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PDE models for American options with counterparty risk and two stochastic factors: Mathematical analysis and numerical solution

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ABSTRACT

In this article we propose new linear and nonlinear partial differential equations (PDEs) models for pricing American options and total value adjustment in the presence of counterparty risk. An innovative aspect comes from the consideration of stochastic spreads, which increases the dimension of the problem. In this setting, we pose new complementarity problems associated to linear and nonlinear PDEs. Moreover, using the mathematical tools of semilinear variational inequalities for parabolic equations, we prove the existence and uniqueness of a solution for these models. For the numerical solution, we mainly combine a semi-Lagrangian time discretization scheme, a fixed point method to cope with nonlinear terms and a finite element method for the spatial discretization, jointly with an augmented Lagrangian active set method to solve the fully discretized complementarity problem. Finally, numerical examples illustrate the expected behaviour of the option prices and the corresponding total value adjustment, as well as the performance of the proposed numerical techniques. Moreover, we compare the numerical results from the PDEs approach with those obtained by applying Monte Carlo techniques.

1. Introduction

Since the last financial crisis, suitable relevant adjustments are taken into account when pricing financial derivatives. Their goal is to mitigate the counterparty risk associated to the derivatives contract. Among all financial derivatives usually traded in the markets, European and American options are some of the most popular. The number of these adjustments is continuously increasing, so that the term *total value adjustment or XVA* is currently used to refer to the set of all of them. The formulation of suitable mathematical models in terms of partial differential equations (PDEs) allows the use of efficient numerical methods to compute the price of the XVA. These methods become competitive with respect to alternative Monte Carlo techniques applied to models which are formulated in terms of expectations. PDEs formulations require the statement of well posed problems to guarantee the existence and uniqueness of a solution. However, the mathematical analysis of these formulations turns out to be more difficult due to the presence of nonlinear PDEs and

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additional inequality constraints arising in the American options pricing problem. It is also important to consider the appropriate stochastic factors that are involved in the modelling of counterparty risk.

As counterparty risk is related to the possible default of different parts in the contract, the intensity of default is an important issue in the modelling. In the previous work [1] we have considered constant intensities of default (from both counterparties, namely the hedger and the investor) to study PDEs models for European and American options, thus involving only one stochastic factor. Also in the setting of constant spread for American options, the numerical solution by means of Monte Carlo techniques has been addressed in [2].

However, default intensities from counterparties do not always exhibit constant behaviour. In particular, if stochastic intensities of default are assumed for both risky counterparties, then a model with three stochastic factors is obtained (see [3]). Following [3], we have analysed PDEs models for European options depending on two stochastic factors in [4].

In [1] we addressed the numerical solution of American options in the case of constant spreads. In the present paper, on one hand we extend to the case of stochastic spreads the work [1], also adding the mathematical analysis of the two factors model for American options. On the other hand, we extend to American options the work developed in [4] for European ones with stochastic spreads. Note that the consideration of American options, instead of European ones, requires a completely new setting and methodology to obtain the existence and uniqueness of a solution. This methodology is based on the mathematical analysis of complementarity problems associated to semilinear parabolic variational inequalities. Also, additional numerical methods are required to cope with the resulting fully discretized complementarity problem. Finally, note that we also extend to the setting of stochastic spreads the use of Monte Carlo techniques proposed in [2] for the case of constant spreads.

Throughout this article, we refer to the risk-free derivative when counterparty risk is not considered, so that classical models for American options apply. Moreover, we use the term risky derivative when counterparty risk is taken into account.

Our approach is based on complementarity problems associated to PDEs operators, that are obtained by means of suitable hedging arguments and Itô's lemma for jump–diffusion processes. This approach extends the classical Black–Scholes inequality for American options without counterparty risk. Thus, different kinds (linear or nonlinear, depending on the assumption of the mark-to-market value at default) of complementarity problems arise. More precisely, a mark-to-market value which is equal to the risk-free derivative leads to a linear complementarity problem involving the value of the risk-free derivative, while a mark-to-market value which is equal to the risky derivative leads to a nonlinear complementarity problem.

In order to state the existence and uniqueness of a solution for the nonlinear complementarity problem we follow the methodology introduced by Jeong and Park [5] for semilinear parabolic variational inequalities, which is based on previous works by Brézis [6,7] for the linear case. We also refer to the works of Stampacchia [8,9] and Kinderlehrer and Stampacchia [10] for details about the analysis of variational inequalities and complementarity problems.

In addition, we propose a set of numerical methods to solve the complementarity problems for both choices of the mark-to-market value. For this purpose, we truncate the unbounded domain and propose suitable conditions at the boundaries of the resulting bounded domain, following ideas from [11]. After this truncation, we propose a time discretization based on the method of characteristics combined with a finite element discretization in the asset and spread variables. For the nonlinear PDE, a fixed point iteration algorithm is proposed. Finally, the augmented Lagrangian active set (ALAS) algorithm is used to solve the discretized complementarity problems.

The plan of the paper is the following. In Section 2, by using hedging arguments we state the complementarity problems. In Section 3, we develop the mathematical analysis of these problems to obtain the existence and uniqueness of a solution. Section 4 presents numerical methods and Section 5 shows some illustrative numerical results. In order to validate these results, some tests have also been solved by the extension of the Monte Carlo techniques described in [2]. Finally, conclusions are presented in Section 6.

2. Mathematical model

In this section, we pose the PDEs models for American options when considering the counterparty risk with two stochastic factors. The main difference with the one factor model introduced in [1] comes from the consideration of stochastic intensities of default instead of constant ones. In order to establish these models, some assumptions and techniques similar to those ones presented in [4] for European options with stochastic spreads will be used, namely self-financing portfolio and non-arbitrage scenarios. Therefore, in forthcoming steps we will address the readers to [4] when necessary.

First, we assume the investor as a risky counterparty and that the intensity of default of the issuer is zero. Therefore, the underlying asset price S_t and the short term CDS spread of the investor h_t satisfy the system of stochastic differential equations:

$$dS_t = (r(t) - q(t))\, S_t\, dt + \sigma^S(t)\, S_t\, dW_t^S\,, \tag{1}$$

$$dh_t = (\mu^h(t) - M^h(t)\sigma^h(t))\, dt + \sigma^h(t)\, dW_t^h\,, \tag{2}$$

where $(r(t) - q(t))$ and $(\mu^h(t) - M^h(t)\sigma^h(t))$ represent the drifts of S_t and h_t, respectively. Moreover, $r(t)$ is the given risk-free interest rate, $q(t)$ is the given asset dividend yield rate, $M^h(t)$ denotes the market price of investor's credit risk,

$\sigma^S(t)$ and $\sigma^h(t)$ are the volatility functions, and W_t^S and W_t^h are two correlated Wiener processes (i.e., $\rho\, dt = dW_t^S\, dW_t^h$), such that ρ is the instantaneous correlation between S_t and h_t. In the present work, we consider r, q, σ^S, σ^h, μ^h and M^h as constant parameters. Note that considering deterministic functions for these parameters does not involve major difficulties.

Next, we assume that the American option is traded between a hedger and an investor, where the latter has nonzero probability of default. From the point of view of the investor, the price of the American option with counterparty risk at time t is denoted by $\widehat{V}_t = \widehat{V}(t, S_t, h_t, J_t^I)$. Thus, it depends on the asset price (S_t), on the spread of the investor (h_t) and on the state of default of the investor at time t (J_t^I), where $J_t^I = 1$ in case of default before or at time t, otherwise $J_t^I = 0$. The risk-free American option value is denoted by $V_t = V(t, S_t)$ and it is the solution of the analogous PDEs problem with a null probability of default.

Note that the intensity of default of the investor can be expressed in terms of the spread as follows:

$$\lambda_t = \frac{h_t}{1-R}, \tag{3}$$

where R is the recovery rate. Moreover, if the investor enters bankrupt then the American option price is given by

$$\widehat{V}(t, S_t, h_t, 1) = RM^+(t, S_t, h_t) + M^-(t, S_t, h_t),$$

where $M(t, S_t, h_t)$ denotes the mark-to-market value close-out, $Z^+ = \max(Z, 0)$ and $Z^- = \min(Z, 0)$. We also define $\Delta\widehat{V}$ as the variation of the derivative value, $\widehat{V}$, when the investor defaults. It is given by

$$\Delta\widehat{V}_t = RM_t^+ + M_t^- - \widehat{V}_t, \tag{4}$$

where $M_t = M(t, S_t, h_t)$. The close-out is understood as the net present value of the residual deal when one party defaults. A detailed discussion about close-out conventions is addressed in [12] and [13]. A risk-free close-out assumes that the surviving counterparty is default-free. From the legal perspective, ISDA document [14] suggests that a replacement close-out should be used, so that the default risk of the survived party is taken into account. This is specially relevant in DVA. For simplicity and following the assumption in the seminal article [15] concerning PDEs modelling, we consider two possibilities for the mark-to-market value of close-out: either equal to the risk-free value or equal to the derivative value including counterparty risk.

With the purpose of pricing the American option with counterparty risk, the option is hedged by a self-financing portfolio Π_t, which is designed to hedge all underlying risk factors. In this case, we can use the same portfolio as in [4] for European options with two factors and consider the following hedging inequality associated to American options (unlike the equality used for European ones):

$$d\widehat{V}_t \le d\Pi_t. \tag{5}$$

Next, by applying Itô's Lemma for jump diffusion processes and the same methodology as in [4], with the difference coming from the consideration of the inequality (5), we can obtain the PDEs model for pricing American options with counterparty risk and two stochastic factors.

More precisely, assuming that $\widehat{V}_t = \widehat{V}(t, S_t, h_t)$, then the function $\widehat{V}$ satisfies the following complementarity problem in the domain $[0, T) \times (0, \infty) \times (0, \infty)$:

$$\begin{cases} \mathcal{L}(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \widetilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta\widehat{V}}{1-R}h - f\widehat{V} \le 0 \\ \widehat{V}(t, S, h) \ge G(S) \\ \mathcal{L}(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(T, S, h) = G(S), \end{cases} \tag{6}$$

where $G(S)$ represents the option payoff, and the differential operator $\widetilde{\mathcal{L}}_{Sh}$ is given by

$$\begin{aligned} \widetilde{\mathcal{L}}_{Sh}V \equiv\ & \frac{1}{2}(\sigma^S)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}(\sigma^h)^2 \frac{\partial^2 V}{\partial h^2} + \rho\sigma^S\sigma^h S \frac{\partial^2 V}{\partial h \partial S} \\ & + (r-q)S\frac{\partial V}{\partial S} + \left(\mu^h - M^h\sigma^h\right)\frac{\partial V}{\partial h}. \end{aligned} \tag{7}$$

Moreover, the term $\Delta\widehat{V}$ is given in terms of the mark-to-market value (see (4)). As for the case of European options in [4], the differential operator $\widetilde{\mathcal{L}}_{Sh}$ can be rewritten in terms of the spread h, due to the following relationship between the drift of the spread $\left(\mu^h - M^h\sigma^h\right)$ and the investor's intensity of default λ:

$$\mu^h - M^h\sigma^h = -\kappa\lambda. \tag{8}$$

Therefore, the differential operator (7) can be rewritten as follows:

$$\mathcal{L}_{Sh}V \equiv \frac{1}{2}(\sigma^S)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}(\sigma^h)^2 \frac{\partial^2 V}{\partial h^2} + \rho\sigma^S\sigma^h S \frac{\partial^2 V}{\partial h \partial S} + (r-q)S\frac{\partial V}{\partial S} - \frac{\kappa}{1-R}h\frac{\partial V}{\partial h}.$$

According to (4) and the possible choices of the mark-to-market value at default, two alternative complementarity problems are obtained:

- If $M = \widehat{V}$, we deduce the following nonlinear complementarity problem:

$$\begin{cases} \mathcal{L}_1(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - f\widehat{V} - h\widehat{V}^+ \leq 0, & \text{in } [0,T)\times(0,\infty)\times(0,\infty) \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}_1(\widehat{V})(\widehat{V}-G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \tag{9}$$

- If $M = V$, the following linear complementarity problem is derived:

$$\begin{cases} \mathcal{L}_2(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - \left(\dfrac{h}{1-R} + f\right)\widehat{V} \\ \qquad\qquad -((1-R)V^+ - V)\dfrac{h}{1-R} \leq 0, \quad \text{in } [0,T)\times(0,\infty)\times(0,\infty) \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}_2(\widehat{V})(\widehat{V}-G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \tag{10}$$

Unlike the case of European options, for the case of American ones the calculus of the XVA value, $U = \widehat{V} - V$, requires the previous computation of the risk-free American option value, V. Thus, the following linear complementarity problem satisfied by V has to be solved:

$$\begin{cases} \mathcal{L}_3(V) = \dfrac{\partial V}{\partial t} + \mathcal{L}_S V - fV \leq 0\,, & \text{in } [0,T)\times(0,\infty) \\ V(t,S) \geq G(S) \\ \mathcal{L}_3(V)(V-G) = 0 \\ V(T,S) = G(S)\,, \end{cases} \tag{11}$$

where the linear operator $\mathcal{L}_S$ is given by

$$\mathcal{L}_S V \equiv \frac{(\sigma^S)^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S}\,.$$

For the numerical solution of (9) and (10) we propose a finite element method, which requires the use of a bounded computational domain. Thus, we choose S_∞ and h_∞ to be large enough real numbers. We define the computational domain $\Omega = (0, S_\infty)\times(0, h_\infty)$, so that the numerical solution is not affected by the choice of S_∞ and h_∞ in the range of values of S and h with financial relevance. Next, we need to impose appropriate boundary conditions on the risky derivative value for the bounded domain Ω. For this purpose, we consider the same boundary conditions for V and $\widehat{V}$ as in the case of European options [4]. More precisely, at $S = 0$ and $S = S_\infty$, the American option value is given by:

$$\begin{cases} \widehat{V}(t,0,h) = V(t,0) = V_0(t)\,, \\ \widehat{V}(t,S_\infty,h) = V(t,S_\infty) = V_\infty(t)\,, \end{cases} \tag{12}$$

where the functions $V_0(t)$ and $V_\infty(t)$ are given by:

$$V_0(t) = \begin{cases} 0\,, & \text{for a call option,} \\ K\,\exp(-f(T-t))\,, & \text{for a put option,} \end{cases} \tag{13}$$

$$V_\infty(t) = \begin{cases} S_\infty - K\,, & \text{for a call option,} \\ 0\,, & \text{for a put option.} \end{cases} \tag{14}$$

If $h = 0$ (which is equivalent to a null intensity of default λ), then the derivative contract has no counterparty risk. Thus, an appropriate choice of the boundary condition is to consider $\widehat{V}(t,S,0) = V(t,S)$.

Finally, in order to impose the boundary condition at $h = h_\infty$, we introduce the matrix

$$A = \frac{1}{2}\begin{pmatrix} (\sigma^S)^2 S^2 & \rho\sigma^S\sigma^h S \\ \rho\sigma^S\sigma^h S & (\sigma^h)^2 \end{pmatrix} \tag{15}$$

and assume the nonhomogeneous Neumann boundary condition for $h = h_\infty$: $(A\nabla\widehat{V}\cdot\vec{n}) = (A\nabla V\cdot\vec{n})$, where $\vec{n}$ denotes the unit outer normal vector to $\partial\Omega$.

In the next section, the existence and uniqueness of a solution of problem (9) are studied. First, in order to obtain a system of PDEs with homogeneous boundary conditions, we introduce the problem which models the XVA. For this purpose, we split up the risky derivative value, $\widehat{V}$, as the sum of the XVA, U, plus the risk-free derivative price, V, i.e. $\widehat{V} = V + U$. By introducing this splitting in (9), the following nonlinear complementarity problem is deduced:

$$\begin{cases} \mathcal{L}_t(U) = \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh}U - fU - h(U+V)^+ \leq -\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\,, \quad \text{in } [0,T)\times\Omega \\ U(t,S,h) \geq G(S) - V(t,S) \\ \left[\mathcal{L}_t(U) - \left(-\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\right)\right]\left[U - (G(S) - V(t,S))\right] = 0 \\ U(T,S,h) = 0 \\ U(t,0,h) = 0 \\ U(t,S_\infty,h) = 0 \\ U(t,S,0) = 0 \\ (A\nabla U\cdot\vec{n})(t,S,h_\infty) = 0\,. \end{cases} \tag{16}$$

The same boundary conditions are considered for the linear problem (10).

3. Mathematical analysis of the nonlinear PDEs problem

In this section, for a given function V, we prove the existence and uniqueness of solution for problem (16). Then, taking into account the existence and uniqueness of a solution V for the classical problem (11) (see [16], for example), we can state the existence and uniqueness of a solution for (9). The mathematical analysis of the linear complementarity problem (10) is much simpler.

As the PDEs problem (16) includes a terminal condition, it is a terminal–boundary value problem. Moreover, the matrix A defined in (15) contains variable coefficients and it becomes degenerate at some boundaries. Next, in order to rewrite problem (16) as an equivalent initial–boundary value problem with a constant matrix, we introduce the time to maturity variable, $\tau = T - t$, as well as the new variables and functions:

$$x = \ln\frac{S}{K}, \qquad u(\tau,x,h) = U(t,S,h), \qquad v(\tau,x) = V(t,S)\,.$$

Note that $x \in (-\infty, x_\infty)$, with $x_\infty = \ln(S_\infty/K)$. In order to get a bounded domain, we introduce the new truncated domain $\widehat{\Omega} = (x_0, x_\infty)\times(0,h_\infty)$.

For the problem formulation, we also introduce:

- The operator $\mathcal{A}$:

$$\mathcal{A}u = -\text{div}\,(\widehat{A}\nabla u), \tag{17}$$

where the matrix $\widehat{A}$ is given by

$$\widehat{A} = \frac{1}{2}\begin{pmatrix} (\sigma^S)^2 & \rho\sigma^S\sigma^h \\ \rho\sigma^S\sigma^h & (\sigma^h)^2 \end{pmatrix}, \tag{18}$$

which is positive definite if and only if $|\rho| < 1$.
- The boundary $\Gamma = \{(x,h) \in \partial\widehat{\Omega} / h \neq h_\infty\}$ and the Hilbert space

$$W = H^1_\Gamma(\widehat{\Omega}) = \{z \in H^1(\widehat{\Omega}) \,/\, z = 0 \text{ on } \Gamma\}\,,$$

which is equipped with the norm:

$$\|z\|^2_{H^1_\Gamma} = \int_{\widehat{\Omega}} |\nabla z|^2\, dx\, dh\,.$$

Moreover, we denote by W^* the dual space of W.
- The operator $\Phi : (0,T]\times H^1_\Gamma(\widehat{\Omega}) \to L^2(\widehat{\Omega})$, which is given by:

$$\Phi(\tau,\varphi)(x,h) = c_0\,\varphi(x,h) + c_1\,\frac{\partial\varphi}{\partial x}(x,h) + c_2(h)\,\frac{\partial\varphi}{\partial h}(x,h) + h(v(\tau,x) + \varphi(x,h))^+\,, \tag{19}$$

for all $\tau \in (0,T]$ and $\varphi \in H^1_\Gamma(\widehat{\Omega})$, where:

$$c_0 = -f\,, \qquad c_1 = -\frac{(\sigma^S)^2}{2} + (r-q)\,, \qquad c_2(h) = -\frac{\kappa}{1-R}h.$$

- Functions ψ and ℓ, which are given by:

$$\psi(\tau,x) = G(Ke^x) - v(\tau,x)\,,$$
$$\ell(\tau,x) = -\frac{\partial v}{\partial \tau}(\tau,x) + \frac{1}{2}(\sigma^S)^2\frac{\partial^2 v}{\partial x^2}(\tau,x) + (r-q-\frac{1}{2}(\sigma^S)^2)\frac{\partial v}{\partial x}(\tau,x) - f v(\tau,x)\,, \tag{20}$$

  and are independent on h.

Then, we end up with the nonlinear complementarity problem in $(0,T)\times\widehat{\Omega}$:

$$\begin{cases} \mathcal{L}_\tau(u) = \dfrac{\partial u}{\partial \tau} + \mathcal{A}u - \Phi(\tau,u) \geq \ell\,, \quad \text{in } (0,T)\times\widehat{\Omega} \\ u \geq \psi \\ [\mathcal{L}_\tau(u) - \ell]\;[u-\psi] = 0 \\ u(0,x,h) = 0 \\ u(\tau,x_0,h) = 0 \\ u(\tau,x_\infty,h) = 0 \\ u(\tau,x,0) = 0 \\ (\widehat{A}\nabla u\cdot\vec{n})(\tau,x,h_\infty) = 0\,. \end{cases} \tag{21}$$

3.1. Variational formulation

In this section, we first use subdifferential calculus tools to formulate the nonlinear complementarity problem (21) in the framework of semilinear parabolic variational inequalities. In this way, we can apply the results from [5] to obtain the existence and uniqueness of a solution of problem (21).

For this purpose, following [5] we introduce the functional space

$$Y = L^2(0,T;W)\cap \mathcal{C}([0,T];L^2(\widehat{\Omega}))\cap W^{1,2}(0,T;W^*)$$

and the operator $\mathcal{H}: Y \to L^2(0,T;W^*)$, defined for each $\tau\in(0,T]$ as follows:

$$\mathcal{H}(u)(\tau,\cdot) = -\frac{\partial u}{\partial \tau}(\tau,\cdot) - \mathcal{A}u(\tau,\cdot) + \Phi(\tau,u(\tau,\cdot)) + \ell(\tau,\cdot)\,. \tag{22}$$

Therefore, problem (21) can be equivalently written in the form:

Find $u\in Y$, such that

$$\mathcal{H}(u) \leq 0\,,\quad u\geq\psi\,,\quad \mathcal{H}(u)(u-\psi) = 0\,, \tag{23}$$

jointly with the initial condition and the homogeneous boundary condition on $h = h_\infty$.

As the function ψ depends on τ (i.e. the obstacle function is time dependent in this problem), for each $\tau\in[0,T]$ we introduce the closed convex set

$$K(\tau) = \{z\in W \,/\, z\geq\psi(\tau,\cdot) \text{ in } \widehat{\Omega}\}\,.$$

We also introduce the indicatrix function $\phi: W\to(-\infty,+\infty]$ associated to the convex set $K(\tau)$ as follows:

$$\phi(z) = \begin{cases} 0\,, & \text{if } z\in K(\tau)\,, \\ +\infty\,, & \text{if } z\notin K(\tau)\,, \end{cases}$$

which is a lower semicontinuous, proper convex function. The subdifferential of ϕ is a maximal monotone multivalued operator denoted by $\partial\phi$, which is defined by:

$$w\in\partial\phi(u) \iff \phi(u)\leq\phi(z) + (w,u-z),\quad \forall z\in W\,,$$

where $(\cdot,\cdot)$ denotes the duality pairing between W^* and W.

In the next proposition, we reformulate the nonlinear complementarity problem (23) in terms of the subdifferential $\partial\phi(u)$.

Proposition 1. *For $u(\tau,\cdot)\in K(\tau)$ and τ a.e. in $(0,T)$, the following conditions are equivalent:*

(P_1) $\mathcal{H}(u)\leq 0\,,\quad u\geq\psi\,,\quad \mathcal{H}(u)(u-\psi) = 0$

(P_2) $\mathcal{H}(u)\in\partial\phi(u)\,.$

Proof.

(a) Let us assume that $u\in K(\tau)$ satisfies (P_2) .

- Let be $\epsilon \in H^1(\widehat{\Omega})$ such that $\epsilon = 0$ on Γ and $\epsilon > 0$ in $\widehat{\Omega}$, so that $z = u + \epsilon \in K(\tau)$. As $\mathcal{H}(u) \in \partial\phi(u)$, then we have

  $$\phi(u) - \phi(z) \leq (\mathcal{H}(u), u - z) = (\mathcal{H}(u), -\epsilon)\,.$$

  Moreover, as $u,\ z \in K(\tau)$ then $\phi(u) = \phi(z) = 0$, so that

  $$(\mathcal{H}(u), \epsilon) \leq 0$$

  for any $\epsilon > 0$. Therefore, $\mathcal{H}(u) \leq 0$ and the first condition in (P_1) is satisfied.
- As $u(\tau, \cdot) \in K(\tau)$ then $u \geq \psi$.
- Next, for a given $\tau \in (0, T)$ we consider the set

  $$\widehat{\Omega}^+(\tau) = \{(x, h) \in \widehat{\Omega} \,/\, u(\tau, x, h) > \psi(\tau, x)\},$$

  which is an open set in $\widehat{\Omega}$. Next, we take $\omega \in H^1_\Gamma(\widehat{\Omega}) \cap L^\infty(\widehat{\Omega})$, such that $\omega \neq 0$ and $\|\omega\|_{H^1_\Gamma(\widehat{\Omega})} = 1$. For $r > 0$, we consider the functions

  $$v_r^\pm(x, h) = \begin{cases} u(\tau, x, h), & \text{if } (x, h) \notin \widehat{\Omega}^+(\tau) \\ u(\tau, x, h) \pm r\omega(x, h), & \text{if } (x, h) \in \widehat{\Omega}^+(\tau)\,. \end{cases}$$

  Let $(x_0, h_0) \in \widehat{\Omega}^+(\tau)$, so that $u(\tau, x_0, h_0) > \psi(\tau, x_0)$ and

  $$v_r^\pm(x_0, h_0) = u(\tau, x_0, h_0) \pm r\omega(x_0, h_0)\,.$$

  Then, $v_r^\pm(x_0, h_0) > \psi(\tau, x_0)$ for r sufficiently small.
  Finally, using the definition of the subdifferential operator and the fact that $v_r^\pm$ and $u(\tau, \cdot)$ belong to $K(\tau)$, we obtain

  $$(\mathcal{H}(u), \pm r\omega)(x_0, h_0) \geq 0\,.$$

  Since $w \neq 0$ and $r > 0$, we get $\mathcal{H}(u) = 0$ in $\widehat{\Omega}^+(\tau)$. Therefore, the third condition in (P_1) is proved.

(b) Assuming that condition (P_1) is satisfied, then we need to prove that

$$(\mathcal{H}(u), u - z) - \phi(u) + \phi(z) \geq 0\,, \quad \forall z \in H^1_\Gamma(\widehat{\Omega})\,.$$

We distinguish two cases:

- If $z \notin K(\tau)$ then $\phi(z) = +\infty$ and $\phi(u) = 0$. Moreover, we have

  $$\left| \int_{\widehat{\Omega}} \mathcal{H}(u)(\psi - z) d\widehat{\Omega} \right| \leq \|\mathcal{H}(u)\|_{H^{-1}(\widehat{\Omega})} \|\psi - z\|_{H^1(\widehat{\Omega})}\,.$$

  Then, the left hand side is finite and

  $$(\mathcal{H}(u), u - z) = (\mathcal{H}(u), u - \psi) + (\mathcal{H}(u), \psi - z)\,,$$

  so that

  $$(\mathcal{H}(u), u - z) - \phi(u) + \phi(z) = +\infty \geq 0\,.$$
- If $z \in K(\tau)$ then $\phi(z) = \phi(u) = 0$, so that

  $$\begin{aligned} \phi(z) - \phi(u) + (\mathcal{H}(u), u - z) &= (\mathcal{H}(u), u - z) \\ &= (\mathcal{H}(u), u - \psi) + (\mathcal{H}(u), \psi - z) = (\mathcal{H}(u), \psi - z) \geq 0\,, \end{aligned}$$

  where the last inequality follows from $\mathcal{H}(u) \leq 0$ and $\psi - z \leq 0$.

Therefore, we have proved that (P_2) holds. □

From Proposition 1, we obtain that problem (23) is equivalent to finding $u \in K(\tau)$ a.e. $\tau \in (0, T]$, such that

$$\frac{\partial u}{\partial \tau} + \mathcal{A}u + \partial\phi(u) \ni \Phi(\cdot, u) + \ell \tag{24}$$

is satisfied jointly with the initial condition and the homogeneous Neumann boundary condition at $h = h_\infty$.

3.2. Existence and uniqueness of a solution

In the previous section, the nonlinear complementarity problem (21) has been equivalently formulated in the form (24), which fits into the framework of the article [5]. From it, we recall the following result to obtain the existence and uniqueness of a solution for semilinear parabolic variational inequalities.

Theorem 1 (*Jeong–Park [5]*). *Let $\mathcal{A}$ be a continuous operator satisfying Garding's inequality and $f(t, x(t))$ be a Lipschitz continuous function in $x(t)$. Assume that $k \in L^2(0,T;V^*)$ and $x_0 \in \overline{D(\phi)}$. Then, the problem*

$$\begin{cases} \dfrac{dx(t)}{dt} + \mathcal{A}x(t) + \partial\phi(x(t)) \ni f(t, x(t)) + k(t), \quad 0 < t \le T \\ x(0) = x_0 \,, \end{cases} \tag{25}$$

has a unique solution $x \in L^2(0,T;V) \cap C([0,T];H)$ and there exists a constant C_1 depending on T such that

$$\|x\|_{L^2 \cap \mathcal{C}} \le C_1 \left(1 + \|x_0\|_H + \|k\|_{L^2(0,T;V^*)} \right).$$

Furthermore, if $k \in L^2(0,T;H)$ then the solution x belongs to $W^{1,2}(0,T;H)$ and satisfies

$$\|x\|_{W^{1,2}(0,T;H)} \le C_1 \left(1 + \|x_0\|_H + \|k\|_{L^2(0,T;H)} \right).$$

In order to apply Theorem 1, we will consider $H = L^2(\Omega)$, $V = H^1_\Gamma(\Omega)$, and the functions $f = \Phi$, $k = \ell$. Next, we prove the following proposition.

Theorem 2. *The following statements are satisfied:*

(a) *The continuous operator $\mathcal{A}$ defined in (17) satisfies Garding's inequality, i.e.:*

$$(\mathcal{A}z, z) \ge \omega_1 \|z\|^2_{H^1_\Gamma(\widehat{\Omega})} - \omega_2 \|z\|^2_{L^2(\widehat{\Omega})} \quad \forall z \in H^1_\Gamma(\widehat{\Omega}) \,, \tag{26}$$

for $\omega_1 > 0$ and $\omega_2 \in \mathbb{R}$.

(b) *$\ell \in L^2(0,T;L^2(\widehat{\Omega})) \subset L^2(0,T;W^*)$.*

(c) *Let $D(\phi) = \left\{ z \in H^1_\Gamma(\widehat{\Omega}) \,/\, \phi(z) < \infty \right\}$ and $u_0 = u(0,x,h)$. Then, $u_0 \in \overline{D(\phi)}$.*

(d) *$\Phi(\tau, \varphi)$ is Lipschitz continuous in the variable φ, i.e.*

$$\|\Phi(\tau, \varphi_1) - \Phi(\tau, \varphi_2)\|_{L^2(\widehat{\Omega})} \le L_G \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})} \,.$$

Therefore, the nonlinear variational inequality (24) has a unique solution $u \in L^2(0,T;H^1_\Gamma(\widehat{\Omega})) \cap \mathcal{C}([0,T];L^2(\widehat{\Omega}))$. In particular $u \in W^{1,2}(0,T;L^2(\widehat{\Omega}))$ and it satisfies

$$\|u\|_{W^{1,2}(0,T;L^2(\widehat{\Omega}))} \le C_1 \left(1 + \|u_0\|_{L^2(\widehat{\Omega})} + \|\ell\|_{L^2(0,T;H^1_\Gamma(\widehat{\Omega}))} \right). \tag{27}$$

Proof.

(a) From the definition of $\mathcal{A}$ in (17), the operator is linear. Moreover, we have

$$\begin{aligned}(\mathcal{A}u, v) &= \int_{\widehat{\Omega}} (-\mathrm{div}\,(\widehat{A}\nabla u)) v \, d\widehat{\Omega} = \int_{\widehat{\Omega}} \widehat{A}\nabla u \, \nabla v \, d\widehat{\Omega} \\ &\le M \|\nabla u\|_{L^2(\widehat{\Omega})} \|\nabla v\|_{L^2(\widehat{\Omega})} = M \|u\|_{H^1_\Gamma(\widehat{\Omega})} \|v\|_{H^1_\Gamma(\widehat{\Omega})} \,.\end{aligned}$$

Therefore, $\mathcal{A}$ is continuous. In order to prove Garding's inequality, we consider that

$$(\mathcal{A}u, u) = \int_{\widehat{\Omega}} (-\mathrm{div}\,(\widehat{A}\nabla u)) u \, d\widehat{\Omega} = \int_{\widehat{\Omega}} \widehat{A}\nabla u \, \nabla u \, d\widehat{\Omega} \ge \lambda_{\min} \|u\|^2_{H^1_\Gamma(\widehat{\Omega})} \,,$$

where $\lambda_{\min} > 0$ is the minimum of the eigenvalues of $\widehat{A}$. Thus, by taking $\omega_1 = \lambda_{\min}$ and $\omega_2 = 0$, we obtain (26).

(b) From the definition of ℓ in (20):

$$\ell(\tau, x) = -\frac{\partial v}{\partial \tau}(\tau, x) + \frac{1}{2}(\sigma^S)^2 \frac{\partial^2 v}{\partial x^2}(\tau, x) + (r - q - \frac{1}{2}(\sigma^S)^2)\frac{\partial v}{\partial x}(\tau, x) - f\, v(\tau, x) \,,$$

where v is the solution of the following complementarity problem

$$\ell \le 0 \,, \quad v \ge \widetilde{G} \,, \quad \ell(v - \widetilde{G}) = 0 \,,$$

with $\widetilde{G}(x) = G(Ke^x)$.

If we consider a put option, then $\widetilde{G}(x) = K(1 - e^x)^+$. As in the region $v > \widetilde{G}$ we have $\ell = 0$, we just consider the region $v = \widetilde{G}$, so that

$$\ell(\tau, x) = -\frac{1}{2}(\sigma^S)^2 Ke^x - (r - q - \frac{1}{2}(\sigma^S)^2)Ke^x - fK + fKe^x = (q - r + f)Ke^x - fK \,.$$

Therefore, in this region we have

$$|\ell(\tau, S)| \le |q - r + f| \, Ke^{x_\infty} + fK \,,$$

so that ℓ is bounded. In particular, we have

$$||\ell||^2_{L^2(0,T;L^2(\widehat{\Omega}))} = \int_0^T \int_{\widehat{\Omega}} |\ell(\tau, x)|^2 \, d\widehat{\Omega} \, d\tau < \infty \, .$$

Analogously, we can proceed in the case of call options. Then, we have shown that $\ell \in L^2(0, T; L^2(\widehat{\Omega}))$. Moreover, since $L^2(\widehat{\Omega}) \subset W^*$ we conclude that $\ell \in L^2(0, T; W^*)$.

(c) It is easy to check that $u(0, x, h) = u_0(x, h) = 0 \geq \psi(\tau, x)$. Thus, $u_0 \in \overline{D(\phi)}$.

(d) For the operator $\Phi : [0, T] \times H^1_\Gamma(\widehat{\Omega}) \to L^2(\widehat{\Omega})$ defined in (19), we deduce that

$$\begin{aligned}
&\left|\Phi(\tau, \varphi_1)(x, h) - \Phi(\tau, \varphi_2)(x, h)\right| \\
&\quad = \Big| h(v(\tau, x) + \varphi_1(x, h))^+ - h(v(\tau, x) + \varphi_2(x, h))^+ \\
&\qquad + c_0\, \varphi_1(x, h) - c_0 \varphi_2(x, h) + c_1 \frac{\partial \varphi_1}{\partial x}(x, h) - c_1 \frac{\partial \varphi_2}{\partial x}(x, h) \\
&\qquad + c_2(h) \frac{\partial \varphi_1}{\partial h}(x, h) - c_2(h) \frac{\partial \varphi_2}{\partial h}(x, h) \Big| \\
&\quad \leq |c_0 + h| \; |\varphi_1 - \varphi_2| + |c_1| \left| \frac{\partial \varphi_1}{\partial x}(x, h) - \frac{\partial \varphi_2}{\partial x}(x, h) \right| \\
&\qquad + |c_2(h)| \left| \frac{\partial \varphi_1}{\partial h}(x, h) - \frac{\partial \varphi_2}{\partial h}(x, h) \right| .
\end{aligned}$$

Then, by integration over $\widehat{\Omega}$, we obtain:

$$\begin{aligned}
&\int_{\widehat{\Omega}} |\Phi(\tau, \varphi_1)(x, h) - \Phi(\tau, \varphi_2)(x, h)|^2 \; dx\, dh \\
&\quad \leq |c_0 + h|^2 \int_{\widehat{\Omega}} |\varphi_1(x, h) - \varphi_2(x, h)|^2 \; dx\, dh \\
&\quad + |c_1|^2 \int_{\widehat{\Omega}} \left| \frac{\partial \varphi_1}{\partial x}(x, h) - \frac{\partial \varphi_2}{\partial x}(x, h) \right|^2 \; dx\, dh \\
&\quad + \left|\tilde{c}_2\right|^2 \int_{\widehat{\Omega}} \left| \frac{\partial \varphi_1}{\partial h}(x, h) - \frac{\partial \varphi_2}{\partial h}(x, h) \right|^2 \; dx\, dh \, .
\end{aligned}$$

In terms of the corresponding norms, we can equivalently write:

$$\begin{aligned}
\|\Phi(\tau, \varphi_1) - \Phi(\tau, \varphi_2)\|_{L^2(\widehat{\Omega})} &\leq L_1 \, \|\nabla \varphi_1 - \nabla \varphi_2\|_{L^2(\widehat{\Omega})} + \tilde{c}_0 \, \|\varphi_1 - \varphi_2\|_{L^2(\widehat{\Omega})} \\
&\leq L_G \, \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}
\end{aligned}$$

where $\tilde{c}_0 = \max\{|c_0 + h| : h \in [0, h_\infty]\}$, $\tilde{c}_2 = \max\{c_2(h) : h \in [0, h_\infty]\}$, $L_1 = \max\{|c_1|, \tilde{c}_2\}$, $L_G = \max\left\{C_0 \tilde{c}_0, L_1\right\}$ and $C_0 > 0$ is the constant associated to the Poincaré–Friedrichs inequality. Then, Φ is Lipschitz continuous in the second variable φ.

Thus, by using Theorem 1 we prove that the nonlinear differential problem (24) has a unique solution $u \in W^{1,2}(0, T; L^2(\widehat{\Omega}))$, which satisfies (27). □

Corollary 1. *There exists a unique solution $u \in Y$ of problem (21).*

Proof. The proof follows from Proposition 1. □

4. Numerical methods

In this section, we describe the different proposed numerical techniques to compute the value of an American option with counterparty risk. In particular, first we need to compute the American option price without counterparty risk, which satisfies the one dimensional model (11). For this purpose, we consider the techniques introduced in [1] for one dimensional problems. Assuming that this model has been previously solved, we will describe the numerical techniques to approximate the solution of the nonlinear problem (9). Note that the linear case (10) can be solved by similar methods.

The numerical discretization is mainly based on the finite element method combined with the method of characteristics. As usual in vanilla options, we consider the maximum value for the asset price S_∞ as four times the strike price. Similarly, we consider the interval $[0, h_\infty]$ for the admissible spread values, where h_∞ is eight times the reference value for the spread.

In order to compute the American option price with counterparty risk (risky derivative) by means of the finite element method, we rewrite problem (9) in the equivalent divergence form and using the time to maturity variable τ as follows:

$$\begin{cases} \mathcal{L}_1(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial \tau} - \operatorname{div}(A\nabla\widehat{V}) + b\cdot\nabla\widehat{V} + f\widehat{V} + h\widehat{V}^+ \geq 0 \qquad \text{in } (0,T]\times\Omega\,, \\ \widehat{V}(\tau,S,h) \geq G(S)\,, \\ \mathcal{L}_1(\widehat{V})(\widehat{V}-G) = 0\,, \\ \widehat{V}(0,S,h) = G(S)\,, \end{cases} \tag{28}$$

where the matrix A is defined in (15) and the vector b is given by:

$$b = \begin{pmatrix} \left((\sigma^S)^2 - (r-q)\right)S \\ \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}h \end{pmatrix}. \tag{29}$$

4.1. Method of characteristics

As for European options in [4], for the time discretization a semi-Lagrangian method – also known as the method of characteristics – is applied (see [17]).

If we rewrite the first order derivatives with respect to the time and spatial variables in terms of the material derivative of $\widehat{V}$ as follows:

$$\frac{D\widehat{V}}{D\tau} = \frac{\partial\widehat{V}}{\partial\tau} + \left((\sigma^S)^2 - (r-q)\right)S\frac{\partial\widehat{V}}{\partial S} + \left(\frac{\rho\sigma^S\sigma^h}{2} + \frac{\kappa}{1-R}h\right)\frac{\partial\widehat{V}}{\partial h}. \tag{30}$$

Then, the first inequality in (28) becomes:

$$\frac{D\widehat{V}}{D\tau} - \operatorname{div}(A\nabla\widehat{V}) + f\widehat{V} + h(\widehat{V})^+ \geq 0\,. \tag{31}$$

Let us introduce a time discretization mesh given by τ^n ($n = 0, 1, \ldots, N_T$), with time steps $\Delta\tau^n = \tau^{n+1} - \tau^n$ that are not necessarily constant. By using the method of characteristics to approximate the material derivative (30), problem (28) is approximated at each time τ^{n+1} by

$$\begin{cases} \mathcal{L}_1^n(\widehat{V}^{n+1}) = \dfrac{\widehat{V}^{n+1} - \widehat{V}^n\circ\chi^n}{\Delta\tau^n} - \operatorname{div}(A\nabla\widehat{V}^{n+1}) + f\widehat{V}^{n+1} + h(\widehat{V}^{n+1})^+ \geq 0\,, \\ \widehat{V}^0(S,h) = 0\,, \\ \widehat{V}^{n+1}(S,h) \geq G(S)\,, \\ \mathcal{L}_1^n(\widehat{V}^{n+1})(\widehat{V}^{n+1} - G) = 0\,, \end{cases} \tag{32}$$

for $n = 0, 1, 2\ldots, N_T - 1$, where the notation $\widehat{V}^n(\cdot) \approx \widehat{V}(\tau^n,\cdot)$ has been considered and $\chi^n(S,h) = \chi((S,h),\tau^{n+1};\tau^n)$. Then, the components of the characteristics function χ are the solution of the terminal value ODE problems

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = \left(\left(\sigma^S\right)^2 - (r-q)\right)\chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \dfrac{d\chi_2}{d\tau} = \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}\chi_2, \\ \chi_2(\tau^{n+1}) = h\,, \end{cases} \tag{33}$$

which can be analytically obtained [4].

As in the finite element method we apply quadrature formulae to compute integrals, we approximate the value of $\widehat{V}^n\circ\chi^n$ in (32) at the quadrature nodes by means of a bilinear interpolation method.

4.2. Fixed point scheme

Due to the presence of the nonlinear function h, after the previous time discretization we obtain the nonlinear problem (32). In order to solve this nonlinearity, a fixed point algorithm is proposed at each time step n. Thus, the function $\widehat{V}^{n+1,k+1}$ is obtained at each iteration k of the fixed point algorithm. More precisely, the global scheme is shown in Algorithm 1.

4.3. Boundary conditions

In Section 2 we have introduced appropriate boundary conditions for problem (9) in order to prove the existence of a solution of (16). Next, we adapt such conditions for the numerical solution. For this purpose, as in [4], we can follow the general theory for second order PDEs developed in [18], which includes Fichera theory [19], to obtain the required

Algorithm 1

Let $N_T > 1$, $n = 0$, $\varepsilon > 0$ and $\widehat{V}^0$ be given
For $n = 1, 2, \ldots, N_T - 1$:

(a) Let $\widehat{V}^{n+1,0} = \widehat{V}^n$, $k = 0$, $e = \varepsilon + 1$
(b) For $k = 0, 1, \ldots$

- Search $\widehat{V}^{n+1,k+1}$ solution of:

$$\begin{aligned} &\left(1 + \Delta\tau^n f\right)\widehat{V}^{n+1,k+1} - \Delta\tau^n \operatorname{div}(A\nabla\widehat{V}^{n+1,k+1}) \\ &\qquad\qquad \geq \widehat{V}^n \circ \chi^n - \Delta\tau^n\, h\,(\widehat{V}^{n+1,k})^+ \\ &\widehat{V}^{n+1,k+1}(S,h) \geq G(S) \\ &\mathcal{L}_1^n(\widehat{V}^{n+1,k+1})(\widehat{V}^{n+1,k+1} - G) = 0 \end{aligned} \tag{34}$$

- Compute the relative error $e = \dfrac{\|\widehat{V}^{n+1,k+1} - \widehat{V}^{n+1,k}\|}{\|\widehat{V}^{n+1,k+1}\|}$

until $e < \varepsilon$.

boundary conditions for problem (34). By using this methodology, we obtain the nonhomogeneous Dirichlet boundary condition for European options on the boundary $S = S_\infty$:

$$\widehat{V}_{\mathcal{E}}^{n+1,k+1}(S_\infty, h) = \frac{-\Delta\tau^n(\widehat{V}_{\mathcal{E}}^{n+1,k})^+ h + \widehat{V}_{\mathcal{E}}^n \circ \chi^n}{1 + \Delta\tau^n f}\,. \tag{35}$$

However, as the American option value is bounded below by its exercise value, we incorporate this condition by imposing

$$\begin{aligned} \widehat{V}^{n+1,k+1}(S_\infty, h) &= \max\left(\widehat{V}_{\mathcal{E}}^{n+1,k+1}(S_\infty, h), G(S_\infty)\right) \\ &= \max\left(\frac{-\Delta\tau^n h(\widehat{V}^{n+1,k})^+ + \widehat{V}^n \circ \chi^n}{1 + \Delta\tau^n f}, G(S_\infty)\right)\,. \end{aligned}$$

Also, by using [18] we can prove that boundary conditions are not required at the boundary $S = 0$.

Next, for the boundaries $h = 0$ and $h = h_\infty$ we apply the same boundary conditions we used for the continuous problem in the mathematical analysis (Section 3). On the boundary $h = 0$ the spread is null, which corresponds to an investor's default probability equal to zero, so that the American option is free of counterparty risk. Thus, we impose

$$\widehat{V}^{n+1,k+1}(S, 0) = V^{n+1,k+1}(S), \qquad \text{for } n = 0, 1, \ldots, N_T\,, \quad k = 0, 1, \ldots\,.$$

On the boundary $h = h_\infty$, the following non homogeneous Neumann boundary condition is assumed:

$$(A\nabla\widehat{V}^{n+1,k+1}) \cdot \vec{n} = (A\nabla V^{n+1,k+1}) \cdot \vec{n}\,,$$

where the matrix A is given by (15).

4.4. Finite element discretization

In order to derive the full discretization of (34) we next apply the finite element method, so that we first state the variational formulation of the problem. For this purpose, we introduce the convex closed subset of $H^1(\Omega)$:

$$\widetilde{\mathcal{K}} = \left\{\varphi \in H^1(\Omega)\,/\,\varphi = \widehat{V}_D^{n+1,k+1} \text{ on } \partial_D\Omega,\ \text{ and } \varphi \geq G(S)\right\},$$

where $\partial_D\Omega$ denotes the part of the boundary of Ω where Dirichlet boundary conditions are posed (i.e. the boundaries $S = S_\infty$ and $h = 0$). Moreover, $\widehat{V}_D^{n+1,k+1}$ denotes the value of $\widehat{V}^{n+1,k+1}$ at these boundaries.

Thus, at each time step $n = 0, 1, 2, \ldots, N_T$ and each fixed point iteration $k = 0, 1, \ldots$, the variational formulation is given by:

Obtain $\widehat{V}^{n+1,k+1} \in \widetilde{\mathcal{K}}$ such that:

$$\begin{aligned} &\int_\Omega (1 + \Delta\tau^n f)\widehat{V}^{n+1,k+1}(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh \\ &+\Delta\tau^n \int_\Omega A\nabla\widehat{V}^{n+1,k+1}\, \nabla(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh \end{aligned}$$

Table 1
Financial data for numerical tests.

$\sigma^S = 0.3$	$\sigma^h = 0.2$	$\rho = 0.2$	$K = 15$	$T = 0.5$
$r = 0.3$	$q = 0.24$	$R = 0.3$	$\kappa = 0.01$	$f = 0.04$

$$
\begin{aligned}
&-\int_{\Gamma_2^{*,+}} (A\nabla V^{n+1,k+1}, n)(\varphi - \widehat{V}^{n+1,k+1})\partial\gamma \ \geq \\
&\qquad\qquad \int_{\Omega} (\widehat{V}^{n} \circ \chi^{n})(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh \\
&-\Delta\tau^n \int_{\Omega} h(\widehat{V}^{n+1,k})^{+}(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh\,, \qquad \forall \varphi \in \widetilde{\mathcal{K}}\,.
\end{aligned}
$$

In order to describe the finite element discretization, we consider a uniform mesh in the domain Ω by first choosing natural numbers $N_S > 0$ and $N_h > 0$. Next, we define the mesh nodes $(S_i = i\Delta S, h_j = j\Delta h)$, for $i = 0, \ldots, N_S + 1$, and $j = 0, \ldots, N_h + 1$. The constant parameters $\Delta S = S_\infty/(N_S+1)$ and $\Delta h = h_\infty/(N_h+1)$ denote the mesh steps in coordinates S and h, respectively. In terms of the uniform mesh, the following piecewise linear Lagrange finite element space W_h and its convex subset $\mathcal{K}_h$ are considered

$$
\begin{aligned}
W_h &= \{\varphi_h \in \mathcal{C}(\Omega) \,/\, \varphi_h|_{T_j} \in \mathcal{P}_1\,,\ \forall T_j \in \mathcal{T}\}\,, \\
\mathcal{K}_h &= \{\varphi_h \in W_h \,/\, \varphi_h = \widehat{V}_D^{n+1,k+1} \text{ on } \partial_D\Omega \text{ and } \varphi_h \geq G(S)\}\,.
\end{aligned}
$$

The fully discretized problem consists of finding $\widehat{V}_h^{n+1,k+1} \in \mathcal{K}_h$, which satisfies the Dirichlet boundary conditions and such that:

$$
\begin{aligned}
&\int_{\Omega} (1 + \Delta\tau^n f)\widehat{V}_h^{n+1,k+1}\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh \\
&+\Delta\tau^n \int_{\Omega} A\nabla\widehat{V}_h^{n+1,k+1}\, \nabla\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh \ \geq \\
&\qquad\qquad \int_{\Omega} \big(\widehat{V}_h^{n} \circ \chi^{n}\big)\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh \\
&-\Delta\tau^n \int_{\Omega} h\big(\widehat{V}_h^{n+1,k}\big)^{+}\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh\,, \qquad \forall \varphi_h \in \mathcal{K}_h.
\end{aligned}
$$

After the time discretization with the method of characteristics and the finite element spatial discretization, the fully discretized problem can be written as a discrete complementarity problem in the form:

$$
\begin{cases}
A_h\widehat{V}_h^{n+1,k+1} \geq b_h^{n+1,k+1}\,, \\
\widehat{V}_h^{n+1,k+1} \geq \Psi_h\,, \\
(A_h\widehat{V}_h^{n+1,k+1} - b_h^{n+1,k+1})(\widehat{V}_h^{n+1,k+1} - \Psi_h) = 0\,,
\end{cases}
\tag{36}
$$

where Ψ_h denotes the discretization of the exercise value function, $G(S)$, which also coincides with the value at maturity. The fully discretized problem involves the computation of the matrix A_h and the vector $b_h^{n+1,k+1}$, the coefficients of which have been obtained by using a quadrature formula based on the midpoints of the edges of the triangles.

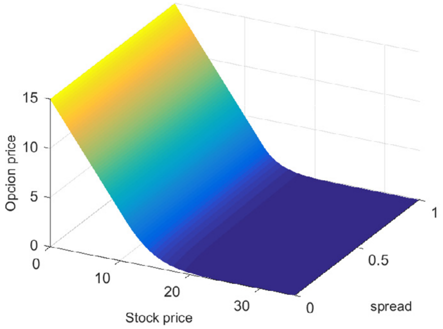
In order to solve problem (36), we use the augmented Lagrangian active set (ALAS) algorithm proposed by Kärkkäinen et al. [20]. The ALAS algorithm has been previously applied in the case of American options with one stochastic factor. A description of the algorithm when applied to other financial problems can be found in [1,21].

5. Numerical examples

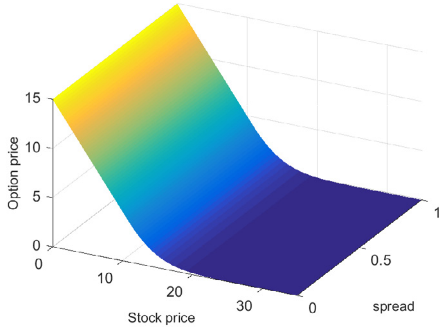
In this section, we mainly show some examples about the behaviour of American options value $\widehat{V}$ when counterparty risk is taken into account and the associated XVA value U. We also show the risk-free value V, when counterparty risk is not considered. More precisely, we represent the risky value and the XVA in terms of the asset and the spread values at the initial time. As in previous articles concerning one dimensional models for American options [1] and two dimensional models for European ones [4], we also compare the solution obtained with the proposed numerical techniques for PDEs models with the results computed with a proposed Monte Carlo technique for an alternative formulation based on expectations. More precisely, as the Monte Carlo based technique we use the classical Longstaff–Schwartz method [22].

In all following tests, the financial data are taken from Table 1. For the first three of them, which are solved by the Lagrange–Galerkin method, the uniform spatial mesh consists of 160 000 nodes ($N_S = N_h = 400$). On the other hand, we use a nonuniform time discretization mesh with nodes $\tau^n = (n/N_T)^2 T$ $(n = 0, 1, \ldots, N_T)$.

All the codes have been developed from scratch in Matlab and all the tests have been carried out on a computer with Intel(R) Xeon(R) CPU E3-1241 3.50 GHz.



(a) Risky derivative value



(b) Risk–free derivative value

Fig. 1. The American put option value (Test 1).

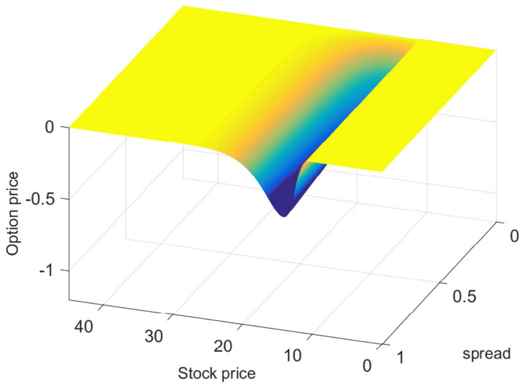


Fig. 2. The total value adjustment (Test 1).

5.1. Test 1: The American put option case with $M = \widehat{V}$

In this first example, we study an American put option sold by an investor when assuming that the mark-to-market value is equal to the value of the risky option. The maturity time is $T = 0.5$ years and we consider $N_T = 700$ time steps. Fig. 1 shows the American option value with the counterparty risk (left side) and the risk-free option value (right side). The difference between them is the XVA, which is represented in Fig. 2. We can observe that the XVA increases, in absolute terms, when the intensity of default from the investor (spread) increases. We also notice how the XVA decreases in the in-the-money area and the value becomes null in the exercise region.

Next, Fig. 3 shows the exercise region associated with the American option, either when considering counterparty risk (left) or without considering it (right). In the first case, the spread value is 0.25. Note that the exercise region is slightly larger when the intensity of default (spread) increases.

The presence of a nonlinear PDE and a complementarity formulation implies that the theoretical proof of the global convergence of the proposed numerical methods seems a very difficult open problem. Therefore, in order to illustrate the practical convergence of the methods we include Table 2. For this purpose, we used uniform time meshes and the empirical convergence ratio is given by:

$$CR = \frac{\left\| \widehat{V}_h - \widehat{V}_{h/2} \right\|_\infty}{\left\| \widehat{V}_{h/2} - \widehat{V}_{h/4} \right\|_\infty},$$

from which we compute the experimental order of convergence $p = \log_2(CR)$.

First, we mention that we have not observed any stability problems. Moreover, as illustrated in Table 2 we obtain a first order convergence, which is the expected one as we use piecewise linear finite element and a first order semi-Lagrangian scheme for the time discretization.

Next, in Table 3 we compare American option prices when using constant and stochastic spreads for a given asset value $S = 18$. In the case of constant spreads, we have taken the different constant values of h indicated in Table 3 and we have applied the model and the methods in [1] for American options with constant spreads. As illustrated by the table, in the case of constant spreads the variation of option prices with respect to the spread h is significantly smaller than

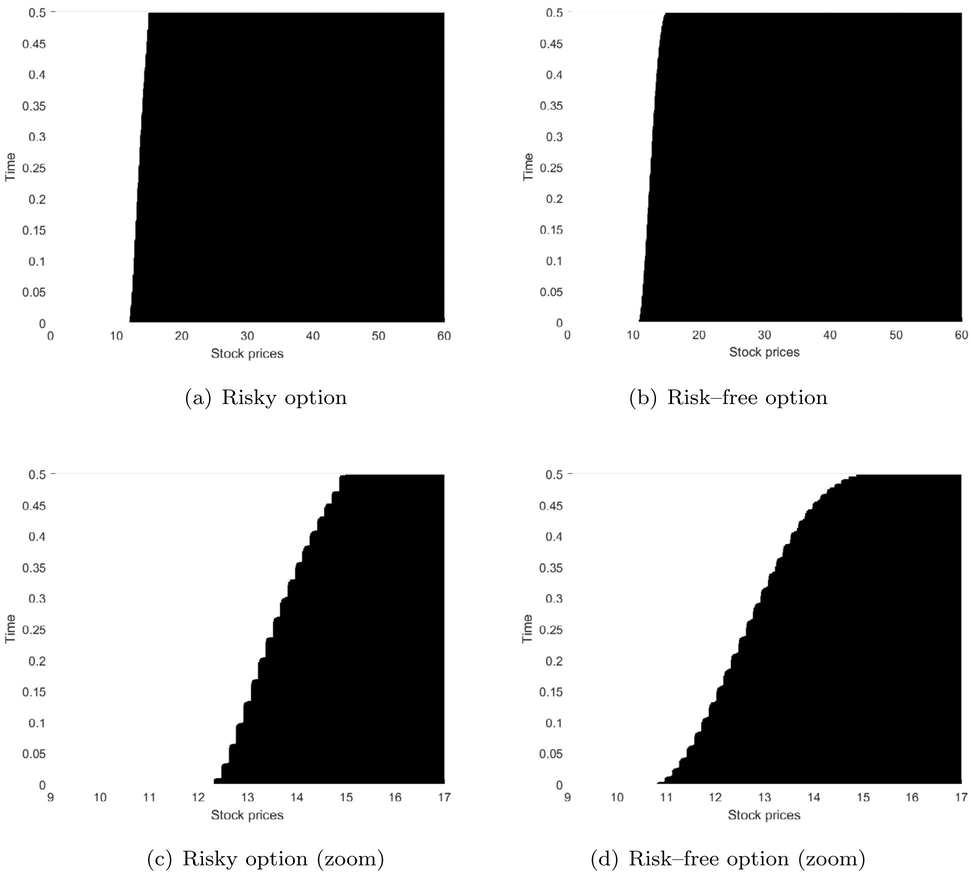


Fig. 3. Exercise regions (white) of Test 1.

Table 2
The experimental order of convergence for American put option (Test 1).

$\Delta S = \Delta h$	$\Delta\tau$	CR	p
2^{-2}	1/12		
2^{-3}	1/24		
2^{-4}	1/48	1.4599	0.5458
2^{-5}	1/96	1.5003	0.5853
2^{-6}	1/192	1.5609	0.6424
2^{-7}	1/384	1.6430	0.7163
2^{-8}	1/768	2.0240	1.0172

Table 3
Comparison between American option values with constant and stochastic spreads, with $M = \widehat{V}$ (Test 1). The parameter values of the problem are: $S = 18$, $K = 15$, $T = 0.5$, $\sigma^S = 0.3$, $\sigma^h = 0.2$, $\rho = 0.2$, $r = 0.3$, $q = 0.24$, $\kappa = 0.01$, $R = 0.3$, $f = 0.04$.

h	MEF constant spread	MEF stochastic spread
0.00	0.382645129	0.388721923
0.05	0.375203220	0.322444826
0.10	0.367972539	0.284762774
0.15	0.360960560	0.266327122
0.20	0.354145830	0.254925921

in the case of stochastic spreads. Therefore, the assumption of stochastic spreads shows more sensitivity of the option price with respect to the spread. For the one dimensional problem, a finite element mesh with 401 nodes and 700 time steps has been used, while for the two dimensional model a mesh with 401 $\times$ 401 nodes and 700 time steps has been considered.

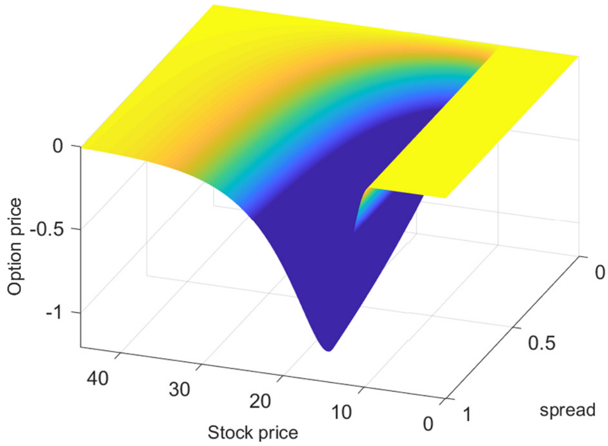


Fig. 4. The total value adjustment (Test 2).

5.2. Test 2: The American put option case with long maturity and $M = \widehat{V}$

In this section, an American put option sold by an investor is also considered. However, in this second test we choose a longer maturity given by $T = 2$ years, and we take $N_T = 1500$ time steps. As expected, in comparison with the American option with a shorter maturity of Test 1, the XVA (Fig. 4) becomes more negative due to a longer exposure to the risk. However, the behaviour of the total value adjustment is similar in both cases.

By comparing the exercise regions (Fig. 5) with those obtained in Test 1, we observe that for a longer maturity the exercise regions become smaller. Nevertheless, we obtain a similar behaviour in both cases, in the sense that the exercise region for a risky option is larger than for a risk-free one.

5.3. Test 3: The American put option case with $M = V$

We show the behaviour of the American option value when the mark-to-market value is the risk-free derivative value. As in Test 1, the maturity time is $T = 0.5$ years and $N_T = 700$, analogously to Test 1, and we can observe a similar behaviour to the one found in that case. Moreover, the value of the option (Fig. 6) and the adjustment value (Fig. 7) are also similar, being slightly more negative for $M = \widehat{V}$.

5.4. Test 4: comparison with Monte Carlo simulations

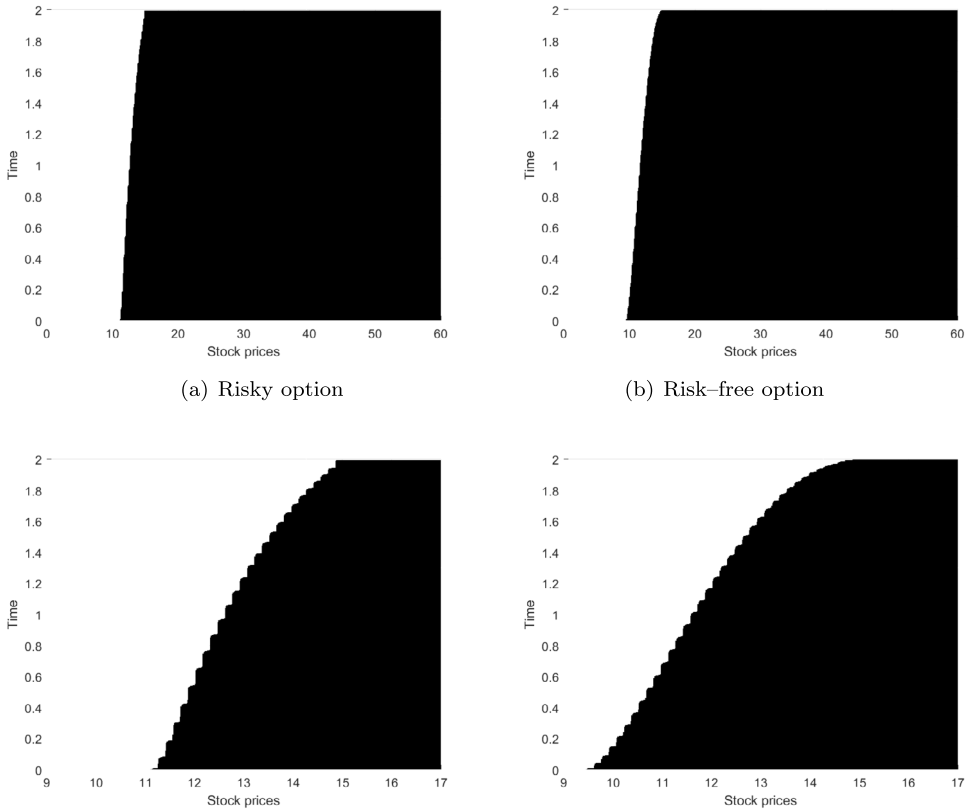
For the case of American options with constant spread (therefore, with just one stochastic factor), in [2] we have also used a formulation based on expectations and its numerical simulation by the Longstaff–Schwartz algorithm. In the present work, we have extended this Monte Carlo like technique to the case of two stochastic factors in order to compare the obtained results with those from the corresponding complementarity problems formulation here proposed.

Note that in [2] we have solved the linear problem (for which the mark-to-market value is $M = V$) by two different simulation algorithms, with an inner Monte Carlo algorithm or using interpolation techniques. Both methods led to similar results, although we have observed that the elapsed CPU time is higher when using the inner iteration algorithm. Therefore, in the present case with two stochastic factors we only compute the XVA when $M = V$ by extending the method that uses interpolation techniques. When the mark-to-market value is $M = \widehat{V}$ we also extend the technique developed for the case of one stochastic factor.

By extending these Monte Carlo techniques to stochastic spreads, in the present work we have also computed two estimators (lower and upper) of the XVA. From these computations, we can observe in Tables 4 and 5 that the XVA values obtained by solving the nonlinear and linear complementarity problems, respectively, belong to the corresponding 99% confidence interval generated with Monte Carlo techniques.

Numerical solutions of complementarity problems have been computed with a mesh of 201×201 nodes and 500 time steps. The elapsed time for such simulation is 55822 seconds. For the Longstaff–Schwartz method, we have implemented $N_P = 1000$ paths and $N_T = 1000$ time instants in the nonlinear case, and only $N_P = 500$ paths in the linear case. In both problems, we have employed three bases [2].

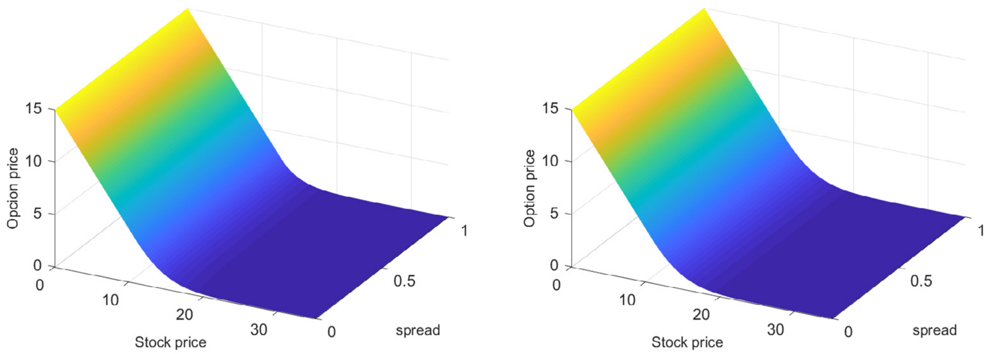
Monte Carlo computing of the risk-free option, which is previous to interpolation, requires 137160 s on a one-dimensional mesh of 100 initial prices, which states the advantage of pricing this kind of options by solving the complementarity problems.



(a) Risky option (b) Risk–free option

(c) Risky option (zoom) (d) Risk–free option (zoom)

Fig. 5. Exercise regions for $T = 2$ (Test 2).



(a) Risky derivative value (b) Risk–free derivative value

Fig. 6. The American put option value, $M = V$ (Test 3).

6. Conclusions

In this article we propose PDEs models for pricing American options including counterparty risk and two stochastic factors corresponding to the asset price and one stochastic spread. The presence of an early exercise in American options leads to formulations in terms of linear and nonlinear complementarity problems, depending on the chosen value for the mark-to-market. Unlike European options, for which the XVA can be directly obtained as the solution of a PDEs problem, the inequality constraints arising for American options require that the XVA has to be obtained as the difference between the risky and the risk-free American option values, i.e. $U = \widehat{V} - V$, which in turn are solutions of respective complementarity problems.

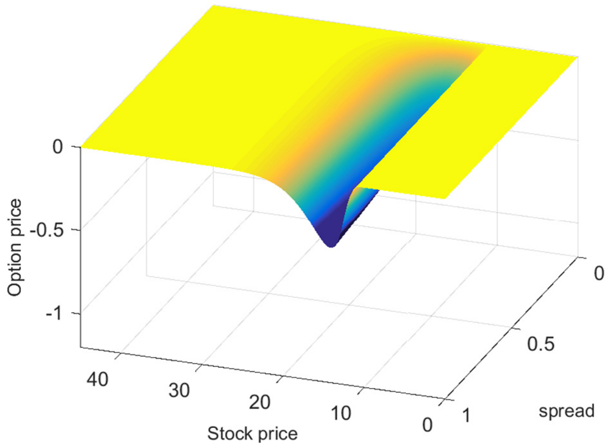


Fig. 7. The total value adjustment (Test 3).

Table 4
The American option value with counterparty risk and $M = \widehat{V}$. The parameter values of the problem are: $K = 15$, $T = 0.5$, $\sigma^S = 0.3$, $\sigma^h = 0.2$, $\rho = 0.2$, $r = 0.3$, $q = 0.24$, $\kappa = 0.01$, $R = 0.3$, $f = 0.04$.

S	h	Finite elements	Lower estimator	Upper estimator	99% confidence interval
0.0	0.00	15.00000000	14.70298010	15.26299737	(14.70298010 , 15.28840736)
27.0		0.014580440	0.00137214	0.02527334	(−0.00022466 , 0.02730209)
3.0	0.05	12.00000000	11.99924144	12.14867741	(11.99700274 , 12.16485122)
9.0		5.999999999	5.99442104	6.33579807	(5.98367098 , 6.36138141)
15.0		1.308562527	1.00503811	1.42993109	(0.88792215 , 1.46313845)
24.0		0.039359070	0.01276533	0.14500936	(0.00370968 , 0.15321851)
9.0	0.10	5.999999999	6.00132959	6.31694827	(5.99355616 , 6.34254985)
12.0		3.064549102	2.92497307	3.49060126	(2.78070521 , 3.52303705)
15.0		1.190349256	1.09738339	1.42209030	(0.97520402 , 1.45608435)
18.0		0.381342801	0.26845411	0.68720345	(0.20919729 , 0.71412533)
27.0		0.007162736	0.00244516	0.03743066	(−0.00000067 , 0.04178265)
18.0	0.15	0.330425270	0.26696364	0.63732297	(0.20669270 , 0.66098768)
21.0		0.085487636	0.07440162	0.43350934	(0.04256772 , 0.45153917)
24.0		0.020619582	0.01324925	0.25670856	(0.00186524 , 0.26792935)
21.0	0.20	0.071796388	0.06151416	0.45182906	(0.03261546 , 0.47117011)
27.0		0.003421215	0.00099960	0.11406895	(−0.00049202 , 0.11881908)
30.0		0.000742742	0.00009576	0.00089707	(−0.00014309 , 0.00258554)

Existence and uniqueness of a solution for the complementarity problem associated to the nonlinear PDE are studied through the theory of nonlinear functional differential problems, which can be applied to parabolic variational inequalities.

In order to compute the risky derivative value, a characteristics-finite element method is combined with an augmented Lagrangian active set technique to handle the additional inequality constraints involved in the formulation. Numerical examples are presented to illustrate and discuss the behaviour of the models and the proposed numerical methods.

Additionally, we can express the American option price in terms of expectations involving optimal stopping times and nonlinear integral relations. In the case of constant intensities of default, this approach has been previously addressed in [2]. In the present article we have extended this methodology in order to compute the American option value when stochastic intensities of default are considered. The proposed techniques involve the computation of lower and upper estimators to build up a confidence interval for the American option price. These estimators are obtained by extending some previous results from [22] and [23].

In this way, we can compare solutions obtained from complementarity problems and from formulations in terms of expectations. The agreement between both solutions is illustrated by numerical results. Comparing the elapsed time consumed by the different methods used to obtain the XVA, we appreciate that Monte Carlo methods require a larger computational time than the finite element techniques.

Finally, we would like to emphasize that this last comparison also illustrates the interest in the statement of well posed PDEs models for the pricing of American options with counterparty risk and two factors, as well as the advantage of the numerical solution of the PDEs with respect to other alternative approaches.

Table 5
The American option value with counterparty risk and $M = V$. The parameter values of the problem are: $K = 15$, $T = 0.5$, $\sigma^S = 0.3$, $\sigma^h = 0.2$, $\rho = 0.2$, $r = 0.3$, $q = 0.24$, $\kappa = 0.01$, $R = 0.3$, $f = 0.04$.

S	h	Finite elements	Lower estimator	Upper estimator	99% confidence interval
0.0	0.00	15.00000000	14.76015652	15.31754947	(14.60199696 , 15.37917273)
27.0		0.014580440	0.00157526	0.03789128	(−0.00047226 , 0.04096308)
3.0	0.05	12.00000000	11.92717148	14.30577857	(11.80632068 , 14.60534869)
9.0		5.999999999	5.77587526	9.03605748	(5.60018068 , 9.24791036)
15.0		1.310644569	1.07192257	3.11079144	(0.94555737 , 3.23919041)
24.0		0.039554986	0.01108702	0.43742176	(0.00046278 , 0.45329951)
9.0	0.10	5.999999999	5.63823578	8.49773982	(5.47119356 , 8.68419050)
12.0		3.066098200	2.78677451	8.07659200	(2.61439440 , 8.32628955)
15.0		1.194868136	1.12406653	2.77078546	(0.99821874 , 2.89558922)
18.0		0.384468906	0.25563850	1.92442123	(0.19352922 , 2.00047634)
27.0		0.007288464	0.00040263	0.02285157	(−0.00033044 , 0.02383561)
18.0	0.15	0.335480535	0.26140404	1.70487679	(0.20130499 , 1.77073497)
21.0		0.087669807	0.06821849	1.09391591	(0.03798815 , 1.13640329)
24.0		0.021348419	0.00759706	0.52814862	(0.00026779 , 0.54853994)
21.0	0.20	0.074863949	0.04667367	0.55078911	(0.02193558 , 0.57228007)
27.0		0.003719381	0.00128050	0.01824716	(−0.00005382 , 0.02095550)
30.0		0.000821669	0.00002521	0.00040568	(−0.00003973 , 0.00089052)

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