Evolution of curves on a surface driven by the geodesic curvature and external force

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Goals

- study a flow of curves on a given two dimensional surface $\mathcal{M}$ in $\mathbb{R}^3$ where normal velocity $V$ of a curve $G$ on $\mathcal{M}$ is a linear function of its geodesic curvature $K_g$ and external force $F$:

$$V = K_g + F$$

A surface curve $G \subset \mathcal{M}$ (left). Its vertical projection to a plane curve
Show how the flow of curves on a given surface driven by the geodesic curvature and external force can be reduced to the flow of curves in the plane driven by the normal velocity

\[ v = \beta(x, k, \nu) \]

where \( k, \nu, x \) are the curvature, tangential angle and position vector of transversally projected planar curve \( \Gamma \)
• Represent the flow of plane curves by a solution to the geometric equation

\[ \partial_t x = \beta \vec{N} + \alpha \vec{T} \]

for the position vector \( x \in \mathbb{R}^2 \) representing a curve \( \Gamma = \text{Image} \, (x) \).

• Reduce the problem to solution of a system of parabolic PDEs for the curvature, angle and local length of a curve. Analyze qualitative behavior of solutions and stability of closed stationary curves on a surface.

• Suggest a suitable tangential velocity functional \( \alpha \) yielding a uniform grid point redistribution along the evolved curve. Compute the flow of curves on various complex surfaces.
Outline

- Transformation of the flow of surface curves to the flow of vertically projected planar curves satisfying $v = \beta(x, k, \nu)$

- Link between the geodesic flow and the edge detection problem in the theory of image segmentation

- Derivation a governing system of PDEs describing the evolution of plane curves satisfying $v = \beta(x, k, \nu)$

- Qualitative aspects of solutions like existence and their limiting behavior. Lyapunov functionals

- Dynamical theory point of view. Closed geodesic curves and their stability.

- Numerical approximation of the geodesic curvature driven flow of surface curves.
Projection of a flow of surface curves to the plane

We consider a flow of surface curves $G_t \subset \mathcal{M}, t \geq 0$, driven by the geodesic curvature $K_g$ and external force

$$\mathcal{V} = K_g + F$$

where $\mathcal{M} = \text{Graph}(\phi)$, $\phi : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a surface in $\mathbb{R}^3$, $\mathcal{V}$ is the normal component of the velocity, $K_g$ is the geodesic curvature of $G_t$ relative to $\mathcal{M}$ and $F$ is the normal component of a gravitational like external force

$$\vec{G} = - (0, 0, \gamma)$$
• the geodesic curvature $K_g$ for a curve $\mathcal{G} = \{(x, \phi(x)) \in \mathbb{R}^3, x \in \Gamma\}$ on a surface $\mathcal{M} = \{(x_1, x_2, \phi(x_1, x_2)) \in \mathbb{R}^3, (x_1, x_2) \in \Omega\}$ can be expressed as a function of the curvature $k$ of its projection to the plane, position vector $x$ and the angle $\nu$.

• The external vector field $\vec{G}$ is assumed to be perpendicular to the plane $\mathbb{R}^2$ and it depends on the vertical coordinate $z = \phi(x)$ only, i.e.

$$\vec{G}(x) = -(0, 0, \gamma)$$

where $\gamma = \gamma(z) = \gamma(\phi(x))$ is a given scalar ”gravity” functional.
• taking the normal component of such an external force we obtain expression for the driving term $\mathcal{F} = \vec{G}.\vec{N}$

\[
\mathcal{F} = -\frac{\gamma(\phi(x))}{\left((1 + |\nabla\phi|^2)(1 + (\nabla\phi.\vec{T})^2)\right)^{\frac{1}{2}}} \nabla\phi.\vec{N}
\]

• $\vec{N} \subset T_x(M)$ is the unit inward normal vector to the surface curve $G_t$ relative to the surface $M$

• $\vec{N}, \vec{T}$ are unit inward normal and tangent vectors to the projected planar curve $\Gamma_t$. 
• the flow of surface curves $\mathcal{G}_t \subset \mathcal{M}$ fulfills $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ iff the normal velocity $v$ of the flow of planar curves $\Gamma_t$, $t \geq 0$, satisfies the geometric equation

$$v = \beta(x, k, \nu) \equiv a(x, \nu) k - b(x, \nu) \nabla x \phi(x).\vec{N}$$

where $\vec{T} = (\cos(\nu), \sin(\nu))$, $\vec{N} = \vec{T}^\perp$, and $a, b : \Omega \subset \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$

$$a(x, \nu) = \frac{1}{1 + (\nabla \phi.\vec{T})^2}$$

$$b(x, \nu) = \frac{1}{1 + |\nabla \phi|^2} \left( \gamma - \frac{\vec{T}^T \nabla^2 \phi \vec{T}}{1 + (\nabla \phi.\vec{T})^2} \right)$$
In image segmentation, detection of image silhouettes plays an important role. An image is represented by a given intensity function $u_0 : \mathbb{R}^2 \rightarrow [0, 1]$.

The problem is to detect edges of the image, i.e. planar curves on which the gradient $\nabla u_0$ is very large.
The idea is to construct an evolving family of planar curves converging to an edge of the image according to the normal velocity

$$\beta(x, k, \nu) = \varepsilon \phi(x) k - \nabla \phi(x) \cdot \vec{N}$$

where $$\phi(x) = h(|\nabla u_0(x)|)$$, $$h$$ is a suitable image contrast function, e.g. $$h(s) = e^{-s}$$

(Caselles et al 1997; Kichenassamy et al. 1996)
The image intensity function $u_0$ (top left) and its density plot (bottom left). 3D plot of Casseles’ functional $\phi$ (bottom-right) and the corresponding vector field $-\nabla \phi(x)$ (top-right)
Governing equations and tangential velocity functional

- An embedded regular plane curve $\Gamma$ can be parameterized by a smooth function $x : S^1 \rightarrow \mathbb{R}^2$, i.e. $\Gamma = \text{Image}(x) := \{x(u), u \in S^1\}$

- We represent the flow of plane curves by a solution $x$ to the geometric equation

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}$$

for the position vector $x \in \mathbb{R}^2$ representing a curve $\Gamma = \text{Image}(x)$.

- The curvature $k$, tangential angle $\nu$ and position vector $x$ satisfies

$$
\begin{align*}
g &= |\partial_u x| \\
\vec{T} &= (\cos \nu, \sin \nu) = \partial_s x = g^{-1} \partial_u x \\
k &= \partial_s x \wedge \partial_s^2 x = g^{-3} \partial_u x \wedge \partial_u^2 x
\end{align*}
$$

Daniel Ševčovič

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Straightforward calculations using Frenet’s formulae yield the fully nonlinear system of governing parabolic PDEs

\[
\begin{align*}
\partial_t k &= \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta \\
\partial_t \nu &= \beta'_k \partial_s^2 \nu + (\alpha + \beta'_\nu) \partial_s \nu + \nabla_x \beta \cdot \vec{T} \\
\partial_t g &= -g k \beta + \partial_u \alpha \\
\partial_t x &= \beta \vec{N} + \alpha \vec{T}
\end{align*}
\]

where \( \beta = \beta(x, k, \nu) \) and \( \alpha \) are the normal and tangential velocities, \( \vec{T} = (\cos(\nu), \sin(\nu)) \), \( \vec{T} \perp \vec{N} \),

\[ g = g(u, t) = |\partial_u x(u, t)| \] is the local length element,

\[ ds = g \, du \] is the arc-length parameterization, \((u, t) \in Q_T = S^1 \times [0, T)\).
A solution is subject to initial and periodic boundary conditions corresponding to an initial curve and is searched in the functional space

\[
E_k = c^{2k+\nu} S^1 \times c^{2k+\nu} S^1 \times c^{1+\nu} S^1 \times (c^{2k+\nu} S^1)^2
\]

where \( k = 0, \frac{1}{2}, 1 \), and \( c^{2k+\nu} = c^{2k+\nu}(S^1) \) is the "little" Hölder space, i.e. the closure of \( C^\infty(S^1) \) in the topology of the Hölder space \( C^{2k+\nu}(S^1) \)
Theorem. Assume that $\Phi_0 = (k_0, \nu_0, g_0, x_0) \in E_1$ where $k_0$ is the curvature, $\nu_0$ is the tangential vector and $g_0 = |\partial u x_0| > 0$ is the local length element of the initial regular curve $\Gamma_0 = \text{Image } (x_0)$. If $\beta = \beta(x, k, \nu)$ is a $C^4$ smooth function such that

$$\min_{\Gamma_0} \beta'_k(x_0, k_0, \nu_0) > 0$$

and $\alpha$ is an admissible tangential velocity functional. Then there exists a unique classical solution $\Phi = (k, \nu, g, x) \in C([0, T], E_1) \cap C^1([0, T], E_0)$ of the governing system of equations defined on some small time interval $[0, T]$, $T > 0$. If $\Phi$ is a maximal solution defined on $[0, T_{\text{max}})$ then either $T_{\text{max}} = +\infty$ or $\liminf_{t \to T_{\text{max}}^-} \min_{\Gamma_t} \beta'_k(x, k, \nu) = 0$ or $T_{\text{max}} < +\infty$ and $\max_{\Gamma_t} |k| \to \infty$ as $t \to T_{\text{max}}$.

- Consequence of the abstract theory of fully nonlinear parabolic equations due to Angenent (1990) and Lunardi.
Review of numerical aspects of tangential velocity functional

- usual choice of the tangential velocity $\alpha = 0$ fails and may lead to serious numerical instabilities

Merging of numerically computed grid points in the case $\alpha = 0$ (left). Impact of a suitable tangential velocity functional $\alpha$ defined as:

$$\partial_s \alpha = k \beta - \langle k \beta \rangle_{\Gamma}$$

... on enhancement of the spatial grids redistribution (right). (Hou et al in ’94, K. Mikula and D. Ševčovič in 1999, 2001).

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Qualitative behavior of solutions

- **General function** $\beta(x, k, \nu)$

$$
\frac{d}{dt} L(\Gamma^t) + \int_{\Gamma^t} k\beta ds = 0, \quad \frac{d}{dt} A(\Gamma^t) + \int_{\Gamma^t} \beta ds = 0
$$

where $L(\Gamma)$ and $A(\Gamma)$ denote the length and enclosed area of $\Gamma$

- **Casseles’ functional in the image segmentation**

$$
\beta(x, k, \nu) = a(\phi)k - b(\phi)\nabla_x \phi(x).\vec{N}
$$

$$
\frac{d}{dt} \int_{\Gamma^t} H(\phi) ds + \int_{\Gamma^t} \frac{H(\phi)}{a(\phi)} \beta^2 ds = 0
$$

where $H = H(\phi)$ is a solution to: $H' = \frac{b}{a} H$. 
• General first integral for the flow driven by the geodesic curvature on a surface

\[ \frac{d}{dt} \mathcal{L}_t + \int_{\mathcal{G}_t} \mathcal{K}_g \mathcal{V} \, dS = 0 \]

It is a Lyapunov functional if \( F = 0 \). Then \( \mathcal{V} \mathcal{K}_g = \mathcal{K}_g^2 \)

• Flow driven by the geodesic curvature \( \mathcal{V} = \mathcal{K}_g + F \) on a surface \( \mathcal{M} = \text{Graph}(\phi) \). Then vertically projected planar curves have the normal velocity \( \beta(x, k, \nu) = a(x, \nu) k - b(x, \nu) \nabla_x \phi(x). \vec{N} \) and

\[ \frac{d}{dt} \int_{\mathcal{G}_t} H(\phi(x)) \, dS + \int_{\mathcal{G}_t} H(\phi(x)) \mathcal{V}^2 \, dS = 0 \]

where \( H' = \gamma H \) (\( \gamma \) is the vertical component of the gravitational like external force \( \vec{G} = -(0, 0, \gamma) \))
Goal is to analyze stationary surface curves with respect to the normal velocity $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$, i.e. surface curves satisfying $\mathcal{K}_g + \mathcal{F} = 0$.

Motivation

- Construction of closed geodesic curves ($\mathcal{K}_g = 0$)
- Analysis of stability of resolved edges in the image

$\mathcal{V} = 0$ on $\mathcal{M}$ iff $\mathcal{v} \equiv \beta(x, k, \nu) = 0$ in the plane

**Definition** A closed $C^2$ smooth planar curve $\overline{\Gamma} = \text{Image}(\overline{x})$ is called a stationary curve with respect to the normal velocity $\beta$ iff $\beta(\overline{x}, \overline{k}, \overline{\nu}) = 0$ on $\overline{\Gamma}$ where $\overline{x}, \overline{k}$ and $\overline{\nu}$ are the position vector, curvature and tangential angle of the curve $\overline{\Gamma}$. 

Daniel Ševčovič
Principle of linearized stability

• Tangential velocity functional in the system of governing equations has no impact on the shape of evolving curves. For $\alpha = 0$ the governing system of equations reduces to:

\[
\begin{align*}
\partial_t k &= g^{-1} \partial_u (g^{-1} \partial_u \beta) + k^2 \beta \\
\partial_t \nu &= g^{-1} \partial_u \beta \\
\partial_t g &= -g k \beta \\
\partial_t x &= \beta \tilde{N}
\end{align*}
\]

• In order to analyze stability of $\tilde{\Gamma}$ we have to investigate the behavior of infinitesimal variations of $k, \nu, g$ and $x$
\(
\delta k, \delta \nu, \delta g \text{ and } \delta x \text{ satisfy the linearized system}
\)

\[
\begin{align*}
\partial_t \delta k &= \bar{g}^{-1} \partial_u (\bar{g}^{-1} \partial_u \delta \beta) + \bar{k}^2 \delta \beta \\
\partial_t \delta \nu &= \bar{g}^{-1} \partial_u \delta \beta \\
\partial_t \delta g &= -\bar{g} \bar{k} \delta \beta \\
\partial_t \delta x &= \delta \beta \bar{N}
\end{align*}
\]

where

\[
\delta \beta = \beta(\bar{x} + \delta x, \bar{k} + \delta k, \bar{\nu} + \delta \nu) - \beta(\bar{x}, \bar{k}, \bar{\nu}) + \text{h.o.t.} = \nabla_x \bar{\beta} \cdot \delta x + \bar{\beta}'_k \delta k + \bar{\beta}'_\nu \delta \nu
\]
The total variation $\delta = \delta \beta$ satisfies the scalar parabolic equation

$$\partial_t \delta = P \partial_u^2 \delta + R \partial_u \delta + Q \delta$$

subject to periodic boundary conditions at $u = 0, 1$ where

$$P = \bar{g}^{-2} \bar{\beta}'_k, \quad R = \bar{g}^{-1} \bar{\beta}'_\nu + \bar{g}^{-1} \bar{\beta}'_k \partial_u \bar{g}^{-1}, \quad Q = \bar{\beta}'_k \bar{\kappa}^2 + \nabla_x \bar{\beta} \cdot \bar{N}$$

Functions $P, Q$ and $R$ are 1-periodic in $u$ and depend on the $\bar{\Gamma}$ only.

**Definition.** A stationary curve $\bar{\Gamma} = \text{Image}(\bar{x})$ is called linearly stable if the zero solution is exponentially asymptotically stable in the space $L^2(S^1)$, i.e. there exist constants $M, \omega > 0$ such that $\| \delta(., t) \|_{L^2(S^1)} \leq Me^{-\omega t} \| \delta(., 0) \|_{L^2(S^1)}$ for any initial condition $\delta(., 0) \in L^2(S^1)$. 
Lemma. Suppose \( P, R, Q \in C^1(S^1), P > 0 \). If \( \int_0^1 \frac{R(u)}{P(u)} \, du = 0 \) then the linear operator \( A : D(A) \subset L^2(S^1, w) \to L^2(S^1, w), D(A) = W^{2,2}(S^1) \), is selfadjoint operator in the weighted Lebesgue space \( L^2(S^1, w) \) with the weight defined as: 
\[
w(u) = P(u)^{-1} \exp(\int_0^u \frac{R(v)}{P(v)} \, dv).
\]

• Notice \( \int_0^1 \frac{R(u)}{P(u)} \, du = 0 \) if and only if \( \int_{\bar{\Gamma}} \frac{\beta'_\nu}{\beta'_k} \, ds = 0 \)

Proposition. Let \( \beta(x, k, \nu) = a(x, \nu)k - b(x, \nu)\nabla \phi.\vec{N} \) where \( a, b \) correspond to projected flow of surface curves and \( \phi(x) \) is \( C^2 \) smooth function. Then
\[
\int_{\bar{\Gamma}} \frac{\beta'_\nu}{\beta'_k} \, ds = 0
\]
for any closed stationary curve \( \bar{\Gamma} = \text{Image}(\bar{x}) \) where \( \bar{\beta} = \beta(\bar{x}, \bar{k}, \bar{\nu}) \).
Theorem. Suppose that \( \bar{\Gamma} \) is a stationary curve with respect to the normal velocity corresponding to the projected flow of surface curves, i.e. \( \bar{\Gamma} \) is a vertical projection of a stationary surface curve \( \mathcal{G} \). Then

- \( \bar{\Gamma} \) is linearly stable if \( \sup_{\bar{\Gamma}} Q < 0 \);
- \( \bar{\Gamma} \) is unstable if \( \int_0^1 Qw \, du > 0 \)

where \( Q = \beta'_k k^2 + \nabla_x \beta.\vec{N} \) and \( w \) is the weight in \( L_w^2(S^1) \)

- Proof follows from selfadjoint property of the parabolic equation for the variation \( \delta \beta \) and analysis of the Rayleigh quotient.
Geodesic flow on Casseles’ functional surface drives evolved curves to the boundary of the image (left). Time evolution of the quantity $Q = \bar{\beta}'_k \bar{k}^2 + \nabla_x \bar{\beta} \cdot \vec{N}$ (right). It eventually becomes negative when $t \to \infty$
Flow on a surface driven by the geodesic curvature

\[ V = K_g \]

- Left: Evolved curve passes through the hill and then selfsimilarly shrinks to a point in finite time.

- Right: Evolved curve passes through both hills and then selfsimilarly shrinks to a point in finite time.
Evolved curve tries to pass through equally high humps. They constitute an obstacle for the evolution. The curve approaches closed geodesic curve on the surface in infinite time.
$V = K_g + F$

A surface flow on a wave-like surface driven by the geodesic curvature and strong external force $F$. Surface curves converge to the stable stationary circular curve with smallest radius (left) and second smallest radius (right).
Intensity function $I_0$ (left) and the Casseles’ functional surface,

$$\phi(x) = h(|\nabla I_0(x)|) \quad \text{(right)}$$
Gravitational-like external force drives the evolved curve towards a narrow valley. Geodesic curvature smoothes evolution in the infinite life-span.
Geodesic flow on Casseles’ functional surface drives evolved curves to the boundary of the image.
Conclusions

- The flow of curves on a given surface can be reduced to a planar flow with the normal velocity depending on the curvature, position and orientation.

- The geometric problem can be transformed to a fully nonlinear parabolic system of equations for the curvature, position, orientation and local length. Local in time existence of smooth solutions.

- Impact of a nontrivial tangential velocity functional on grid points redistribution has been emphasized.

- Various first integrals decreasing along trajectories have been constructed and analyzed. Closed stationary curves have been identified. Criterion for their linearized stability has been derived.

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