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Option Pricing in Illiquid Markets with Jumps

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ABSTRACT

The classical linear Black–Scholes model for pricing derivative securities is a popular model in the financial industry. It relies on several restrictive assumptions such as completeness, and friction-less of the market as well as the assumption on the underlying asset price dynamics following a geometric Brownian motion. The main purpose of this paper is to generalize the classical Black–Scholes model for pricing derivative securities by taking into account feedback effects due to an influence of a large trader on the underlying asset price dynamics exhibiting random jumps. The assumption that an investor can trade large amounts of assets without affecting the underlying asset price itself is usually not satisfied, especially in illiquid markets. We generalize the Frey–Stremme nonlinear option pricing model for the case the underlying asset follows a Lévy stochastic process with jumps. We derive and analyze a fully nonlinear parabolic partial-integro differential equation for the price of the option contract. We propose a semi-implicit numerical discretization scheme and perform various numerical experiments showing the influence of a large trader and intensity of jumps on the option price.

1. Introduction

Over recent decades, the Black–Scholes model and its generalizations become widely used in financial markets because of its simplicity and existence of the analytic formula for pricing European style options. According to the classical theory developed by Black, Scholes and Merton the price $V(t, S)$ of an option in a stylized financial market at the time $t \in [0, T]$ and the underlying asset price S can be computed as a solution to the linear Black–Scholes parabolic equation:

$$\frac{\partial V}{\partial t}(t, S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(t, S) + rS\frac{\partial V}{\partial S}(t, S) - rV(t, S) = 0, \ \ t \in [0, T), S > 0. \quad (1)$$

Here $\sigma > 0$ is the historical volatility of the underlying asset driven by the geometric Brownian motion, $r > 0$ is the risk-free interest rate of zero-coupon bond. A solution is subject to the terminal pay-off condition $V(T, S) = \Phi(S)$ at maturity $t = T$. Evidence from stock markets observations indicates that this model is not the most realistic one

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since it assumes that the market is liquid, complete, frictionless and without transaction costs. We also recall that the linear Black–Scholes equation provides a solution corresponding to a perfectly replicated portfolio which need not be a desirable property. In the last two decades some of these assumptions have been relaxed in order to model, for instance, the presence of transaction costs (see, e.g., Kwok (2008) and Avellaneda, Levy, and Paras (1995)), feedback and illiquid market effects due to large traders choosing given stock-trading strategies Schönbucher and Wilmott (2000), Frey and Stremme (1997), Frey and Patie (2002), risk from the unprotected portfolio Jandaka and Ševovi (2005). In all aforementioned generalizations of the linear Black–Scholes Equation (1) the constant volatility σ is replaced by a nonlinear function $\tilde{\sigma}(S\partial_S^2 V)$ depending on the second derivative $\partial_S^2 V$ of the option price itself. In the class of generalized Black–Scholes equation with such a nonlinear diffusion function, an important role is played by the nonlinear Black–Scholes model derived by Frey and Stremme in (Jandaka and Ševovi 2005) (see also (Frey and Stremme 1997) (Frey 1998)). In this model, the asset dynamics takes into account the presence of feedback effects due to a large trader choosing his/her stock-trading strategy (see also Schönbucher and Wilmott 2000). The diffusion coefficient is again non-constant:

$$\tilde{\sigma}(S\partial_S^2 V)^2 = \sigma^2\big(1 - \rho S\partial_S^2 V\big)^{-2}, \tag{2}$$

where $\sigma, \rho > 0$ are constants.

Another important direction in generalizing the original Black–Scholes equation arises from the fact that the sample paths of a Brownian motion are continuous, but the realized stock price of a typical company exhibits random jumps over the intraday scale, making the price trajectories discontinuous. In the classical Black–Scholes model the underlying asset price process is assumed to follow a geometric Brownian motion. However, the empirical distribution of stock returns exhibits fat tails. Several alternatives have been proposed in the literature for the generalization of this model. The models with jumps and diffusion can, at least in part, solve the problems inherent to the linear Black–Scholes model, and they have also an important role in the options market. While in the Black–Scholes model the market is complete, implying that every pay-off can be perfectly replicated, in jump-diffusion models there is no perfect hedge and this way the options are not redundant. It turns out that the option price can be computed from the solution $V(t, S)$ of the following partial integro-differential (PIDE) Black–Scholes equation:

$$\frac{\partial V}{\partial t}(t, S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(t, S) + rS\frac{\partial V}{\partial S}(t, S) - rV(t, S)$$
$$+ \int_{\mathbb{R}} V(t, S + H(z, S)) - V(t, S) - H(z, S)\frac{\partial V}{\partial S}(t, S)\nu(\mathrm{d}z) = 0, \tag{3}$$

where $H(z, S) = S(e^z - 1)$ and ν is the so-called Lévy measure characterizing the underlying asset process with random jumps in time and space. Note that, if $\nu = 0$ then (3) reduces to the classical linear Black–Scholes Equation (1).

The novelty and main purpose of this paper is to take into account both directions of generalizations of the Black–Scholes equation. The assumption that an investor can trade large amounts of the underlying asset without affecting its price is no longer true, especially in illiquid markets. Therefore, we will derive, analyze, and perform the numerical

computation of the model. We relax the assumption of liquid market following the Frey–Stremme model under the assumption that the underlying asset price follows a Lévy stochastic process with jumps. We will show that the corresponding PIDE nonlinear equation has the form:

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\sigma^2}{(1-\rho S \partial_S \phi)^2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \\
&+ \int_{\mathbb{R}} V(t, S + H(t,z,S)) - V(t,S) - H(t,z,S)\frac{\partial V}{\partial S} \nu(\mathrm{d}z) = 0,
\end{aligned}
\tag{4}
$$

where the function $H(t, z, S)$ may depend, e.g., on the large trader strategy function $\phi = \phi(t, S)$. This function may depend on the delta $\partial_S V$ of the price V, if $\rho > 0$.

The paper is organized as follows. In the next Section 2 we recall known facts regarding exponential Lévy models. We also recall important classes of Lévy measures with finite and infinite activity. Section 3 is devoted to the derivation of the novel option pricing model taking into account feedback effects of a large trader on the underlying asset following jump-diffusion Lévy process. We show that the price of an option can be computed from a solution to a fully nonlinear partial integro-differential equation (PIDE) (4). We also derive a formula for the trading strategy function ϕ which minimizes the variance of the tracking error. Next, in Section 4 we present a semi-implicit numerical discretization scheme for solving the resulting nonlinear PIDE. The scheme is based on finite difference approximation. In Section 5 we present numerical results when considering the Variance–Gamma process. We also perform sensitivity numerical analysis of a solution with respect to model parameters ρ and ν.

2. Preliminaries, definitions and motivation

We consider a stylized economy with two traded assets, a risky asset, usually a stock with a price S_t, and a risk-less asset, typically a bond with a price B_t which is taken as numeraire. The bond market is assumed to be perfectly elastic as bonds are assumed to be more liquid when compared to stocks. In this economy, there are two types of traders: the reference traders and the program traders. The program traders are also referred to as portfolio insurers since they use dynamic hedging strategies to hedge portfolio against jumps in stock prices. They are either single traders or a group of traders acting together. It is assumed that their trades influence the stock price equilibrium. The reference traders can be considered as representative traders of many small agents. We assume they act as price takers. Typically, it is assumed that $\tilde{D}(t, Y_t, S_t)$ is the reference trader demand function which depends on the income process Y_t or some other fundamental state variable influencing the reference trader demand. The aggregate demand of program traders is denoted by $\varphi(t, S_t) = \xi\phi(t, S_t)$, where ξ is the number of written identical securities that the program traders are trying to hedge and $\phi(t, S_t)$ is the demand per unit of the security being hedged. For simplicity, we assume that ξ is the same for every program trader. The general case where different securities are considered can be found, e.g., in (Ronnie Sircar and Papanicolaou 1998). Assume the supply of a stock with the price $\tilde{S}_0$ is constant. Let $D(t, Y, S) = \frac{\tilde{D}(t,Y,S)}{\tilde{S}_0}$ denote the quantity demanded by a reference trader per unit of supply. Then, the total demand relative to the supply at time t is given by $G(t, Y, S) = D(t, Y, S) + \rho\phi(t, S)$, where $\rho = \frac{\xi}{\tilde{S}_0}$

and $\rho\phi(t, S)$ is the proportion of the total supply of the stock that is being traded by program traders. In order to obtain the market equilibrium the variables Y and S should satisfy $G(t, Y, S) = 1$. Assume that the function G is monotone with respect to Y and S variables, and it is sufficiently smooth. Then, we can solve the implicit equation $G(t, Y_t, S_t) = 1$ to obtain $S_t = \psi(t, Y_t)$ where ψ is a sufficiently smooth function. Following (Ronnie Sircar and Papanicolaou 1998), we assume that the stochastic process Y_t has the following dynamics:

$$dY_t = \mu(t, Y_t)dt + \eta(t, Y_t)dW_t.$$

Then, by using Itô's lemma for the process $S_t = \psi(t, Y_t)$ we obtain

$$dS_t = \left(\partial_t \psi + \mu \partial_y \psi + \frac{\eta^2}{2} \partial_y^2 \psi\right)dt + \eta \partial_y \psi dW_t \equiv b(t, S_t)S_t dt + v(t, S_t)S_t dW_t. \tag{5}$$

It means that S_t follows a geometric Brownian motion with a nonconstant volatility function $v(t, S) = \eta(t, Y)\partial_Y \psi(t, Y)/\psi(t, Y)$ where $Y = \psi^{-1}(t, S)$. Following the argument used in the derivation of the original Black–Scholes equation, we obtain a generalization of the Black–Scholes partial differential equation with a nonconstant volatility function $\sigma = v(t, S)$. In this paper, we follow Frey and Stremme's approach (cf. Frey and Stremme 1997; Frey 1998). The idea is to prescribe a dynamics for the underlying stock price instead of deriving it by using the market equilibrium and dynamics for the income process Y_t as it is done, e.g., in (Ronnie Sircar and Papanicolaou 1998). This way Frey and Stremme derived the same stock price dynamics as in (Ronnie Sircar and Papanicolaou 1998) corresponding to a situation where the demand function is of logarithmic type, where $\gamma = \frac{\sigma}{\eta_0}$, and the income process Y_t follows a geometric Brownian motion, i.e.

$$\partial_Y D(Y, S) = \gamma \frac{1}{Y}, \; \partial_S D(Y, S) = -\frac{1}{S}, \; dY_t = \mu_0 Y_t dt + \eta_0 Y_t dW_t, \tag{6}$$

$$v(t, S) = \eta(t, Y)\frac{\partial_Y \psi(t, Y)}{\psi(t, Y)} = -\frac{\eta_0 Y}{S}\frac{\gamma \frac{1}{Y}}{-\frac{1}{S} + \rho \frac{\partial \phi}{\partial S}} = \frac{\sigma}{1 - \rho S \frac{\partial \phi}{\partial S}}.$$

Assuming the delta hedging strategy with $\phi(t, S) = \partial_S V(t, S)$ and inserting the volatility function $v(t, S)$ into (5) we obtain the generalized Black–Scholes equation with the nonlinear diffusion function of the form (2).

Our main goal is to extend the Frey–Stremme model to an underlying asset following a Lévy process. Next, we recall the basic properties of Lévy jump-diffusion processes.

2.1. *Exponential Lévy models*

Let $X_t, t \geq 0$, be a stochastic process. The measure $v(A)$ of a Borel set $A \in \mathcal{B}(\mathbb{R})$ defined by $v(A) = \mathbb{E}[J_X([0, 1] \times A)]$ where $J_X([0, t] \times A) = \#\{s \in [0, t] : \Delta X_s \in A\}$ is the Poisson random measure. It gives the mean number, per unit of time, of jumps whose amplitude belongs to the set A. Recall that the Lévy-Itô decomposition provides a representation of X_t which can be interpreted as a combination of a Brownian motion

with a drift ω and an infinite sum of independent compensated Poisson processes with variable jump sizes (see Cont and Tankov 2004), i.e.,

$$dX_t = \omega dt + \sigma dW_t + \int_{|x| \geq 1} x J_X(dt, dx) + \int_{|x| < 1} x \widetilde{J}_X(dt, dx),$$

where $\widetilde{J}_X([0, t] \times A) = J_X([0, t] \times A) - t\nu(A)$ is the compensation of J_X.

Any Lévy process is a strong Markov process, the associated semigroup is a convolution semigroup. Its infinitesimal generator $L : u \mapsto L[u]$ is a nonlocal partial integro-differential operator given by (see (Applebaum 2009)):

$$
\begin{aligned}
L[u](x) &= \lim_{h \to 0^+} \frac{\mathbb{E}[u(x + X_h)] - u(x)}{h} \\
&= \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial u}{\partial x} + \int_{\mathbb{R}} \left[u(x + y) - u(x) - y 1_{|y| \leq 1} \frac{\partial u}{\partial x}(x) \right] \nu(dy),
\end{aligned}
\tag{7}
$$

which is well defined for any compactly supported function $u \in C_0^2(\mathbb{R})$.

Let $S_t, t \geq 0$, be a stochastic process representing an underlying asset process under a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}$ represents the price history up to the time t. If the market is arbitrage-free then there is an equivalent measure $\mathbb{Q}$ under which discounted prices of all traded financial assets are $\mathbb{Q}-$ martingales. This result is known as the fundamental theorem of asset pricing (see (Cont and Tankov 2004)). The measure $\mathbb{Q}$ is also known as the risk-neutral measure. We consider the exponential Lévy model in which the risk-neutral price process S_t under $\mathbb{Q}$ is given by $S_t = e^{rt + X_t}$, where X_t is a Lévy process under $\mathbb{Q}$ with the characteristic triplet (σ, γ, ν). Then, the arbitrage-free market hypothesis imposes that $\hat{S}_t = S_t e^{-rt} = e^{X_t}$ is a martingale, which is equivalent to the following conditions imposed on the triplet (σ, γ, ν):

$$\int_{|y| \geq 1} e^y \nu(dy) < \infty, \ \gamma \in \mathbb{R}, \ \gamma = -\frac{\sigma^2}{2} - \int_{-\infty}^{+\infty} \left(e^y - 1 - y 1_{|y| \leq 1} \right) \nu(dy). \tag{8}$$

The risk-neutral dynamics of S_t under $\mathbb{Q}$ is given by

$$dS_t = rS_t dt + \sigma S_t dW_t + \int_{\mathbb{R}} (e^y - 1) S_t \widetilde{J}_X(dt, dy). \tag{9}$$

The exponential price process $e^{X_t}, t \geq 0$, is also a Markov process with the state space $(0, \infty)$ and the infinitesimal generator:

$$L^S[V](S) = \lim_{h \to 0} \frac{\mathbb{E}[V(S e^{X_h})] - V(S)}{h} = rS \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \tag{10}$$

$$+ \int_{\mathbb{R}} \left[V(S e^y) - V(S) - S(e^y - 1) \frac{\partial V}{\partial S} \right] \nu(dy) \tag{11}$$

(see (Cont and Tankov 2004)). Recall that a Lévy process is called the Lévy type stochastic integral if it has the following representation:

$$dX_t = \omega dt + \sigma dW_t + \int_{|x| \geq 1} K(t, x) J_X(dt, dx) + \int_{|x| < 1} H(t, x) \widetilde{J}_X(dt, dx).$$

An important result that will be needed later is the following variant of Itô's lemma.

Theorem 2.1. (Applebaum 2009) *Let $f \in C^{1,2}([0, T] \times \mathbb{R})$ and $H, K \in C([0, T] \times \mathbb{R})$. Let $X_t, t \geq 0$, be a Lévy stochastic process. Then*

$$
\begin{aligned}
\mathrm{d}f(t, X_t) = {} & \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}[X_t, X_t] \\
& + \int_{|x| \geq 1} f(t, X_t + K(t, x)) - f(t, X_t) J_X(\mathrm{d}t, \mathrm{d}x) \\
& + \int_{|x| < 1} f(t, X_t + H(t, x)) - f(t, X_t) \tilde{J}_X(\mathrm{d}t, \mathrm{d}x) \\
& + \int_{|x| < 1} f(t, X_t + H(t, x)) - f(t, X_t) - H(t, x)\frac{\partial f}{\partial x}(t, X_t)\nu(\mathrm{d}x)\mathrm{d}t.
\end{aligned}
\tag{12}
$$

A classical example of a Lévy process is the jump-diffusion model introduced by Merton in (Merton 1976). It has the following dynamics:

$$
\mathrm{d}X_t = \left(b + \int_{|x| < 1} x\nu(\mathrm{d}x)\right)\mathrm{d}t + \int_{|x| \geq 1} x J_X(\mathrm{d}t, \mathrm{d}x) + \sigma \mathrm{d}W_t + \int_{|x| < 1} x\tilde{J}_X(\mathrm{d}t, \mathrm{d}x).
$$

Then, by applying Itô's lemma to $S_t = e^{X_t}$ we obtain:

$$
\mathrm{d}S_t = (b + \sigma^2/2)S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t + S_t\int_{\mathbb{R}} (e^x - 1) J_X(\mathrm{d}t, \mathrm{d}x).
$$

In financial applications, exponential Lévy models are of several types. In this paper we are concerned with the so-called jump-diffusion models in which we represent the log-price as a Lévy process with a non-zero diffusion part ($\sigma > 0$) and a jump process with either finite activity with ($\nu(\mathbb{R}) < \infty$) or infinite activity ($\nu(\mathbb{R}) = \infty$).

In the context of financial modelling, a jump-diffusion model was proposed by Merton in (Merton 1976). The random jump variables are normally distributed with the mean m and variance δ^2. Its Lévy density is given by:

$$
\nu(\mathrm{d}x) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\delta^2}}\mathrm{d}x.
\tag{13}
$$

Another popular and frequently used model is the so-called double exponential model which was introduced by Kou (2002). In this model the distribution of jumps have a Lévy measure of the form:

$$
\nu(\mathrm{d}x) = \lambda\left(\theta\lambda^+ e^{-\lambda^+ x}1_{x>0} + (1-\theta)\lambda^- e^{\lambda^- x}1_{x<0}\right)\mathrm{d}x,
\tag{14}
$$

where λ is the intensity of jumps, θ is the probability of having a positive jump and $\lambda^{\pm} > 0$ correspond to the level of decay of the distribution of positive and negative jumps. This implies that the distribution of jumps is asymmetric and the tails of the distribution of returns are semi-heavy.

Among examples of infinite activity Lévy processes used in the financial modelling there are, e.g., the Variance Gamma (see Madan, Carr, and Chang 1998), Normal

Inverse Gaussian (NIG) (see Barndorff-Nielsen and Levendorski 2001) or CGMY processes. The Variance Gamma process is a process of infinite activity $v(\mathbb{R}) = \infty$ and finite variation $\int_{|x| \leq 1} |x| v(dx) < \infty$. Its Lévy measure is given by:

$$v(dx) = \frac{1}{\kappa |x|} e^{Ax - B|x|} dx \text{ with } A = \theta/\sigma^2 \text{ and } B = \sigma^{-2} \sqrt{\theta^2 + 2\sigma^2/\kappa}, \tag{15}$$

where the parameters σ and θ are related to volatility and drift of the Brownian motion and κ is a parameter related to the variance of the subordinator, in this case, the Gamma process (see Cont and Tankov 2004).

All aforementioned examples of jump-diffusion models have Lévy measures belonging to the class of the so-called admissible activity Lévy measures.

Definition 2.2. A Lévy measure v is called an admissible activity Lévy measure if

$$0 \leq \frac{v(dz)}{dz} \leq h(z) \equiv C|z|^{-\alpha} \left(e^{D^- z} 1_{z \geq 0} + e^{D^+ z} 1_{z < 0} \right) e^{-\mu z^2}, \tag{16}$$

for any $z \in \mathbb{R}$ and shape parameters $\alpha \geq 0$, $D^{\pm} \in \mathbb{R}$ and $\mu \geq 0$.

Remark 1. Note that the additional conditions $\int_{\mathbb{R}} \min(z^2, 1) v(dz) < \infty$ and $\int_{|z| > 1} e^z v(dz) < \infty$ are satisfied provided that v is an admissible Lévy measure with shape parameters $\alpha < 3$, and, either $\mu > 0, D^{\pm} \in \mathbb{R}$, or $\mu = 0$ and $D^- + 1 < 0 < D^+$.

For the Merton model, we have $\alpha = 0, D^{\pm} = 0$ and $\mu = 1/(2\delta^2) > 0$. In the Kou model $\alpha = \mu = 0, D^+ = \lambda^-, D^- = -\lambda^+$. As for the Variance Gamma process we have $\alpha = 1, \mu = 0, D^{\pm} = A \pm B$.

3. Feedback effects under jump-diffusion underlying asset price dynamics

Let us suppose that a large trader uses a stock-holding strategy α_t and S_t is a cadlag process (right continuous with limits to the left). Henceforth, we shall identify S_t with S_{t-}. We assume S_t has the following dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + \rho S_t d\alpha_t + \int_{\mathbb{R}} S_t (e^x - 1) J_X(dt, dx). \tag{17}$$

It can be viewed as a perturbation of the classical jump-diffusion model. Indeed, if a large trader does not trade then $\alpha_t = 0$ or the market liquidity parameter ρ is set to zero then the stock price S_t follows the classical jump-diffusion model.

In what follows, we will assume the following structural hypothesis:

Assumption 3.1. *Assume the trading strategy* $\alpha_t = \phi(t, S_t)$ *and the parameter* $\rho \geq 0$ *satisfy* $\rho L < 1$, *where* $L = \sup_{S > 0} \left| S \frac{\partial \phi}{\partial S} \right|$.

Next, we show an explicit formula for the dynamics of S_t satisfying (17) under certain regularity assumptions made on the stock-holding function $\phi(t, S)$.

Proposition 3.2. *Suppose that the stock-holding strategy $\alpha_t = \phi(t, S_t)$ satisfies Assumption 3.1 where $\phi \in C^{1,2}([0, T] \times \mathbb{R}^+)$. If the process $S_t, t \geq 0$, satisfies the implicit stochastic Equation (17) then the process S_t satisfies the following SDE:*

$$dS_t = b(t, S_t)S_t dt + v(t, S_t)S_t dW_t + \int_{\mathbb{R}} H(t, x, S_t)J_X(dt, dx), \tag{18}$$

where
$$b(t, S) = \frac{1}{1 - \rho S \frac{\partial \phi}{\partial S}(t, S)} \left(\mu + \rho \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}v(t, S)^2 S^2 \frac{\partial^2 \phi}{\partial S^2} \right) \right), \tag{19}$$

$$v(t, S) = \frac{\sigma}{1 - \rho S \frac{\partial \phi}{\partial S}(t, S)}, \tag{20}$$

$$H(t, x, S) = S(e^x - 1) + \rho S[\phi(t, S + H(t, x, S)) - \phi(t, S)]. \tag{21}$$

Proof: We can rewrite the SDE (18) for S_t, in the following way:

$$dS_t = \left(b(t, S_t)S_t + \int_{|x|<1} H(t, x, S_t)v(dx) \right) dt + v(t, S_t)S_t dW_t$$
$$+ \int_{|x|\geq 1} H(t, x, S_t)J_X(dt, dx) + \int_{|x|<1} H(t, x, S_t)\tilde{J}_X(dt, dx).$$

Since $\phi(t, S)$ is assumed to be a smooth function then, by applying Itô formula (12) to the process $\phi(t, S_t)$, we obtain

$$d\alpha_t = \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}v(t, S_t)^2 S_t^2 \frac{\partial^2 \phi}{\partial S^2} \right) dt + \frac{\partial \phi}{\partial S} dS_t$$
$$+ \int_{\mathbb{R}} \phi(t, S_t + H(t, x, S_t)) - \phi(t, S_t) - H(t, x, S_t)\frac{\partial \phi}{\partial S}(t, S_t)J_X(dt, dx). \tag{22}$$

Now, inserting the differential $d\alpha_t$ into (17), we obtain

$$dS_t = \mu S_t dt + \sigma S_t dW_t + \int_{\mathbb{R}} S_t(e^x - 1)J_X(dt, dx) + \rho S_t \frac{\partial \phi}{\partial S} dS_t$$
$$+ \rho S_t \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}v(t, S_t)^2 S_t^2 \frac{\partial^2 \phi}{\partial S^2} \right) dt \tag{23}$$
$$+ \rho S_t \int_{\mathbb{R}} \phi(t, S_t + H(t, x, S_t)) - \phi(t, S_t) - H(t, x, S_t)\frac{\partial \phi}{\partial S}(t, S_t)J_X(dt, dx).$$

Rearranging terms in (23) we conclude

$$(1 - \rho S_t \frac{\partial \phi}{\partial S}(t, S_t))\mathrm{d}S_t = (\mu S_t + \rho S_t (\frac{\partial \phi}{\partial t} + \frac{1}{2}v(t, S_t)^2 S_t^2 \frac{\partial^2 \phi}{\partial S^2}))\mathrm{d}t$$

$$+ \sigma S_t \mathrm{d}W_t - \rho S_t \int_{\mathbb{R}} H(t, x, S_t)\frac{\partial \phi}{\partial S}(t, S_t)J_X(\mathrm{d}t, \mathrm{d}x) \qquad (24)$$

$$+ \int_{\mathbb{R}} S_t(e^x - 1) + \rho S_t(\phi(t, S_t + H(t, x, S_t)) - \phi(t, S_t))J_X(\mathrm{d}t, \mathrm{d}x).$$

Comparing terms in (18) and (24) we end up with expressions (19), (20), and the implicit equation for the function H:

$$H(t, x, S) = \frac{1}{1 - \rho S \frac{\partial \phi}{\partial S}(t, S)}(S(e^x - 1) + \rho S(\phi(t, S + H(t, x, S)) - \phi(t, S)))$$

$$- \frac{1}{1 - \rho S \frac{\partial \phi}{\partial S}(t, S)}\rho S \frac{\partial \phi}{\partial S}(t, S)H(t, x, S). \qquad (25)$$

Simplifying this expression for H we conclude (21), as claimed. □ ∎

The function H is given implicitly by Equation (21). If we expand its solution H in terms of a small parameter ρ, i.e., $H(t, x, S) = H^0(t, x, S) + \rho H^1(t, x, S) + O(\rho^2)$ as $\rho \to 0$, we conclude the following proposition:

Proposition 3.3. *Assume ρ is small. Then the first order approximation of the function $H(t, x, S)$ reads as follows:*

$$H(t, x, S) = S(e^x - 1) + \rho S(\phi(t, Se^x) - \phi(t, S)) + O(\rho^2) \quad as \rho \to 0. \qquad (26)$$

Proposition 3.4. *Assume that the asset price process $S_t = e^{X_t + rt}$ fulfills SDE (18) where the Lévy measure v is such that $\int_{|x| \geq 1} e^{2x}v(\mathrm{d}x) < \infty$. Denote by $V(t, S)$ the price of a derivative security given by*

$$V(t, S) = \mathbb{E}\left[e^{-r(T-t)}\Phi(S_T)|S_t = S\right] = e^{-r(T-t)}\mathbb{E}\left[\Phi(Se^{r(T-t)+X_{T-t}})\right]. \qquad (27)$$

Assume that the pay-off function Φ is a Lipschitz continuous function and the function ϕ has a bounded derivative. Then $V(t, S)$ is a solution to the PIDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}v(t, S)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$+ \int_{\mathbb{R}} V(t, S + H(t, x, S)) - V(t, S) - H(t, x, S)\frac{\partial V}{\partial S}(t, S)v(\mathrm{d}x) = 0, \qquad (28)$$

where $v(t, S)$ and $H(t, x, S)$ are given by (20) and (21), respectively.

Proof: The asset price dynamics of S_t under the $\mathbb{Q}$ measure is given by

$$dS_t = rS_t dt + v(t, S_t)S_t dW_t + \int_{\mathbb{R}} H(t, x, S_t)\tilde{J}_X(dt, dx). \tag{29}$$

If we apply Itô's lemma to $V(t, S_t)$ we obtain $d(V(t, S_t)e^{-rt}) = a(t)dt + dM_t$ where

$$a(t) = \frac{\partial V}{\partial t} + \frac{1}{2}v(t, S_t)^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t \frac{\partial V}{\partial S} - rV$$

$$+ \int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - H(t, x, S_t)\frac{\partial V}{\partial S}(t, S_t)v(dx),$$

$$dM_t = e^{-rt}S_t v(t, S_t)\frac{\partial V}{\partial S}dW_t + e^{-rt}\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t)\tilde{J}_X(dt, dx).$$

Our goal is to show that M_t is a martingale. Consequently, we have $a \equiv 0$ a.s., and V is a solution to (28) (see Proposition 8.9 of (Cont and Tankov 2004)). To prove the term $\int_0^T e^{-rt}\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t)\tilde{J}_S(dt, dy)$ is a martingale it is sufficient to show that

$$\mathbb{E}\left[\int_0^T e^{-2rt}\left(\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t)v(dx)\right)^2 dt\right] < \infty. \tag{30}$$

Since $\sup_{0 \le t \le T}\mathbb{E}[e^{X_{T-t}}] < \infty$ and the pay-off function Φ is Lipschitz continuous, $V(t, S)$ is Lipschitz continuous as well with some Lipschitz constant $C > 0$. As the function $\phi(t, S)$ has bounded derivatives we obtain

$$S|\phi(t, S + H(t, x, S)) - \phi(t, S)| \le S\left|\frac{\partial\phi}{\partial S}\right||H(t, x, S)| \le L|H(t, x, S)|$$

(see Assumption 3.1). Since $H(t, x, S) = S(e^x - 1) + \rho S(\phi(t, S + H(t, x, S)) - \phi(t, S))$ we obtain $|H(t, x, S)|^2 \le S^2(e^x - 1)^2/(1 - \rho L)^2$. As V is Lipschitz continuous with the Lipschitz constant $C > 0$ we have

$$\mathbb{E}\left[\int_0^T e^{-2rt}\left(\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t)v(dx)\right)^2 dt\right]$$

$$\le \frac{C^2}{(1-\rho L)^2}\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} e^{-2rt}|S_t|^2(e^x - 1)^2 v(dx)dt\right] < \infty,$$

because $\sup_{t \in [0,T]}\mathbb{E}[S_t^2] < \infty$. Here $C_0 = \int_{\mathbb{R}}(e^x - 1)^2 v(dx) < \infty$ due to the assumptions made on the measure v. It remains to prove that $\int_0^T e^{-rt}S_t v(t, S_t)\frac{\partial V}{\partial S}(t, S_t)dW_t$ is a martingale. Since $S\frac{\partial\phi}{\partial S}(t, S)$ is assumed to be bounded we obtain

$$0 < v(t, S) = \frac{\sigma}{1 - \rho S\frac{\partial\phi}{\partial S}(t, S)} \le \frac{\sigma}{1 - \rho L} \equiv C_1 < \infty.$$

Therefore $\mathbb{E}\left[\int_0^T e^{-2rt}(\frac{\partial V}{\partial S}(t, S_t)\nu(t, S_t)S_t)^2 \mathrm{d}t\right] \leq C^2 C_1^2 \int_0^T e^{-2rt}\mathbb{E}[S_t^2]\mathrm{d}t < \infty$ because S_t is a martingale. Hence M_t is a martingale as well. As a consequence, $a \equiv 0$ and so V is

a solution to PIDE (28), as claimed. □ □

Remark 2. If $\rho = 0$ then $H(t, x, S) = S(e^x - 1)$ and Equation (28) reduces to:

$$
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV
$$
$$
+ \int_{\mathbb{R}} V(t, Se^x) - V(t, S) - S(e^x - 1)\frac{\partial V}{\partial S}(t, S)\nu(\mathrm{d}x) = 0, \tag{31}
$$

which is the well-known classical PIDE. If there are no jumps ($\nu = 0$) and a trader follows the delta hedging strategy, i.e., $\phi(t, S) = \partial_S V(t, S)$, then Equation (28) reduces to the Frey–Stremme option pricing model:

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\sigma^2}{\left(1 - \rho S \partial_S^2 V\right)^2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{32}
$$

(cf. Frey 1998). Finally, if $\rho = 0$ and $\nu = 0$ Equation (28) reduces to the classical linear Black–Scholes equation.

For simplicity, we assume the interest rate is zero, $r = 0$. Then, the function $V(t, S)$ is a solution to the PIDE:

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\nu(t, S)^2 S^2\frac{\partial^2 V}{\partial S^2}
$$
$$
+ \int_{\mathbb{R}} V(t, S + H(t, x, S)) - V(t, S) - H(t, x, S)\frac{\partial V}{\partial S}(t, S)\nu(\mathrm{d}x) = 0. \tag{33}
$$

Let us define the tracking error of a trading strategy $\alpha_t = \phi(t, S_t)$ as follows: $e_T^M := \Phi(S_T) - V_0 = V(T, S_T) - V_0 - \int_0^T \alpha_t \mathrm{d}S_t$.

By applying Itô's formula to $V(t, S_t)$ and using (33) we obtain

$$
V(T, S_T) - V_0 = V(T, S_T) - V(0, S_0) = \int_0^T \mathrm{d}V(t, S_t)
$$
$$
= \int_0^T \frac{\partial V}{\partial S}\mathrm{d}S_t + \int_0^T \frac{\partial V}{\partial t} + \frac{1}{2}\nu(t, S_t)^2 S_t^2\frac{\partial^2 V}{\partial^2 S}\mathrm{d}t
$$
$$
+ \int_0^T\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - H(t, x, S_t)\frac{\partial V}{\partial S}J_X(\mathrm{d}t, \mathrm{d}x)
$$
$$
= \int_0^T \frac{\partial V}{\partial S}\mathrm{d}S_t - \int_0^T\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - H(t, x, S_t)\frac{\partial V}{\partial S}\nu(\mathrm{d}x)\mathrm{d}t
$$
$$
+ \int_0^T\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - H(t, x, S_t)\frac{\partial V}{\partial S}J_X(\mathrm{d}t, \mathrm{d}x)
$$
$$
= \int_0^T \frac{\partial V}{\partial S}\mathrm{d}S_t + \int_0^T\int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - H(t, x, S_t)\frac{\partial V}{\partial S}\tilde{J}_X(\mathrm{d}t, \mathrm{d}x).
$$

Using expression (29) for the dynamics of the asset price S_t (with $r = 0$), the tracking error e_T^M can be expressed as follows:

Remark 3. For the delta hedging strategy $\alpha_t = \phi(t, S_t) = \frac{\partial V}{\partial S}(t, S_t)$ the tracking error function e_T^M can be expressed as follows:

$$e_T^M = \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - H(t, x, S_t)\frac{\partial V}{\partial S}(t, S_t)\tilde{J}_X(\mathrm{d}t, \mathrm{d}x).$$

Clearly, the tracking error for the delta hedging strategy need not be zero for $v0$.

$$
\begin{aligned}
e_T^M &= V(T, S_T) - V_0 - \int_0^T \alpha_t \mathrm{d}S_t = \int_0^T \left(\frac{\partial V}{\partial S}(t, S_t) - \alpha_t\right)\mathrm{d}S_t \\
&\quad + \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - H(t, x, S_t)\frac{\partial V}{\partial S}\tilde{J}_X(\mathrm{d}t, \mathrm{d}x) \\
&= \int_0^T v(t, S_t)S_t\left(\frac{\partial V}{\partial S} - \alpha_t\right)\mathrm{d}W_t \\
&\quad + \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t, x, S_t)) - V(t, S_t) - \alpha_t H(t, x, S_t)\tilde{J}_X(\mathrm{d}t, \mathrm{d}x).
\end{aligned}
\tag{34}
$$

Next, we propose a criterion that can be used to find the optimal hedging strategy.

Proposition 3.5. *The trading strategy $\alpha_t = \phi(t, S_t)$ of a large trader minimizing the variance $\mathbb{E}\left[(\epsilon_T^M)^2\right]$ of the tracking error is given by the implicit equation:*

$$
\begin{aligned}
\phi(t, S_t) &= \beta^\rho(t, S_t)[v(t, S_t)^2 S_t^2 \frac{\partial V}{\partial S}(t, S_t) \\
&\quad + \int_{\mathbb{R}} (V(t, S_t + H(t, x, S_t)) - V(t, S_t))H(t, x, S_t)v(\mathrm{d}x)],
\end{aligned}
\tag{35}
$$

where $\beta^\rho(t, S_t) = 1/[v(t, S_t)^2 S_t^2 + \int_{\mathbb{R}} H(t, x, S_t)^2 v(\mathrm{d}x)]$ *and* $H(t, x, S) = S(e^x - 1) + \rho S[\phi(t, S + H(t, x, S)) - \phi(t, S)]$.

Proof: Using expression (34) for the tracking error ϵ_T^M and Itô's isometry we obtain

$$
\begin{aligned}
\mathbb{E}\left[(\epsilon_T^M)^2\right] &= \mathbb{E}\left[\int_0^T v(t, S_t)^2 S_t^2\left(\frac{\partial V}{\partial S}(t, S_t) - \alpha_t\right)^2 \mathrm{d}t\right] \\
&\quad + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} (V(t, S_t + H(t, x, S_t)) - V(t, S_t) - \alpha_t H(t, x, S_t))^2 v(\mathrm{d}x)\mathrm{d}t\right].
\end{aligned}
$$

The minimizer α_t of the above convex quadratic minimization problem satisfies the first order necessary conditions $\mathrm{d}(\mathbb{E}\left[\epsilon_T^2\right], \alpha_t) = 0$, that is,

$$0 = -2\mathbb{E}\left[\int_0^T (v(t, S_t)^2 S_t^2 \left(\frac{\partial V}{\partial S}(t, S_t) - \alpha_t\right)\right.$$

$$\left. + \int_{\mathbb{R}} H(t, x, S_t)(V(t, S_t + H(t, x, S_t)) - V(t, S_t) - \alpha_t H(t, x, S_t))v(dx))\omega_t dt\right]$$

for any variation ω_t. Thus the tracking error minimizing strategy α_t is given by (35).

□   ▢

Remark 4. The optimal trading strategy minimizing the variance of the tracking error need not satisfy the structural Assumption 3.1. For instance, if $v = 0$ then the tracking error minimizer is just the delta hedging strategy $\phi = \partial_S V$. In the case of a call or put option its gamma, i.e., $\partial_S^2 V(t, S)$ becomes infinite as $t \to T$ and $S = K$. Given a level $L > 0$ we can however minimize the tracking error $\mathbb{E}[\epsilon_T^2]$ under the additional constraint $\sup_{S>0}\left|S\frac{\partial\phi}{\partial S}(t, S)\right| \le L$. That is we can solve the following convex constrained nonlinear optimization problem

$$\min_{\phi} \; \mathbb{E}[\epsilon_T^2] \quad s.t. \quad |S\partial_S\phi| \le L$$

instead of the unconstrained minimization problem proposed in Proposition 3.5.

Remark 5. Notice that, if $v = 0$ and $\rho \ge 0$, the trading strategy α_t reduces to the Black–Scholes delta hedging strategy, i.e., $\alpha_t = \frac{\partial V}{\partial S}(t, S_t)$. If $v \ne 0$ and $\rho = 0$, then the optimal trading strategy becomes $\alpha_t = \phi^0(t, S_t)$ where

$$\phi^0(t, S_t) = \beta^0(t, S_t)\left(\sigma^2 S_t^2 \frac{\partial V}{\partial S}(t, S_t) + \int_{\mathbb{R}} S_t(e^x - 1)(V(t, S_t e^x) - V(t, S_t))v(dx)\right),$$

where $\beta^0(t, S_t) = 1/\left[\sigma^2 S_t^2 + \int_{\mathbb{R}} S_t^2(e^x - 1)^2 v(dx)\right]$.

We conclude this section by the following proposition providing the first order approximation of the tracking error minimizing trading strategy for the case when the parameter $\rho \ll 1$ is small. In what follows, we derive the first order approximation of $\phi^\rho(t, S_t)$ in the form $\phi^\rho(t, S_t) = \phi^0(t, S_t) + \rho\phi^1(t, S_t) + O(\rho^2)$ as $\rho \to 0$.

Clearly, the first order Taylor expansion for the volatility function $v(t, S)$ has the form:

$$v(t, S)^2 = \frac{\sigma^2}{(1 - \rho S\partial_S\phi)^2} = \sigma^2 + 2\rho\sigma^2 S\frac{\partial\phi^0}{\partial S}(t, S) + O(\rho^2), \; \text{as } \rho \to 0.$$

With regard to Proposition 3.3 (see (26)) we have $H(t, x, S) = H^0(t, x, S) + \rho H^1(t, x, S) + O(\rho^2)$, where

$$H^0(t, x, S) = S(e^x - 1), \qquad H^1(t, x, S) = S[\phi^0(t, Se^x) - \phi^0(t, S)]. \tag{36}$$

The function β^ρ can be expanded as follows: $\beta^\rho(t, S) = \beta^0(t, S) + \rho\beta^{(1)}(t, S) + O(\rho^2)$,

$$\beta^0(t,S) = 1/[\sigma^2 S^2 + S^2 \int_{\mathbb{R}} (e^x - 1)^2 v(dx)],$$

$$\beta^{(1)}(t,S) = -(\beta^0(t,S))^2 \left[2\sigma^2 S^3 \frac{\partial \phi^0}{\partial S}(t,S) + 2S^2 \int_{\mathbb{R}} (e^x - 1)[\phi^0(t,Se^x) - \phi^0(t,S)]v(dx) \right].$$
(37)

Using the first order expansions of the functions v^2, β^ρ and H we obtain the following results.

Proposition 3.6. *For small values of the parameter $\rho \ll 1$, the tracking error variance minimizing strategy $\alpha_t = \phi^\rho(t,S_t)$ is given by*

$$\phi^\rho(t,S_t) = \phi^0(t,S_t) + \rho\phi^{(1)}(t,S_t) + O(\rho^2), \ as \rho \to 0,$$
(38)

where
$$\phi^{(1)}(t,S) = \beta^0(t,S) \left[2\sigma^2 S^3 \frac{\partial V}{\partial S}(t,S) \frac{\partial \phi^0}{\partial S}(t,S) \right.$$
$$+ \int_{\mathbb{R}} \left(V(t,Se^x) - V(t,S) + \frac{\partial V}{\partial S}(t,Se^x)H^0(t,x,S) \right) H^1(t,x,S)v(dx) \right]$$
$$+ \beta^{(1)}(t,S) \left[\sigma^2 S^2 \frac{\partial V}{\partial S}(t,S) + \int R(V(t,Se^x) - V(t,S))H^0(t,x,S)v(dx) \right]$$

and the functions H^0, H^1, β^0 and $\beta^{(1)}$ are defined as in (36) and (37).

4. Implicit-explicit numerical discretization scheme

The aim of this section is to propose a full time-space discretization scheme for solving the nonlinear PIDE (28). The method of discretization is based on a finite difference approximation of all derivatives occurring in (28) and approximation of the integral term by means of the trapezoidal integration rule on a truncated domain.

In order to solve (28), we transform it into a nonlinear parabolic PIDE defined on $\mathbb{R}$. Indeed, using the standard transformations $V(t,S) = e^{-r\tau}u(\tau,x), \phi(t,S) = \psi(\tau,x)$ where $\tau = T - t, x = \ln(\frac{S}{K})$ we conclude that $V(t,S)$ is a solution to (28) if and only if the function $u(\tau,x)$ solves the following nonlinear parabolic equation:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2} \frac{\sigma^2}{(1-\rho\frac{\partial \psi}{\partial x})^2} \frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2} \frac{\sigma^2}{(1-\rho\frac{\partial \psi}{\partial x})^2} \right) \frac{\partial u}{\partial x}$$
$$+ \int_{\mathbb{R}} u(\tau, x + \xi(\tau,z,x)) - u(\tau,x) - H(T-\tau,z,Ke^x)\frac{1}{K}e^{-x}\frac{\partial u}{\partial x}(\tau,x)v(dz),$$
(39)

$$u(0,x) = h(x) \equiv \Phi(Ke^x), \quad (\tau,x) \in [0,T] \times \mathbb{R},$$
(40)

and

$$H(t,z,S) = S(e^z - 1) + \rho S[\phi(t, S + H(t,z,S)) - \phi(t,S)],$$
(41)

$$\xi(\tau, z, x) = \ln(1 + K^{-1}e^{-x}H(T - \tau, z, Ke^x)). \tag{42}$$

4.1. *A numerical scheme for solving nonlinear pides with finite activity Lévy measures*

We first consider the case when the Lévy measure ν has a finite activity, i.e., $\nu(\mathbb{R}) < \infty$. Let us denote

$$\lambda = \int_{\mathbb{R}} \nu(dz), \quad \text{and} \quad \omega(\tau, x) = \int_{\mathbb{R}} H(T - \tau, z, S_0 e^x) \frac{1}{S_0} e^{-x} \nu(dz).$$

We have $\lambda < \infty$. Observe that (39) is equivalent to

$$\frac{\partial u}{\partial \tau} = \frac{1}{2} \frac{\sigma^2}{(1 - \rho \frac{\partial \psi}{\partial x})^2} \frac{\partial^2 u}{\partial^2 x} + \left(r - \frac{1}{2} \frac{\sigma^2}{(1 - \rho \frac{\partial \psi}{\partial x})^2} - \omega \right) \frac{\partial u}{\partial x} - \lambda u$$

$$+ \int_{\mathbb{R}} u(\tau, x + \xi(\tau, z, x)) \nu(dz). \tag{43}$$

We proceed to solve (43) by means of the semi-implicit finite difference scheme proposed in (Voltchkova 2005). The idea is to separate the right-hand side into two parts: the differential part and the integral part.

Let $u_i^j = u(\tau_j, x_i), \tau_j = j\Delta\tau, x_i = z_i = i\Delta x$ for $i = -N + 1, \cdots, N - 1$ and $j = 1, \cdots, M$. We approximate the differential part implicitly except of $\psi(\tau, x)$

$$\left(\frac{\partial u}{\partial x} \right)_i^j \approx \begin{cases} \frac{u_{i+1}^{j+1} - u_i^{j+1}}{\Delta x}, & \text{if } \frac{(\sigma_i^j)^2}{2} - r + \omega_i^j < 0, \\ \frac{u_i^{j+1} - u_{i-1}^{j+1}}{\Delta x}, & \text{if } \frac{(\sigma_i^j)^2}{2} - r + \omega_i^j \geq 0, \end{cases} \quad \sigma_i^j = \frac{\sigma}{1 - \rho D\psi_i^j},$$

$$\left(\frac{\partial^2 u}{\partial x^2} \right)_i^j \approx \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2},$$

$$\left(\frac{\partial u}{\partial \tau} \right)_i^j \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}. \quad \left(\frac{\partial \psi}{\partial x} \right)_i^j \approx \frac{\psi_{i+1}^j - \psi_i^j}{\Delta x} = D\psi_i^j.$$

As for the integral operator, first, we have to truncate the integration domain to a bounded interval $[B_l, B_r]$. We approximate this integral by choosing integers K_l and K_r such that $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_l + 1/2)\Delta x]$. Then

$$\int_{B_l}^{B_r} u(\tau_j, x_i + \xi(\tau_j, z_i, x_i)) \nu(dz) \approx \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) \nu_k, \tag{44}$$

where $\nu_k = \frac{1}{2} (\nu(z_{k+1/2}) + \nu(z_{k-1/2})) \Delta x$. Analogously,

$$\omega_i^j \approx \frac{e^{-x_i}}{K} \sum_{k=K_l}^{K_r} H(T - \tau_j, z_k, Ke^{x_i}) \nu_k, \quad \text{and} \quad \lambda \approx \sum_{k=K_l}^{K_r} \nu_k,$$

where $\xi(\tau, z, x)$ is given as in (42).

Inserting the finite difference approximations of derivatives of u into (43) we obtain

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{1}{2}(\sigma_i^j)^2 \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2} - \lambda u_i^{j+1}$$
$$+ \left(r - \frac{1}{2}(\sigma_i^j)^2 - \omega_i^j\right)\frac{u_{i+1}^{j+1} - u_i^{j+1}}{\Delta x} + \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(\tau_j, z_k, x_i))v_k, \tag{45}$$

provided that $\frac{1}{2}(\sigma_i^j)^2 - r + \omega_i^j < 0$. Similarly, we can derive a differential equation for the case $\frac{1}{2}(\sigma_i^j)^2 - r + \omega_i^j \geq 0$. If we define coefficients $\beta_{i\pm}^j$, and β_i^j as follows:

$$\beta_{i\pm}^j = -\frac{\Delta\tau}{2(\Delta x)^2}(\sigma_i^j)^2 - \frac{\Delta\tau}{\Delta x}\left(r - \frac{1}{2}(\sigma_i^j)^2 - \omega_i^j\right)^{\pm}, \tag{46}$$

$$\beta_i^j = 1 + \Delta\tau\lambda - (\beta_{i-}^j + \beta_{i+}^j), \tag{47}$$

where $(a)^+ = \max(a, 0)$, $(a)^- = \min(a, 0)$. Then, the tridiagonal system of linear equations for the solution, reads as follows:

$$u_i^0 = h(x_i), \text{ for } i = -N+1, \cdots, N-1,$$
$$u_i^{j+1} = g(\tau_{j+1}, x_i), \text{ for } i = -N+1, \cdots, -N/2 - 1,$$
$$\beta_{i+}^j u_{i+1}^{j+1} + \beta_i^j u_i^{j+1} + \beta_{i-}^j u_{i-1}^{j+1} = u_i^j + \Delta\tau \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(\tau_j, z_k, x_i))v_k, \tag{48}$$
$$\text{for } i = -N/2 + 1, \cdots, N/2 - 1,$$
$$u_i^{j+1} = g(\tau_{j+1}, x_i), \text{ for } i = N/2, \cdots, N-1,$$

where

$$\xi(\tau_j, z_k, x_i) = \ln(1 + K^{-1}e^{-x_i}H(T - \tau_j, z_k, Ke^{x_i})),$$

and g is a function of points x_i lying outside the localization interval. Following Proposition 4.3.1 in (Voltchkova 2005) the recommended choice is $g(\tau, x) = h(x + r\tau) = \Phi(Ke^{r\tau+x})$. The term $u(\tau_j, x_i + \xi(\tau_j, z_k, x_i))$ entering the sum in the right-hand side of (48) is approximated by means of the first order Taylor series expansion:

$$u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) \approx u_i^j + \frac{u_{i+1}^j - u_i^j}{\Delta x}\xi(\tau_j, z_k, x_i).$$

4.2. Numerical scheme for solving nonlinear pides with infinite activity Lévy measures

Next, we consider the case when the Lévy measure has an infinite activity, e.g., the Variance Gamma process where its Lévy density explodes at zero and $v(\mathbb{R}) = \infty$. Equation (39) is equivalent to

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\sigma^2}{(1-\rho\frac{\partial \psi}{\partial x})^2}\frac{\partial^2 u}{\partial^2 x} + \left(r - \frac{1}{2}\frac{\sigma^2}{(1-\rho\frac{\partial \psi}{\partial x})^2} - \omega\right)\frac{\partial u}{\partial x}$$
$$+ \int_{\mathbb{R}} u(\tau, x + \xi(\tau, z, x)) - u(\tau, x)v(\mathrm{d}z). \tag{49}$$

Equation (49) differs from (43) as the term $u(\tau, x)$ is contained in an integral part because $\lambda = \int_{\mathbb{R}} v(\mathrm{d}z) = \infty$. Proceeding similarly as for discretization of (43) we can solve (49) numerically by means of a semi-implicit finite difference scheme. If the coefficients $\beta_{i\pm}^{j}$ are defined as in (46) and $\beta_i^j = 1 - (\beta_{i-}^j + \beta_{i+}^j)$, then the solution vector $u^j = (u_{-N+1}^j, \cdots, u_{N-1}^j)^T, j = 0, \cdots, M$, is a solution to the following tridiagonal system of linear equations:

$$u_i^0 = h(x_i), \text{ for } i = -N+1, \cdots, N-1,$$
$$u_i^{j+1} = g(\tau_{j+1}, x_i), \text{ for } i = -N+1, \cdots, -N/2 - 1,$$
$$\beta_{i+}^j u_{i+1}^{j+1} + \beta_i^j u_i^{j+1} + \beta_{i-}^j u_{i-1}^{j+1} = u_i^j + \Delta\tau \sum_{k=K_l}^{K_r} \left(u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) - u(\tau_j, x_i)\right)v_k,$$
$$\text{ for } i = -N/2 + 1, \cdots, N/2 - 1,$$
$$u_i^{j+1} = g(\tau_{j+1}, x_i), \text{ for } i = N/2, \cdots, N-1. \tag{50}$$

The term $u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) - u(\tau_j, x_i)$ entering the sum in the right-hand side of (50) is again approximated by means of the first order Taylor series expansion, i.e.,

$$u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) - u(\tau_j, x_i) \approx \frac{u_{i+1}^j - u_i^j}{\Delta x}\xi(\tau_j, z_k, x_i).$$

5. Numerical results

In this section, we present results of numerical experiments using the finite difference scheme described in Section 4 for the case of a European put option, i.e., $\Phi(S) = (K - S)^+$. As for the Lévy process, we consider the Variance Gamma process with parameters $\theta = -0.33, \sigma = 0.12, \kappa = 0.16$, and other option pricing model parameters: $r = 0, K = 100, T = 1$. Numerical discretization parameters were chosen as follows: $\Delta x = 0.01, \Delta t = 0.005$. Since the Variance Gamma process has infinite activity, we employ numerical discretization scheme described in Section 4.2. In what follows, we present various option prices computed by means of the finite-difference numerical scheme described in Section 4 for the linear Black–Scholes ($\rho = 0$) and the Frey–Stremme model ($\rho > 0$) and their jump-diffusion PIDE generalizations.

In Figure 1 we show a comparison of European put option prices between the classical PIDE and the linear Black–Scholes model, and comparison between the classical PIDE and the Frey–Stremme PIDE model for the case when the large trader's
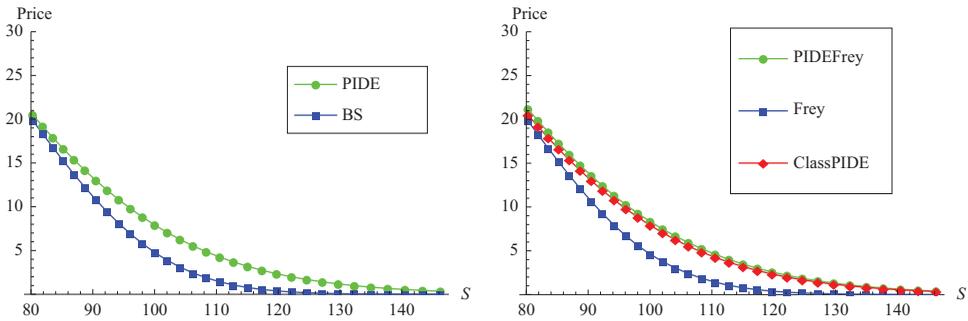
Figure 1. Comparison of European put option prices between the classical PIDE and the linear Black–Scholes model (left). Comparison between the classical PIDE and the Frey–Stremme PIDE model (right).
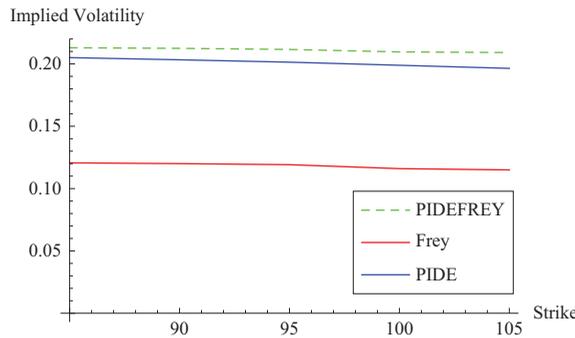

Figure 2. Comparison of implied volatilities between the Frey–Stremme model, classical PIDE and Frey–Stremme PIDE generalization.

influence is small, $\rho = 0.001$. In Figure 2 we plot dependence of the implied volatilities as decreasing functions of the strike price K for the Frey–Stremme model and its PIDE generalizations. We can observe that the implied volatilities for the Frey–Stremme PIDE model is always higher when varying the strike price of the European Put option.

Numerical values of option prices for various models and parameter settings are summarized in Tables 1 and 2. Numerical results confirm our expectation that assuming risk arising from sudden jumps in the underlying asset process yields a higher option price when comparing to the Frey–Stremme model option price.

In Figure 3 (left) we compare European put option prices $V(0, S)$ computed by means of the Black–Scholes and Frey–Stremme models depending on the parameter ρ measuring the influence of a large trader. We can observe that the price of the European put option increases with respect to ρ, as expected. Furthermore, the price computed from the Frey–Stremme PIDE model is larger than the one obtained from the linear Black–Scholes equation. Moreover, the price computed from Frey–Stremme PIDE model is higher than the one computed by means of the nonlinear Frey–Stremme model. This is due to the fact that the jump part of the underlying asset process enhances risk, and, consequently increases the option price. Figure 3 (right) shows

Table 1. European put option prices $V(0, S)$ for the Black-Scholes and Frey–Stremme models with $\rho = 0.001$ and their PIDE generalizations.

S	B-S $v = 0, \rho = 0$	F-S $v = 0, \rho \neq 0$	B-S PIDE $v \neq 0, \rho = 0$	F-S PIDE $v \neq 0, \rho \neq 0$
61.8783	38.1217	38.1258	38.2297	38.8234
67.032	32.9691	32.9763	33.4319	34.1889
72.6149	27.3972	27.4207	28.4887	29.4425
78.6628	21.4275	21.5118	23.5224	24.6911
85.2144	15.2547	15.4835	18.6979	20.0701
92.3116	9.42895	9.85754	14.2078	15.7321
100.	4.78444	5.32697	10.243	11.8282
108.329	1.88555	2.34727	6.95353	8.48304
117.351	0.550422	0.814477	4.41257	5.77178
127.125	0.114716	0.216426	2.60009	3.70615
137.713	0.016615	0.043112	1.41444	2.2351

Table 2. European put option prices $V(0, S)$ for the Frey–Stremme and Frey–Stremme PIDE models for various values of ρ.

| S	F-S $\rho = 0.1$	F-S PIDE $\rho = 0.1$	F-S $\rho = 0.2$	F-S PIDE $\rho = 0.2$	F-S $\rho = 0.3$	F-S PIDE $\rho = 0.3$
61.8783	38.1257	38.4958	38.1258	38.8234	38.1373	39.2259
67.032	32.9759	33.7763	32.9763	34.1889	33.019	34.6865
72.6149	27.4191	28.9293	27.4207	29.4425	27.5623	30.049
78.6628	21.5061	24.0698	21.5118	24.6911	21.8893	25.4118
85.2144	15.4688	19.3477	15.4835	20.0701	16.2645	20.896
92.3116	9.83127	14.9344	9.85754	15.7321	11.0916	16.6367
100.	5.29421	10.9999	5.32697	11.8282	6.8043	12.7672
108.329	2.31882	7.68096	2.34727	8.48304	3.68338	9.4005
117.351	0.797286	5.05246	0.814477	5.77178	1.72932	6.61053
127.125	0.209195	3.11214	0.216426	3.70615	0.693804	4.41995
137.713	0.040995	1.78547	0.043112	2.2351	0.234949	2.79821
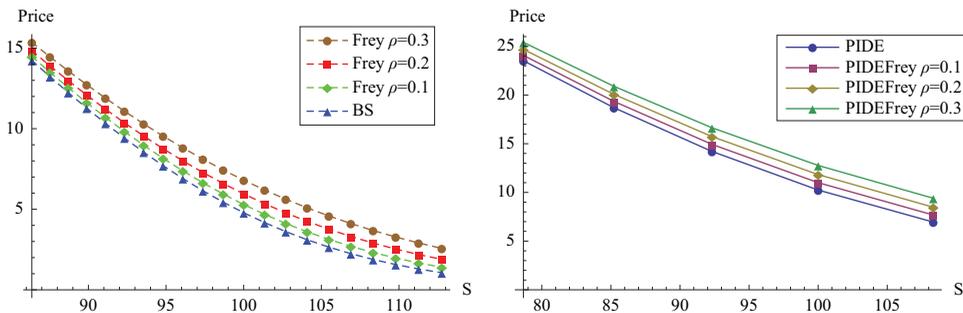
Figure 3. Comparison of European put option prices for the Black–Scholes and the Frey–Stremme models (left) and Frey–Stremme PIDE model for various ρ.

a comparison of the option prices for the Black–Scholes and Frey–Stremme PIDE model for various values of ρ.

6. Conclusion

In this paper we investigated a novel nonlinear option pricing model generalizing the Frey–Stremme model under the assumption that the underlying asset price follows a Lévy stochastic process. We derived the fully nonlinear PIDE for pricing options under influence of a large trader. We also proposed the hedging strategy minimizing the variance of the tracking error. We derived a semi-implicit finite difference numerical approximation scheme for solving the nonlinear PIDE. We presented various numerical experiments illustrating the influence of the large trader under the Lévy process with jumps.

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