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THE EARLY EXERCISE BOUNDARY FOR THE AMERICAN PUT NEAR EXPIRY: NUMERICAL APPROXIMATION

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ABSTRACT. It is well known [**11**] that the early exercise boundary for the American put approaches the strike price at expiry with infinite velocity. This causes difficulties in developing efficient and accurate numerical procedures and consequently trading strategies, during the volatile period near expiry. Based on the work of D. Ševčovič [**10**] for the American call with dividend, an integral equation is derived for the free boundary for the American put which leads to an accurate numerical procedure and an interesting, and accurate, asymptotic solution for the early exercise boundary near expiry.

1. Introduction. Many different, but equivalent, integral equations have been derived for the American put [**1**], [**3**], [**6**, pp. 384–386], [**7**], [**8**], [**9**], some of which lead to an analysis of the free boundary near expiry [**1**], [**7**], [**8**]. In [**9**] a survey of both theoretical and computational work on the American put is presented. In this note we shall derive an alternative integral equation which will provide an accurate numerical method for calculating the early exercise boundary near expiry and, in addition, derive an analytical asymptotic approximation. These numerical and analytical approximations will be compared with the binomial and trinomial methods along with the other approximations mentioned above.

2. Integral equation for the American put. We shall price the American put using the Black-Scholes equation. With the Black-Scholes model of stock prices, the American put option $P(S,t)$ then

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obeys the following parabolic PDE,

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \quad S > S_f(t)$$

$$\left.\begin{aligned} P(S,t) &= E - S \\ \frac{\partial P}{\partial S}(S,t) &= -1 \end{aligned}\right\} S = S_f(t)$$

$$P(S,t) \longrightarrow 0 \quad \text{as} \quad S \longrightarrow \infty$$

$$P(S,T) = \max(E - S, 0)$$

where E is the exercise price, T is the expiry time, $S_f(t)$ is the free boundary separating the holding and early exercise regions, $S(t)$ is the time-dependent stock price, r is the risk-free interest rate, and σ is the volatility.

Note that at expiry $S_f(T) = E$. It is well known, however, that the early exercise boundary approaches expiry with infinite velocity (Van Moerbeke [**11**]), leading to difficulties in accurate pricing and in trading strategies during this extremely volatile period.

In deriving the integral equation for the free boundary, we employ ideas from Ševčovič's [**10**] work on the American call with dividend which was seen to favorably compare with other approaches [**12**]. We shall focus on the differences with this paper and skip over the details available there. We begin, as usual, by a series of substitutions that will simplify our analysis. Let

$$x = \log\left(\frac{S}{\varrho(\tau)}\right)$$

$$\tau = \frac{\sigma^2}{2}(T - t)$$

where $\varrho(\tau) = S_f(t)$. Note that $\varrho(0) = S_f(T) = E$. This substitution along with the transformation

$$\begin{aligned} \Pi(x,\tau) &= P - S\frac{\partial P}{\partial S} \\ &= P - P_x \end{aligned}$$

yields

$$\frac{\partial \Pi}{\partial \tau} = \frac{\partial^2 \Pi}{\partial x^2} + a(\tau)\frac{\partial \Pi}{\partial x} - k\Pi, \quad x > 0$$

$$\Pi(0,\tau) = E \tag{2.1}$$

$$\frac{\partial \Pi}{\partial x}(0,\tau) = -kE \tag{2.2}$$

$$\Pi(x,\tau) \longrightarrow 0 \quad \text{as} \quad x \to \infty$$

$$\Pi(x,0) = 0 \quad \text{for} \quad x > 0$$

where

$$a(\tau) = \frac{\dot{\varrho}(\tau)}{\varrho(\tau)} + k - 1$$

$$k = \frac{2r}{\sigma^2}.$$

Note that Π synthesizes a portfolio with a one put holding and $-(\partial P/\partial S)$ units of the underlying stock. (Interestingly enough, the transformation Π can be used to derive the Black-Scholes PDE with no arbitrage condition $d\Pi = r\Pi\, dt$ and the fact that P has a self-replicating strategy, see [**2**], [**4**], [**6**], [**12**]).

We define the Fourier sine and cosine transforms as

$$F_s(f)(\omega) = \int_0^\infty f(x)\sin(\omega x)\, dx$$

$$F_c(f)(\omega) = \int_0^\infty f(x)\cos(\omega x)\, dx.$$

Let

$$p(\omega,\tau) = F_s(\Pi(\cdot,\tau))(\omega)$$

$$q(\omega,\tau) = F_c(\Pi(\cdot,\tau))(\omega).$$

With this set of transformations, one obtains a system of ODEs

$$\frac{d}{d\tau}p = -a(\tau)\omega q - (k+\omega^2)p + E\omega$$

$$\frac{d}{d\tau}q = a(\tau)\omega p - (k+\omega^2)q + E(k - a(\tau))$$

that can be solved via the variation of parameters formula. One can solve explicitly for p and q using the initial condition obtained from the Fourier sine and cosine transforms of equation (2.1) (for more details see Ševčovič [**10**]). Supposing smoothness of solution Π up to the boundary, one can conclude that Π must satisfy

$$0 = kE + \frac{\partial \Pi}{\partial x}(0, \tau). \tag{2.3}$$

Then the transformed boundary condition (2.2) along with the inverse Fourier transform gives the following integral equation for the free boundary in terms of the new variable $\eta(\tau)$

$$\varrho(\tau) = Ee^{-(k-1)\tau} e^{2\sqrt{\tau}\eta(\tau)}$$

where

$$\eta(\tau) = -\sqrt{-\log\left[\sqrt{\pi}k\tau^{1/2}e^{k\tau}\left(1 - \frac{F(\tau)}{\sqrt{\pi}}\right)\right]} \tag{2.4}$$

$$g(t, \theta) = \frac{1}{\cos\theta}[\eta(\tau) - \sin\theta\eta(\tau\sin^2\theta)] \tag{2.5}$$

$$F(\tau) = 2\int_0^{\pi/2} e^{-k\tau\cos^2\theta - g^2(\tau,\theta)}\{\sqrt{\tau}\sin\theta + g(\tau, \theta)\tan\theta\}\, d\theta. \tag{2.6}$$

Equations (2.4)–(2.6) define an implicit problem for η which will be the basis of our analysis in the next sections.

3. Approximation of the free boundary near expiry. One can try to solve problem (2.4)–(2.6) recursively. Beginning with an initial guess $S_0(t) = E$ for the free boundary, equations (2.4)–(2.6) become

$$\begin{aligned}
\eta_0(\tau) &= \frac{k-1}{2}\sqrt{\tau} \\
g_0(\tau, \theta) &= \cos\theta\eta_0(\tau) \\
F_0(\tau) &= 2\int_0^{((k+1)/2)\sqrt{\tau}} e^{-u^2}\, du = \sqrt{\pi}\ \mathrm{erf}\left(\frac{k+1}{2}\sqrt{\tau}\right)
\end{aligned}$$

where $\operatorname{erf}(z)$ is the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds.$$

Substituting g_0 and F_0 into η yields

$$\eta_1(\tau) = -\sqrt{-\log\left[\sqrt{\pi} k \tau^{1/2} e^{k\tau}\left(1 - \operatorname{erf}\left(\frac{k+1}{2}\sqrt{\tau}\right)\right)\right]}.$$

Since the error function is of order $O(\sqrt{\tau})$,

$$\eta_{1\mathrm{approx}}(\tau) \approx -\sqrt{-\log[\sqrt{\pi} k \tau^{1/2} e^{k\tau}]}.$$

The estimate for η_2 is more involved. In order to compute g_1 we first compute

$$\begin{aligned}
&\eta_1(\tau \sin^2\theta) \\
&= -\sqrt{-\log\left[\sqrt{\pi} k \tau^{1/2} \sin\theta e^{k\tau \sin^2\theta}\left(1 - \frac{2}{\sqrt{\pi}} \int_0^{((k+1)/2)\sqrt{\tau}\sin\theta} e^{-u^2}\, du\right)\right]} \\
&= -\left(-\log\left[(\sqrt{\pi} k \tau^{1/2} e^{k\tau}) \sin\theta e^{-k\tau\cos^2\theta}\left\{1 - \frac{2}{\sqrt{\pi}} \int_0^{((k+1)/2)\sqrt{\tau}} e^{-u^2}\, du\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \frac{2}{\sqrt{\pi}} \int_{((k+1)/2)\sqrt{\tau}\sin\theta}^{((k+1)/2)\sqrt{\tau}} e^{-u^2}\, du\right\}\right]\right)^{1/2}.
\end{aligned}$$

After some algebra one obtains

$$\eta_1(\tau \sin^2\theta) = \eta_1(\tau)\sqrt{1 + \frac{\log G(\tau,\theta)}{-\eta_1^2(\tau)}} \tag{3.1}$$

where

$$G(\tau,\theta) = \sin\theta e^{-k\tau\cos^2\theta}\left(1 + \frac{(2/\sqrt{\pi}) \int_{((k+1)/2)\sqrt{\tau}\sin\theta}^{((k+1)/2)\sqrt{\tau}} e^{-u^2}\, du}{1 - (2/\sqrt{\pi}) \int_0^{((k+1)/2)\sqrt{\tau}} e^{-u^2}\, du}\right).$$

Now if the second term in the square root of equation (3.1) is small, we say it may Taylor expand. This term blows up at $\theta = 0$. If $\theta \gg \sqrt{\tau}$ one has

$$\eta_1(\tau \sin^2 \theta) \approx \eta_1(\tau)\left[1 - \frac{\log G(\tau, \theta)}{2\eta_1^2(\tau)}\right]$$

and

$$\begin{aligned} g_1(\tau, \theta) &\approx -\frac{1}{\cos\theta}\left[\eta_1(\tau) - \sin\theta\,\eta_1(\tau)\left(1 - \frac{\log G(\tau,\theta)}{2\eta_1^2(\tau)}\right)\right] \\ &\approx -\eta_1(\tau)\left[\frac{1-\sin\theta}{\cos\theta}\right]. \end{aligned}$$

Substituting this into F and then F into η yields the result

$$\eta_2(\tau) \sim -\sqrt{-\log[2\sqrt{\pi}k\tau^{1/2}e^{k\tau}]} \tag{3.2}$$

which is our conjecture for the behavior of $\eta(\tau)$ near expiry. This is a much more interesting behavior at $\tau = 0$ than for the American call with dividend for which η is replaced with a constant obtained from satisfying a transcendental equation. Kuske and Keller [**7**] also derive an asymptotic solution from an integral equation (their coefficient inside the logarithmic term of equation (3.2) differs from our value of 2). To make the above rigorous, one must show that $F(\tau)$ is bounded as $\tau \to 0^+$. More precisely, we need to find a $\delta(\tau) > 0$ such that, if

$$F(\tau) = \int_0^{\delta(\tau)} + \int_{\delta(\tau)}^{\pi/2} = I_1(\tau) + I_2(\tau) \tag{3.3}$$

then $I_1(\tau) \to 0$ as $\tau \to 0^+$ for $\delta(\tau) \gg \sqrt{\tau}$. We shall verify this in the next section.

Before proceeding to numerical simulations we give an alternative derivation of the above conjecture based on the ansatz

$$\eta(\tau) \sim -a|\log\tau|^p \tag{3.4}$$

as suggested by the above analysis. We begin by stating some obvious properties for $g(\tau, \theta)$ which follow from the condition $\eta(\tau) \to -\infty$ as $\tau \to 0$ [**11**]. Note that $g(\tau, \theta) \to 0$ as $\theta \to (\pi/2)$ and that $g(\tau, 0) = \eta(\tau)$.

Also by the mean value theorem $g(\tau,\theta) = (1/2)\sqrt{\tau}\cos\theta\xi'(\tilde{\tau})$ where $\tau\sin^2\theta < \tilde{\tau} < \tau$ and $\xi(\tau) = 2\sqrt{\tau}\eta(\tau)$.

With the assumption (3.4),

$$g(\tau,\theta) = -\frac{1}{\cos\theta}|\log\tau|^p\left[1-\sin\theta\left(1+\frac{\log\sin^2\theta}{\log\tau}\right)^p\right].$$

If $\log(\sin^2\theta)/\log\tau \ll 1$ or equivalently $\theta \gg \sqrt{\tau}$ as before, then

$$\begin{aligned} g(\tau,\theta) &\approx -\frac{a}{\cos\theta}|\log\tau|^p(1-\sin\theta) \\ &\approx \eta(\tau)\left(\frac{1-\sin\theta}{\cos\theta}\right) \end{aligned}$$

Formally substituting the above expression for g into F and making the substitution $u = -\eta(\tau)(1-\sin\theta)/\cos\theta$ yields, for small τ,

$$\begin{aligned} -\frac{F(\tau)}{2} &\approx -\int_0^{\pi/2} e^{-g^2(\tau,\theta)}\tan\theta g(\tau,\theta)\,d\theta \\ &= \int_0^{-\eta} e^{-u^2}\frac{\eta^2-u^2}{\eta^2+u^2}\,du \\ &= \int_0^{-\eta} e^{-u^2}\frac{du}{1+(u/\eta)^2} - \frac{1}{\eta^2}\int_0^{-\eta} e^{-u^2}\frac{u^2}{1+(u/\eta)^2}\,du \\ &= \int_0^{-\eta} e^{-u^2}\left(1-\left(\frac{u}{\eta}\right)^2+\cdots\right)du \\ &\quad -\frac{1}{\eta^2}\int_0^{-\eta} e^{-u^2}u^2\left(1-\left(\frac{u}{\eta}\right)^2+\cdots\right)du. \end{aligned}$$

Using the following asymptotic expansion as $\eta \to -\infty$

$$\begin{aligned} \int_0^{-\eta} e^{-u^2}\,du &= \int_0^{\infty} e^{-u^2}\,du - \int_{-\eta}^{\infty} e^{-u^2}\,du \\ &\sim \frac{\sqrt{\pi}}{2} + \frac{1}{2\eta}e^{-\eta^2}\left(1-\frac{1}{2\eta^2}+\cdots\right) \end{aligned}$$

one obtains $F(\tau) \sim -\sqrt{\pi} - (1/\eta)e^{-\eta^2}(1+O(1/\eta^2))$. This once again yields the conjectured approximation (3.2), namely,

$$\eta(\tau) \sim -\sqrt{-\log[2\sqrt{\pi}k\tau^{1/2}e^{k\tau}]}.$$

4. Verification of asymptotic conjecture (3.2) to the integral equation. Now we show that the ansatz $\eta(\tau) \sim -\sqrt{-\log \tau^{1/2}}$ implies $I_1(\tau) = o(1)$ where $I_1(\tau)$ is given by equation (3.3). Thus one obtains (3.2) as the asymptotic solution to the integral equation (2.4)–(2.6). With the substitution $s = \tau \sin^2 \theta$, $F(\tau)$ is

$$F(\tau) = \int_0^\tau e^{-k(\tau-s)-(A^2(\tau,s)/4(\tau-s))}\left\{1 + \frac{A(\tau,s)}{2(\tau-s)}\right\}\frac{ds}{\sqrt{\tau-s}}$$
$$= \int_0^{\bar\delta(\tau)} + \int_{\bar\delta(\tau)}^\tau = I_1(\tau) + I_2(\tau)$$

where

$$A(\tau,s) = 2\sqrt{\tau}\eta(\tau) - 2\sqrt{s}\eta(s)$$

and $\bar\delta(\tau)$ is to be suitably chosen. Note that $\bar\delta = \tau\sin^2\delta$ (δ is in terms of θ integration; see equations (2.6) and (3.3)).

Let $\xi(\tau) = 2\sqrt{\tau}\eta(\tau)$ and note that ξ is negative, decreasing and $\xi \to 0$ as $\tau \to 0^+$ with the above choice of $\eta(\tau)$. For small τ, $|\xi(\tau) - \xi(s)| \le 2|\xi(\tau)| = -2\xi(\tau)$. This yields

$$|I_1(\tau)| \le \int_0^{\bar\delta(\tau)} 1\left(1 - \frac{\xi(\tau)}{\tau - s}\right)\frac{ds}{\sqrt{\tau-s}}$$
$$\le \tau^{1/2}\left[\frac{\bar\delta}{\tau} + O\left(\frac{\bar\delta}{\tau}\right)^2\right] - \xi\tau^{-1/2}\left[\frac{\bar\delta}{\tau} + O\left(\frac{\bar\delta}{\tau}\right)^2\right]$$

if $(\bar\delta/\tau) \ll 1$. With $\bar\delta(\tau) = \tau^{3/2}$ the leading term in the second expression above becomes $2\sqrt{\tau}\eta$. Thus

$$|I_1(\tau)| \le \tau + O(\tau^{3/2}) - \xi(\tau)[1 + O(\tau^{1/2})].$$

This gives the desired result, $I_1(\tau) = o(1)$.

The estimate for $I_2(\tau)$ only requires slight modifications from our previous analysis. When $\bar\delta(\tau) = \tau^{3/2}$, then $\delta(\tau) = \arcsin \tau^{1/4} \approx \tau^{1/4}$. Using the same substitution as before, $u = -\eta(\tau)(1 - \sin\theta)/\cos\theta$, one obtains

$$I_2(\tau) \sim -\sqrt{\pi} + \frac{1}{\eta\gamma}e^{-(\eta\gamma)^2}\left(1 + O\left(\frac{1}{\eta\gamma}\right)^2\right)$$

where $\gamma(\tau) = ((1-\tau^{1/4})/\sqrt{1-r^{1/2}}) = 1 + O(\tau^{1/4})$. Thus the leading order term of $F(\tau)$ is $-\sqrt{\pi}$, and this gives the asymptotic solution (3.2).

5. Numerical simulations: comparison with the analytical asymptotic solution (3.2). In what follows we consider the following two sets of parameters: $\sigma = .4$, $r = .1$, $E = 50$ and $\sigma = .25$, $r = .1$, $E = 10$. With these parameters we shall compare how well the binomial method, trinomial method, integral equation (2.4)–(2.6), and asymptotic approximation (3.2) predict the position of the early exercise free boundary. Also we shall compare how well our asymptotic solution fares with other approximations obtained by different authors. All of these results regarding these two sets of parameters are summarized in Tables 1 and 2, respectively.

We begin with the binomial and trinomial methods to obtain accurate data for the free boundary. The position of the early exercise boundary obtained from the trinomial method is recorded in the fourth column of Tables 1 and 2. Only values that differ in the fourth decimal place from the binomial method are indicated. For both the binomial and trinomial methods a depth of 1,000 subdivisions was used. Results from the trinomial tree were computed using the software package "Option Calculator" developed by Srivastava et al. at Carnegie Mellon University. Tables 1 and 2 show that the data from the binomial and trinomial trees agree quite well. For $\sigma = .4$, $r = .1$, $E = 50$, these values agreed to the tenth of a cent for less than one hour to expiry. For values up to 2.6 weeks they agreed to the cent (see Table 1). Table 2 corresponds to the parameters $\sigma = .25$, $r = .1$, $E = 10$. Here the binomial and trinomial methods agree even better. Both sets of data match to the tenth of a cent up to 2.6 weeks before expiry. Since the binomial and trinomial methods match, we shall carry out the rest of the comparisons with the binomial method.

Figures 1a–2 compare the free boundary calculation using the binomial method with the integral equation (2.4)–(2.6) for values of $T - t$ from .876 hours to 2.6 weeks before expiry. Following the recursion outlined in Section 3, four iterations of the integral equation were used. There was a slight discrepancy between these methods. For small times ($T - t < .876$) there was agreement to 2 and 3 decimal places for the two sets of parameters, respectively (see Tables 1 and 2). As we move further from expiry, the values deviate even further.

TABLE 1. Free boundary position for the set of parameters: $\sigma = .4$, $E = 50$, $r = .1$. The fourth column corresponds to the trinomial method. Only entries that differ in the fourth decimal place from the binomial method are indicated. A depth of 1000 steps was used for the binomial and trinomial trees.

$T - t$	Integral equation	Binomial method	Trinomial method	Asymp. solution	M.B.W. approx.
.000005	49.8458	49.8479		49.8469	49.8548
.00001	49.7884	49.7904	49.7905	49.7901	49.8010
.00002	49.7102	49.7125		49.7128	49.7277
.00003	49.6520	49.6545		49.6554	49.6733
.00004	49.6039	49.6066		49.6081	49.6283
.00005	49.5623	49.5651	49.5652	49.5671	49.5893
.00006	49.5251	49.5281	49.5282	49.5305	49.5545
.00007	49.4913	49.4946		49.4973	49.5230
.00008	49.4602	49.4635	49.4636	49.4667	49.490
.00009	49.4312	49.4346	49.4347	49.4382	49.4667
.00010 (.876 hrs)	49.4040	49.4077	49.4076	49.4115	49.4413
.00020	49.1911	49.1957	49.1958	49.2029	49.2424
.00030	48.0345	49.0400	49.0401	49.0501	49.0963
.00040	48.9063	48.9124	49.9127	48.9252	48.9766
.00050	48.7958	48.8029		48.8179	48.8735
.00060	48.6979	48.7053	48.7055	48.7229	48.7820
.00070	48.6092	48.6174		48.6371	48.6993
.00080	48.5279	48.5362	48.5365	48.5584	48.6234
.00090	48.4524	48.4615	48.4616	48.4856	48.5530
.00100 (8.76 hrs)	48.3819	48.3915		48.4176	48.4871
.00200	47.8379	47.8511	47.8513	47.8964	47.9794
.00300	47.4466	47.4620	47.4625	47.5249	47.6137
.00400	47.1312	47.1489	47.1494	47.2279	47.3188
.00500	46.8631	46.8836	46.8833	46.9771	47.0678
.00600	46.6278	46.6489	46.6497	46.7584	46.8475
.00700	44.4169	46.4403	46.4404	46.5636	46.6499
.00800	46.2251	46.2490	46.2500	46.3874	46.4700
.00900	46.0486	46.0750	46.0746	46.2262	46.3044
.01000 (3.65 days)	45.8848	45.9115	45.9122	46.0773	46.1507
.02000	45.6580	44.6950	44.6962	44.9906	44.9968
.03000	43.8140	43.8595	43.8597	44.2762	44.1991

TABLE 1. CONTINUED.

$T-t$	Integral equation	Binomial method	Trinomial method	Asymp. solution	M.B.W. approx.
.04000	43.1558	43.2070	43.2075	43.7431	43.5747
.05000 (2.6 wks)	42.6111	42.6681	42.6672	43.3211	43.0562

TABLE 2. Free boundary position for the set of parameters: $\sigma = .25$, $E = 10$, $r = .1$. The fourth column corresponds to the trinomial method. Only entries that differ in the fourth decimal place from the binomial method are indicated. A depth of 1000 steps was used for the binomial and trinomial trees.

$T-t$	Integral equation	Binomial method	Trinomial method	Asymp. solution	M.B.W. approx.
.000005	9.9815	9.9820		9.9816	9.9827
.00001	9.9746	9.9751		9.9749	9.9762
.00002	9.9653	9.9659		9.9657	9.9675
.00003	9.5844	9.9590		9.9589	9.9611
.00004	9.9528	9.9533		9.9533	9.9557
.00005	9.9478	9.9484		9.9485	9.9511
.00006	9.9435	9.9440		9.9442	9.9470
.00007	9.9395	9.9400		9.9403	9.9433
.00008	9.9358	9.9364		9.9367	9.9399
.00009	9.9324	9.9329		9.9334	9.9367
.00010 (.876 hrs)	9.9292	9.9297	9.9298	9.9303	9.9337
.00020	9.9042	9.9048	9.9049	9.9059	9.9104
.00030	9.8858	9.8866		9.8882	9.8934
.00040	9.8709	9.8717		9.8737	9.8794
.00050	9.8580	9.8589		9.8613	9.8674
.00060	9.8466	9.8476		9.8504	9.8567
.00070	9.8363	9.8373		9.8405	9.8472
.00080	9.8268	9.8279	9.8280	9.8315	9.8383
.00090	9.8181	9.8192	9.8193	9.8231	9.8302
.00100 (8.76 hrs)	9.8099	9.8111		9.8154	9.8225
.00200	9.7470	9.7487		9.7561	9.7639
.00300	9.7019	9.7039		9.7143	9.7218
.00400	9.6657	9.6679		9.6811	9.6880

TABLE 2. CONTINUED.

$T-t$	Integral equation	Binomial method	Trinomial method	Asymp. solution	M.B.W. approx.
.00500	9.6349	9.6375	9.6374	9.6533	9.6593
.00600	9.6080	9.6106	9.6107	9.6293	9.6341
.00700	9.5839	9.5868		9.6079	9.6115
.00800	9.5620	9.5650	9.5651	9.5888	9.5910
.00900	9.5418	9.5452	9.5451	9.5713	9.5721
.01000 (3.65 days)	9.5232	9.5265	9.5266	9.5553	9.5546
.02000	9.3841	9.3885	9.3886	9.4416	9.4236
.03000	9.2891	9.2940	9.2941	9.3712	9.3334
.04000	9.2155	9.2206	9.2207	9.3222	9.2630
.05000 (2.6 wks)	9.1550	9.1600	9.1601	9.2865	9.2047

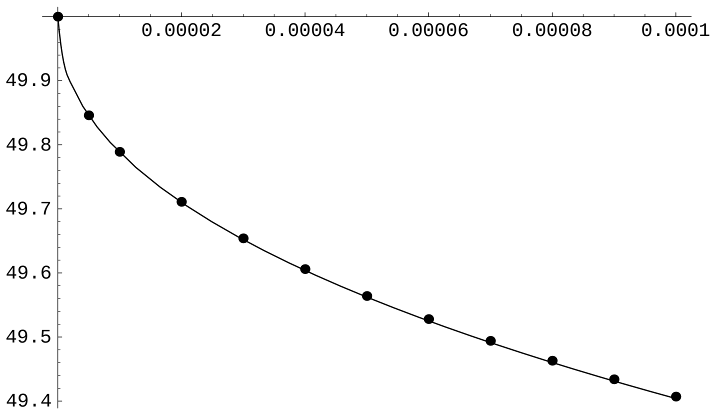

FIGURE 1a. Profile of $S_f(T-t)$ obtained from the binomial method and integral equation (2.4)–(2.6) for $\sigma = .4$, $r = .1$, $E = 50$, $T-t = .876$ hrs. The solid curve corresponds to four iterations of the integral equation.

Next we examine how accurately our asymptotic approximation matches the data from the binomial method. Near expiry at about one hour, the asymptotic approximation matches the data from the binomial method (see Figures 3a and 4a). At 8.76 hours with $\sigma = 0.4$, $r = .1$, $E = 50$, we see that the asymptotic approximation gives an overestimate for the free boundary for a fixed value of time (see Fig-

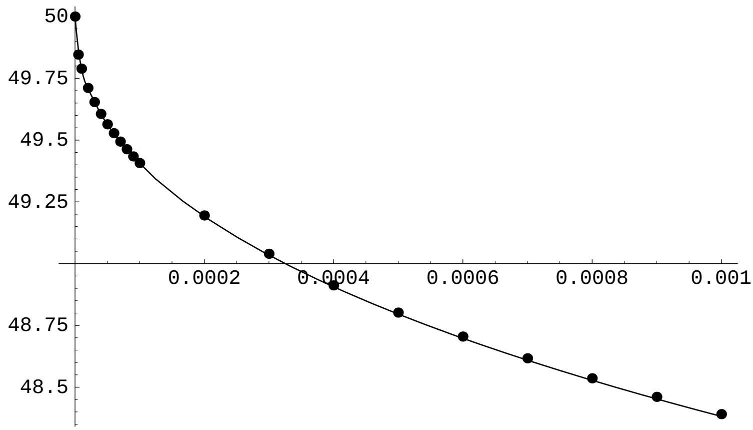

FIGURE 1b. Binomial method versus integral equation for $\sigma = .4$, $r = .1$, $E = 50$, $T - t = 8.76$ hrs.

ure 3b). At 8.76 hours, the asymptotic approximation is off by 3 cents (see Table 1). Similarly, with $\sigma = 0.25$, $r = .1$, $E = 10$ at 8.76 hours the approximation gives an overestimate but of only .4 cents (see Table 2).

Now we compare our asymptotic solution with MacMillan, Barone-Adesi and Whaley's [**1**], [**6**, pp. 384–386], [**8**] numerical approximation of the American put free boundary. They apply a transformation that results in a Cauchy-Euler equation that can be solved analytically. For times very close to expiry, one can see that our approximation of the free boundary matches the data from the binomial and trinomial methods more accurately. For example, in Figure 5a where $\sigma = 0.4$, $r = .1$,

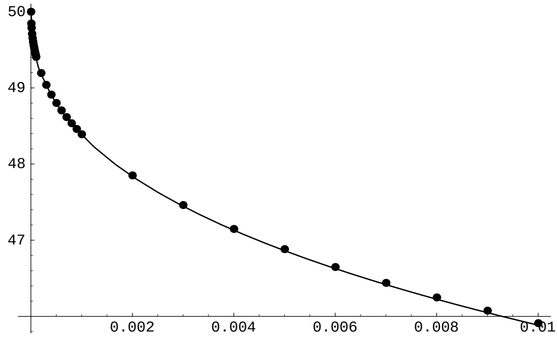

FIGURE 1c. Binomial method vs. integral equation for $\sigma = .4$, $r = .1$, $E = 50$, $T - t = 3.65$ days.

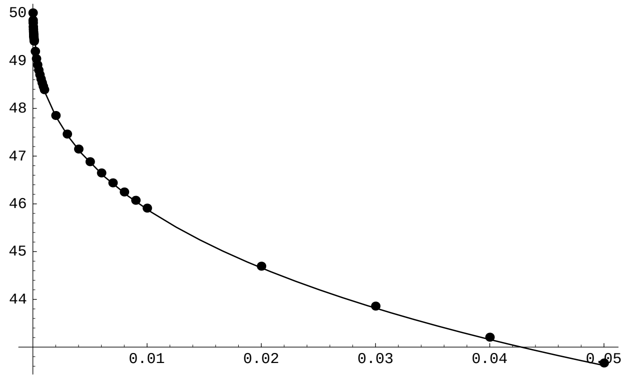

FIGURE 1d. Binomial method vs. integral equation for $\sigma = .4$, $r = .1$, $E = 50$, $T-t = 2.6$ wks.

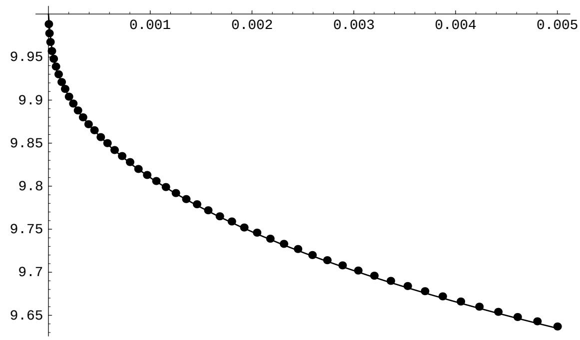

FIGURE 2. Binomial method vs. integral equation for $\sigma = .25$, $r = .1$, $E = 10$, $T-t = 1.825$ days.

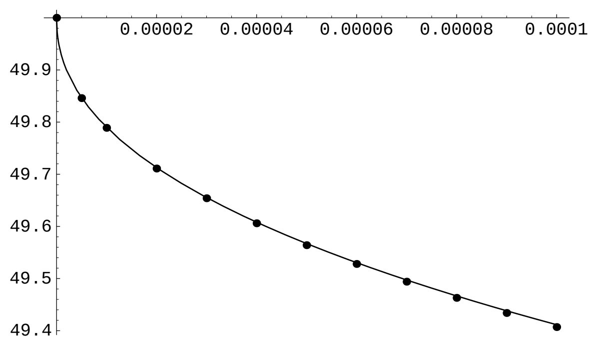

FIGURE 3a. Asymptotic approximation (solid curve) vs. binomial method approximation of $S_f(T-t)$ for $\sigma = .4$, $r = .1$, $E = 50$, $T-t = .876$ hrs.

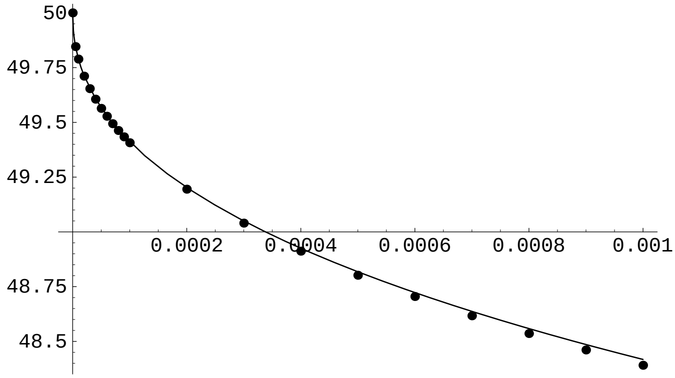

FIGURE 3b. Asymptotic approximation vs. binomial method for $\sigma = .4$, $r = .1$, $E = 50$, $T - t = 8.76$ hrs.

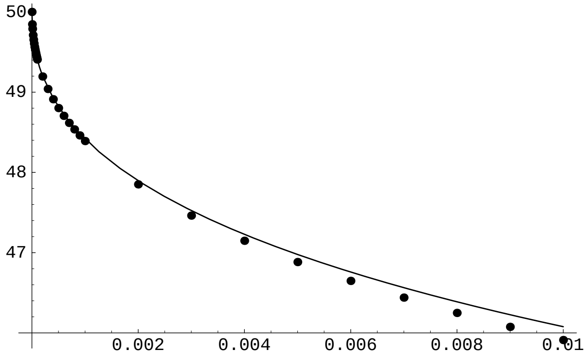

FIGURE 3c. Asymptotic approximation vs. binomial method for $\sigma = .4$, $r = .1$, $E = 50$, $T - t = 3.65$ days.

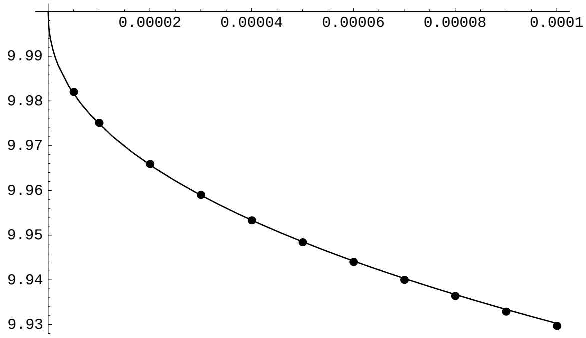

FIGURE 4a. Asymptotic approximation vs. binomial method for $\sigma = .25$, $r = .1$, $E = 10$, $T - t = .876$ hrs.

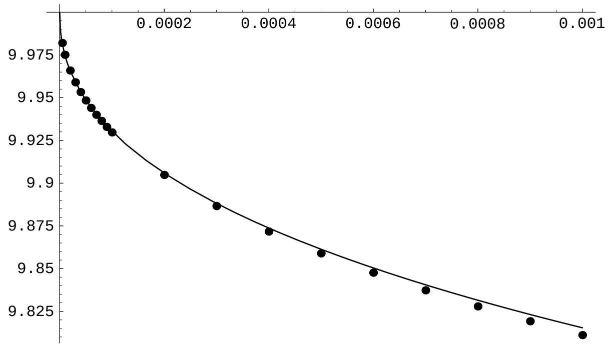

FIGURE 4b. Asymptotic approximation vs. binomial method for $\sigma = .25$, $r = .1$, $E = 10$, $T-t = 8.76$ hrs.

$E = 50$, our approximation is off by .4 cents at .876 hours while their approximation is off by 3 cents (see Table 1). Similarly, for $\sigma = 0.25$, $r = .1$, $E = 10$ at .876 hours before expiry, our approximation differs from the binomial method by 0.06 cents while theirs differs by .4 cents (see Table 2).

6. Concluding remarks. Since the American put approaches the strike price with infinite velocity, it is difficult to obtain efficient and accurate numerical procedures for evaluating this option near expiry. An integral equation (2.4)–(2.6) is derived for the early exercise boundary which not only gives an accurate numerical procedure for evaluating

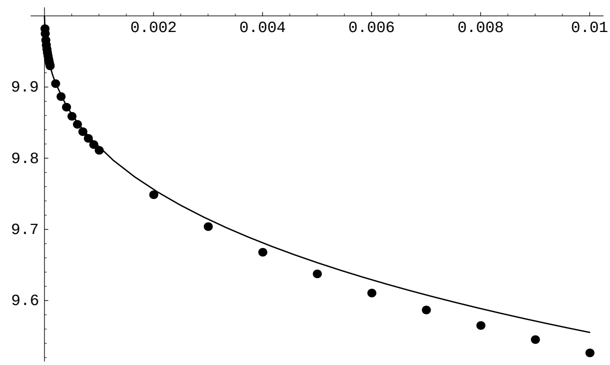

FIGURE 4c. Asymptotic approximation vs. binomial method for $\sigma = .25$, $r = .1$, $E = 10$, $T-t = 3.65$ days.

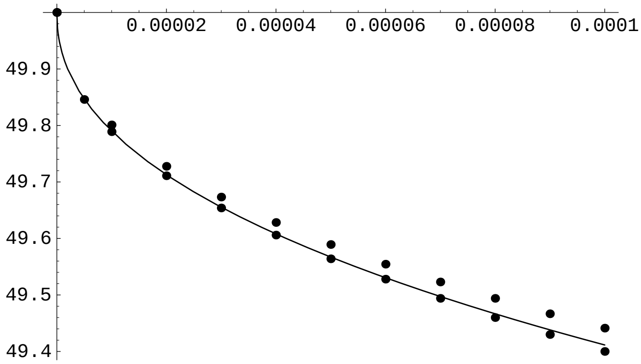

FIGURE 5a. MBW approximation vs. the asymptotic solution (3.2) for $\sigma = .4$, $r = .1$, $E = 50$, $T-t = 8.76$ hrs. MBW data lies above the asymptotic approximation (solid line) and above the data from the binomial method.

the free boundary but also enables us to derive an asymptotic solution near expiry (3.2).

The asymptotic approximation (3.2) fits the data obtained from the binomial and trinomial trees near expiry. Two sets of parameters were used in this paper that involved the volatility, strike price and risk-free interest rate. Depending upon which set of data, our asymptotic approximation agrees to the cent with the binomial method from 1 hour to 8 hours to expiry (see Tables 1 and 2). Also our asymptotic solution approximated the position of the free boundary better than MacMillan, Barone-Adesi and Whaley's numerical approximation for

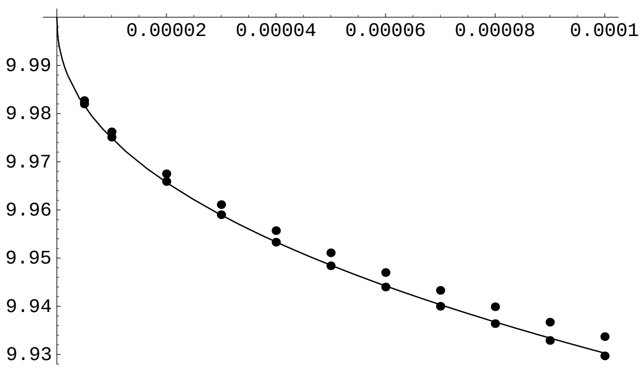

FIGURE 5b. MBW approximation vs. the asymptotic solution (3.2) for $\sigma = .25$, $r = .1$, $E = 10$, $T-t = 8.76$ hrs.

times close to expiry.

Future work that is required from this note is quite clear. In order to capture the exercise boundary for longer times τ, an asymptotic series needs to be proposed. Our original asymptotic solution (3.2) would then correspond to the first term of such an expansion. It would then be interesting to calculate higher order terms and see how far from expiry they predict the position of the free boundary.

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