left(1 - C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right);$$

TABLE 2 American call option prices obtained from the numerical solution of the nonlinear model with variable transaction costs for different meshes.

(a) American bid call option prices, $V_{\mathrm{Bid}_{\mathrm{vtc}}}$

	$n = 250, m = 200$				$n = 500, m = 800$		
S	V_{BinMin}	$V_{\mathrm{Bid}_{\mathrm{vtc}}}$	V_{BinMax}	S	V_{BinMin}	$V_{\mathrm{Bid}_{\mathrm{vtc}}}$	V_{BinMax}
40	0.0320	0.0513	1.3405	40	1.4511	1.6594	2.8670
42	0.1075	0.3252	1.8846	42	2.0137	2.3869	3.6039
44	0.2901	0.8232	2.5527	44	2.6979	3.2309	4.4371
46	0.6535	1.5097	3.3483	46	3.5064	4.1868	5.3645
48	1.2675	2.3859	4.2711	48	4.4382	5.2488	6.3833
50	2.1740	3.4244	5.3175	50	5.4897	6.4133	7.4889
52	3.3738	4.6126	6.4817	52	6.6553	7.6764	8.6772
54	4.8304	5.9521	7.7555	54	7.9270	9.0342	9.9423
56	6.4862	7.4377	9.1295	56	9.2959	10.4824	11.2798
58	8.2809	9.0643	10.5943	58	10.7532	12.0179	12.6832
60	10.1635	10.8273	12.1397	60	12.2892	13.6385	14.1481

(b) American ask call option prices, $V_{\mathrm{Ask}_{\mathrm{vtc}}}$

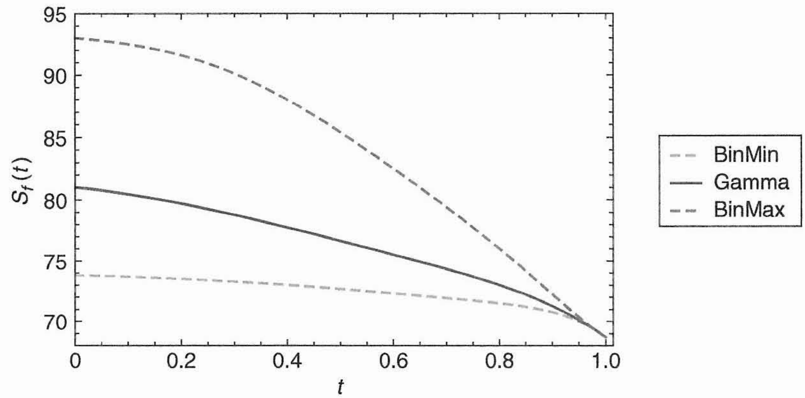
	$n = 250, m = 200$				$n = 500, m = 800$		
S	V_{BinMin}	$V_{\mathrm{Ask}_{\mathrm{vtc}}}$	V_{BinMax}	S	V_{BinMin}	$V_{\mathrm{Ask}_{\mathrm{vtc}}}$	V_{BinMax}
40	1.4511	1.6594	2.8670	40	1.4420	1.6692	2.8519
42	2.0137	2.3869	3.6039	42	2.0027	2.3945	3.5870
44	2.6979	3.2309	4.4371	44	2.6851	3.2412	4.4187
46	3.5064	4.1868	5.3645	46	3.4922	4.2134	5.3450
48	4.4382	5.2488	6.3833	48	4.4231	5.2601	6.3627
50	5.4897	6.4133	7.4889	50	5.4742	6.4300	7.4678
52	6.6553	7.6764	8.6772	52	6.6395	7.6922	8.6557
54	7.9270	9.0342	9.9423	54	7.9115	9.2167	9.9211
56	9.2959	10.4824	11.2798	56	9.2812	11.0264	11.2586
58	10.7532	12.0179	12.6832	58	10.7393	12.2017	12.6628
60	12.2892	13.6385	14.1481	60	12.2763	13.6505	14.1283

Comparison with the option prices V_{BinMin} and V_{BinMax} computed by means of the binomial tree method for the constant volatilities σ_{min} and σ_{max}.

the upper bound price V_{BinMax}, meanwhile, corresponds to the solution with higher constant volatility:

$$\hat{\sigma}^2_{\mathrm{max}} = \sigma^2\left(1 - \underline{C_0}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right).$$

FIGURE 3 An early exercise boundary function $S_f(t)$, $t \in [0, T]$, computed for a model with variable transaction costs (solid line, Gamma) compared with an early exercise boundary computed by means of binomial trees with constant volatilities $\sigma_{\min}$ (dashed bottom curve) and $\sigma_{\max}$ (dashed top curve).



Similarly, as for the ask price $V_{\mathrm{Ask}_{\mathrm{vtc}}}$, the lower bound V_{BinMin} corresponds to the solution of the binomial tree method with lower volatility,

$$\hat{\sigma}^2_{\min} = \sigma^2\left(1 + \underline{C_0}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right),$$

whereas the upper bound V_{BinMax} corresponds to the solution with higher constant volatility:

$$\hat{\sigma}^2_{\max} = \sigma^2\left(1 + C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right).$$

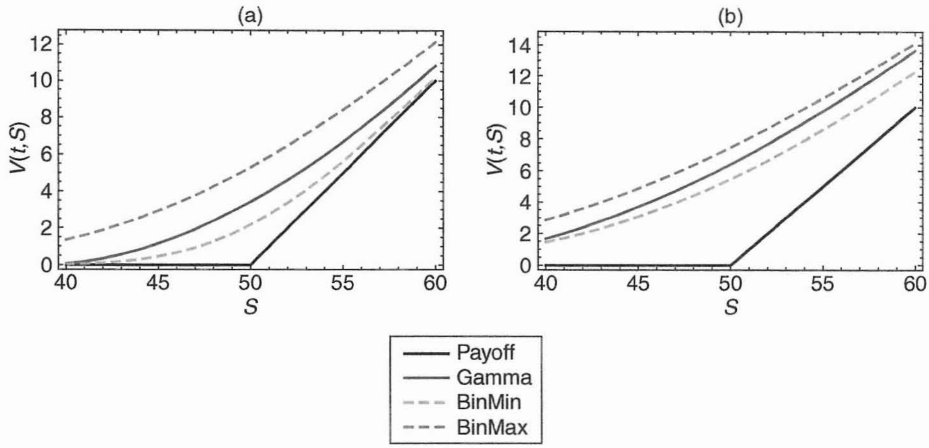
With regard to Ševčovič and Žitňanská (2016), for a European-style option, one can derive the following lower and upper bounds by using the parabolic comparison principle:

$$V_{\sigma_{\min}}(S, t) \leqslant V_{vtc}(t, S) \leqslant V_{\sigma_{\max}}(t, S), \quad S > 0,\ t \in [0, T].$$

In the case of American-style options, analogous inequalities for the numerical solution can be observed in Table 2.

In Table 3, we present a comparison of the results obtained by our method based on a solution to the Gamma variational inequality, in which we considered the constant volatilities $\sigma_{\min}$ and $\sigma_{\max}$, and those obtained by the well-known method based on binomial trees for American-style call options (see Kwok 1998). The difference in the prices is of the order of the mesh size $h = L/n$.

FIGURE 4 A graph of American (a) bid and (b) ask call option prices $V(t, S)$, $S \in [40, 60]$, at $t = 0$ computed by means of the nonlinear Black–Scholes model with variable transaction costs, with mesh size $n = 500$, $m = 800$, compared with solutions $V_{\sigma_{\min}}$ and $V_{\sigma_{\max}}$ calculated by binomial trees with constant volatilities $\sigma_{\min}$ and $\sigma_{\max}$.



In Figure 3, we present the free boundary function $S_f(t)$ obtained using our method, with the variable transaction costs function C for the bid option value, compared with the binomial trees with $\sigma_{\min}$ and $\sigma_{\max}$. In Figure 4, we plot the graphs of the solution $V_{\text{vtc}}(t, S)$ at $t = 0$ for both bid and ask prices. We also plot the prices obtained by the binomial tree method with constant lower volatility $\sigma_{\min}$ and higher volatility $\sigma_{\max}$, respectively.

6 CONCLUSIONS

In this paper, we investigated a novel nonlinear generalization of the Black–Scholes equation for pricing American-style call options, assuming variable transaction costs for trading the underlying assets. In this way, we presented a model that addresses a more realistic financial framework than the classical Black–Scholes model. From a mathematical point of view, we analyzed a problem that consists of a fully nonlinear parabolic equation in which the nonlinear diffusion coefficient depends on the second derivative of the option price. Further, for the American call option, we transformed the nonlinear complementarity problem into the so-called Gamma variational inequality. We solved the Gamma variational inequality by means of the PSOR method and presented an effective numerical scheme for discretizing the Gamma variational inequality. Then, we performed numerical computations using the model

TABLE 3 Ask call option values $V_{\mathrm{Ask_{vtc}}}$ of the numerical solution of the model under constant volatilities $\sigma = \sigma_{\min}$ (left) and $\sigma = \sigma_{\max}$ (right) compared with the prices computed by binomial trees with $n = 100$ and $n = 200$ nodes, respectively.

	$\sigma = \sigma_{\min}$					$\sigma = \sigma_{\max}$			
	$n = 250, m = 200$		$n = 500, m = 800$			$n = 250, m = 200$		$n = 500, m = 800$	
S	$V_{\mathrm{Ask_{vtc}}}$	V_{BinMin}	$V_{\mathrm{Ask_{vtc}}}$	V_{BinMin}	S	$V_{\mathrm{Ask_{vtc}}}$	V_{BinMax}	$V_{\mathrm{Ask_{vtc}}}$	V_{BinMax}
40	1.4737	1.4511	1.4634	1.4420	40	2.8827	2.8670	2.8663	2.8519
42	2.2417	2.0137	2.110	2.002	42	3.6273	3.6039	3.5923	3.5870
44	2.7156	2.6979	2.7025	2.6851	44	4.4618	4.4371	4.4067	4.4187
46	3.5287	3.5064	3.5193	3.4922	46	5.3945	5.3645	5.3561	5.3450
48	4.4572	4.4382	4.4498	4.4231	48	6.4095	6.3833	6.3515	6.3627
50	5.5019	5.4897	5.4996	5.4742	50	7.5002	7.4889	7.4710	7.4678
52	6.6993	6.6553	6.6684	6.6395	52	8.7049	8.6772	8.6682	8.6557
54	7.9537	7.9270	7.9350	7.9115	54	9.9765	9.9423	9.9326	9.9211
56	9.3367	9.2959	9.3145	9.2812	56	11.3071	11.2798	11.2742	11.2586
58	10.8015	10.7532	10.7683	10.7393	58	12.7103	12.6832	12.6790	12.6628
60	12.3369	12.2892	12.3189	12.2763	60	14.1640	14.1481	14.1374	14.1283

with variable transaction costs and compared our results with lower and upper bounds computed by means of the binomial tree method with constant volatilities. Finally, we presented a comparison of early exercise boundary functions.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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