# Risk approach in pension planning

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# Three pillars



| 1st | 2nd | 3rd |
|---|---|---|
| PAYG | Private pension funds | Suppl. pension savings |
| mandatory | mandatory | voluntary |

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# Three pillars



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# Three pillars



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2nd pillar (9%) → Growth / Balanced / Conservative fund


# Three pillars



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|---|---|---|
| PAYG | Private pension funds | Suppl. pension savings |
| mandatory | mandatory | voluntary |
| 9% | 9% | |

| | Growth | Balanced | Conservative fund |
|---|---|---|---|
| stocks | $\leq 80\%$ | $\leq 50\%$ | $0\%$ |
| bonds | $\geq 20\%$ | $\geq 50\%$ | $100\%$ |

# Problem definition



Want to achieve a particular sum?
How to reach this?

Minimalize risk, insecureness,
associated with
decisions,
aiming the target terminal wealth...


# Problem definition



Future pensioner with the expected retirement time $T$

deposits once a year $\tau > 0$
of his yearly salary $w_t$ (with growth rate $\varrho_t$)
to pension funds $j \in \{1, 2, ..., J\}$ with returns $r_t^j$.

**Accumulated sum at time $t$ :** $A_t$


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**More important:** $y_t = A_t / w_t$

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**Accumulated sum at time $t$ :** $A_t$

**More important:** $y_t = A_t / w_t$

Denote $s_t^j = \frac{1+r_t^j}{1+\rho_t}$, $\mathbf{s}_t = [s_t^1, s_t^2, s_t^3]^\top$, $\mathbf{y}_t = [y_t^1, y_t^2, y_t^3]^\top$

$$\begin{aligned}
&\mathbf{y}_0^\top \mathbf{1} = \tau \\
&\mathbf{y}_t^\top \mathbf{1} = \mathbf{y}_{t-1}^\top \mathbf{s}_t + \tau\,, \qquad t \in \{1, ..., T-1\}, \\
&\mathbf{y}_T^\top \mathbf{1} = \mathbf{y}_{T-1}^\top \mathbf{s}_T \\
&\mathbf{y}_t \geq 0 \qquad t \in \{1, ..., T\}\,.
\end{aligned}$$


# Tree representation



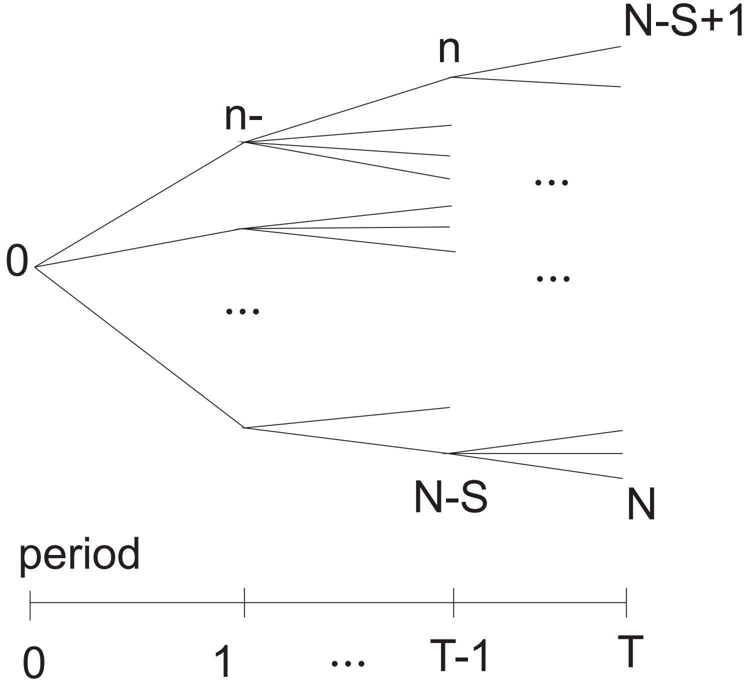


Figure 1: Tree representation

set of all nodes
$\mathcal{N} = \{0, ..., N\}$

set of nonterminal nodes without root
$\mathcal{N}_0 = \{1, ..., N-S\}$

set of terminal nodes
$\mathcal{T} = \{N-S+1, ...., N\}$

stages
$\xi(n) \in \{0, ..., T\}$

predecessor $n_-$
set of succ. $\{n\}^+$

values in node $n$: $s_n^1, s_n^2, s_n^3$


# Risk measures



How to measure the insecureness of the value of savings?


# Risk measures



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**Static risk measures**

insecureness
of the terminal
wealth

$y_T$

# Risk measures



How to measure the insecureness of the value of savings?

**Static risk measures**

insecureness
of the terminal
wealth

$y_T$

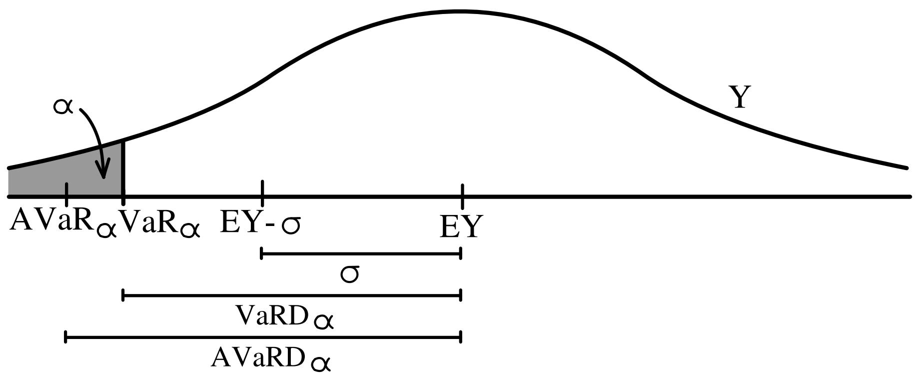
**Dynamic risk measures**

insecureness
of all intermediate
wealths

$y_1, \ldots, y_T$

# Risk measures






value-at-risk $VaR_\alpha$ : Prob$(Y \geq VaR_\alpha) \geq 1-\alpha$

average $VaR$ : $AVaR_\alpha(Y) = \mathbb{E}(Y|Y \leq VaR_\alpha)$.

EXAMPLES of **RISK MEASURES**:
- variance $\sigma^2$, standard deviation $Stdev(Y)$
- value-at-risk dev. $VaRD_\alpha(Y) = \mathbb{E}(Y) - VaR_\alpha(Y)$
- average VaR dev. $AVaRD_\alpha(Y) = \mathbb{E}(Y) - AVaR_\alpha(Y)$
- mean absolute dev., lower semi-variance, ...

Goal: $\max VaR_\alpha, \max AVaR_\alpha, \min AVaRD_\alpha, ...$






## Terminal risk

$$\min AVaRD_\alpha(\mathbf{y}_T^\top \mathbf{1})$$

$$\begin{aligned} &\mathbf{y}_0^\top \mathbf{1} = \tau \\ &\mathbf{y}_t^\top \mathbf{1} = \mathbf{y}_{t-1}^\top \mathbf{s}_t + \tau\,, t \in \{1, ..., T-1\}, \\ &\mathbf{y}_T^\top \mathbf{1} = \mathbf{y}_{T-1}^\top \mathbf{s}_T \\ &\mathbf{y}_t \geq 0 \in \{1, ..., T\} \end{aligned}$$

$$\mathbb{E}(\mathbf{y}_T^\top \mathbf{1}) \geq \mu\,.$$

$$\min AVaRD_\alpha(\mathbf{y}_T^\top \mathbf{1})$$

$$\begin{aligned} &\mathbf{y}_0^\top \mathbf{1} = \tau \\ &\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau\,, \qquad n \in \mathcal{N}_0, \\ &\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n \qquad n \in \mathcal{T}, \\ &\mathbf{y}_n \geq 0 \qquad n \in \mathcal{N} \end{aligned}$$

$$\sum_{m \in \mathcal{T}} p_m (\mathbf{y}_m^\top \mathbf{1}) \geq \mu\,.$$

Rockafellar & Uryasev:

$AVaR_\alpha(X) = \max_{a \in \mathbb{R}} \{a - \frac{1}{\alpha}\mathbb{E}[X-a]^-\}$, $[g]^- = \max\{-g, 0\}$


# Terminal risk



## Terminal risk

$$\min AVaRD_\alpha(\mathbf{y}_T^\top \mathbf{1})$$

$$\mathbf{y}_0^\top \mathbf{1} = \tau$$
$$\mathbf{y}_t^\top \mathbf{1} = \mathbf{y}^\top \ldots \mathbf{s}_t + \tau, \; t \in \{1, \ldots, T-1\}$$
$$\mathbf{y}_T^\top \mathbf{1} \ldots$$
$$\mathbf{y}_t \geq \ldots$$
$$\mathbb{E}(\mathbf{y} \ldots$$

$$\min AVaRD_\alpha(\mathbf{y}_T^\top \mathbf{1})$$

$$\mathbf{y}_0^\top \mathbf{1} = \tau$$
$$\mathbf{y}^\top \mathbf{1} = \mathbf{y}^\top \ldots \mathbf{s} \ldots + \tau \quad n \in \mathcal{N}_0,$$

$$\min_{y,a,z} \left( \sum_{m\in\mathcal{T}} p_m(\mathbf{y}^\top \mathbf{1}) - a + \frac{1}{\alpha} \sum_{m\in\mathcal{T}} p_m z_{m-N+S} \right)$$

$$-a + \mathbf{y}^\top \mathbf{1} + z_{m-N+S} \geq 0, m \in \mathcal{T}$$
$$z_{m-N+S} \geq 0, m \in \mathcal{T}$$

$$\mathbf{y}_0^\top \mathbf{1} = \tau$$
$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau\,, \qquad n \in \mathcal{N}_0,$$
$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n \qquad n \in \mathcal{T},$$
$$\mathbf{y}_n \geq 0 \qquad n \in \mathcal{N}$$

$$\sum_{m\in\mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) \geq \mu\,.$$

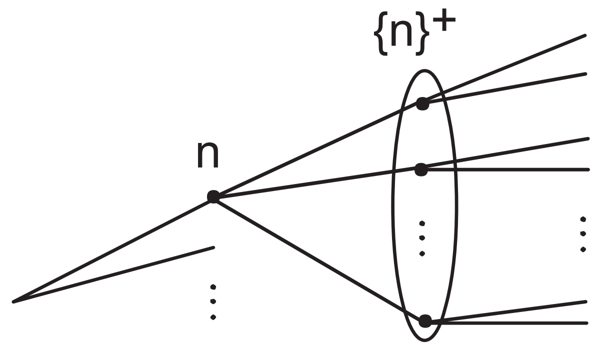
Rock…

$AVa\ldots \}$


# Intermediate risk



## Intermediate risk




$$\min \sum_{n\in\mathcal{N}\setminus\mathcal{T}} AVaRD_\alpha(\mathbf{y}_{\{n+\}})$$

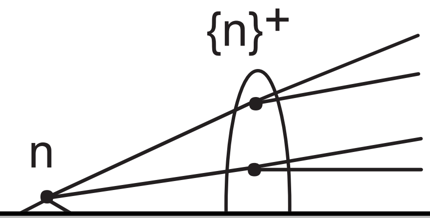
$$\mathbf{y}_0^\top \mathbf{1} = \tau$$
$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau\,, \qquad n\in\mathcal{N}_0,$$
$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n \qquad n\in\mathcal{T},$$
$$\mathbf{y}_n \geq 0 \qquad n\in\mathcal{N}$$

$$\sum_{m\in\mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) \geq \mu\,.$$






## Intermediate risk




$$\min_{a,z,y} \sum_{n\in\mathcal{N}\setminus\mathcal{T}} \left( \sum_{k\in\{n\}^+} (pc(k)\mathbf{y}_k^\top \mathbf{1}) - a_n + \frac{1}{\alpha} \sum_{k\in\{n\}^+} pc(k) z_{kn} \right)$$

$$-a_n + \mathbf{y}_k^\top \mathbf{1} + z_{kn} \geq 0, \qquad k \in \{n\}^+, n \in \mathcal{N}\setminus\mathcal{T}$$

$$z_{kn} \geq 0, \qquad k \in \{n\}^+, n \in \mathcal{N}\setminus\mathcal{T}$$

$$\mathbf{y}_0^\top \mathbf{1} = \tau$$

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau\,, \qquad n \in \mathcal{N}_0,$$

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n \qquad n \in \mathcal{T},$$

$$\mathbf{y}_n \geq 0 \qquad n \in \mathcal{N}$$

$$\sum_{m\in\mathcal{T}} p_m(\mathbf{y}_m^\top \mathbf{1}) \geq \mu\,.$$

conditional prob.

$$pc(k) = \frac{p(k)}{\sum_{l\in\{n\}^+} p(l)}$$

for all $k \in \{n\}^+, n \in \mathcal{N}\setminus\mathcal{T}$




# Numerical implementation

$$
\begin{aligned}
&\min_{\mathbf{x}=(a,z,y)} \mathbf{c}^\top \mathbf{x}\\
&\mathbf{A}_{ineq}\mathbf{x} \le \mathbf{b}_{ineq}\\
&\mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq}\\
&\mathbf{y} \ge 0, z \ge 0\,.
\end{aligned}
$$

| | Terminal risk | Intermed. risk |
|---|---|---|
| vars | $1+S+J(1+N)$ | $1+N-S+N+J(1+N)$ |
| $\mathbf{A}_{ineq}$ | $(1+S)\times vars$ | $(1+N)\times vars$ |
| #nonzero | $(2J+2)S$ | $JS+(J+2)N$ |
| $\mathbf{A}_{eq}$ | $(1+N)\times vars$ | $(1+N)\times vars$ |
| #nonzero | $(1+2N)J$ | $(1+2N)J$ |

binary tree, 40 time stages $\longrightarrow N \sim 10^{12}, S \sim 5*10^{11}$

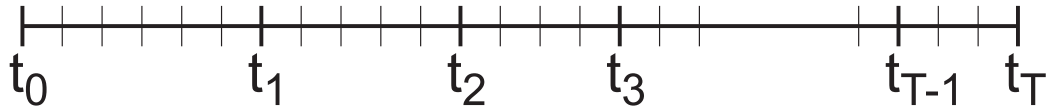
20 time stages $\longrightarrow N \sim 10^{6}, S \sim 5*10^{5}$

ternary tree, 20 time stages $\longrightarrow N \sim 10^{9}, S \sim 10^{9}$

# Numerical implementation



possible periods $\neq$ tree periods




$$l_k = t_k - t_{k-1}$$

reg. contrib. $\tau$ is transferred $l_k$-times during $[t_{k-1}, t_k]$

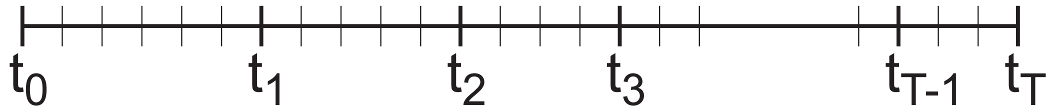
and

redistributed to funds $1, ..., J$ according to weights in time $t_{k-1}$

# Numerical implementation



possible periods $\neq$ tree periods

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and
redistributed to funds $1, ..., J$ according to weights in time $t_{k-1}$

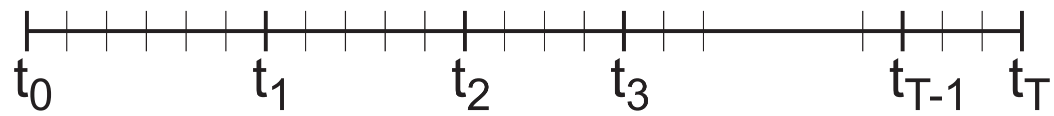
$\tau_n := [\tau_n^1, ..., \tau_n^J]^\top$ for $n \rightarrow \{n\}^+$, period $[t_{\xi(n)}, t_{\xi(n)+1}]$
$(\tau_n^\top \mathbf{1} = \tau)$

$$\frac{\tau_n^j}{\tau} = \frac{y_n^j}{\mathbf{y}_n^\top \mathbf{1}}$$

# Numerical implementation



possible periods $\neq$ tree periods

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$(\tau_n^\top \mathbf{1} = \tau)$

$$\frac{\tau_n^j}{\tau} = \frac{y_n^j}{\mathbf{y}_n^\top \mathbf{1}}$$

$+$ nonlinear constraint:

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau_{n-}^\top \sum_{i=0}^{l_{\xi(n)}-1} (\mathbf{s}_n)^{i/l_{\xi(n)}} \qquad \text{for all } n \in \mathcal{N}_0$$

# Numerical implementation



linear program $\longrightarrow$ Matlab built-in function *linprog*
+ the *sparse* function

Avoid the nonlinear constraint: an iterative algorithm

1. fix the starting point $\tau_n^j = \tau/J$ for all $n, j$.
2. solve linprog with fixed $\tau_n^j$
3. obtain optimal $y_n^j$ for all $n, j$.
4. calculate new $\tau_n^j$ (see previous slide)
5. repeat until accuracy is met

Stopping criterion: $\epsilon = |\mathbf{c}^\top \mathbf{x} - \mathbf{c}^\top \tilde{\mathbf{x}}| \leq 0.001$



# Numerical implementation - data



| Fund type | Stocks | Bonds and money market instruments |
|---|---|---|
| Growth Fund (1) | up to 80% | at least 20% |
| Balanced Fund (2) | up to 50% | at least 50% |
| Conservative Fund (3) | no stocks | 100% |

Table 1: Limits for investment for the pension funds in Slovak Republic.

| | Return | StDev |
|---|---|---|
| S&P | 0.1028 | 0.1690 |
| bonds | 0.0516 | 0.0082 |

Table 2: Historical return and its standard deviation for the S&P index and $10$-years government bonds (Jan 1996 - June 2002).


# Numerical implementation - data



| Period | 2006-08 | 2009-14 | 2015-21 | 2022-24 | 2025-50 |
|---|---|---|---|---|---|
| wage growth $(1+\varrho_t)$ | 1.075 | 1.070 | 1.065 | 1.060 | 1.050 |

Table 3: The expected wage growth in Slovak Republic. Source: Slovak Savings Bank (SLSP).

- reg. contrib. $\tau = 9\%$
- $6$ decision (tree) periods with lengths $[l_1, ..., l_6] = [10, 8, 7, 4, 4, 7]$, regarding the law restrictions on fund choice
- additional constraint $y_n^1 = 0$ for all $n$ in stages $4, 5$
- the last period is ommitted from the optimization (here $y_n^1 = y_n^2 = 0$) $\Rightarrow T = 5$
- $\mu = 4, 4.5, 5$ (goal after $33$ years)
- $\alpha = 0.05$

# Numerical implementation - scenarios



## Scenario tree generation:

node $n$ : $[s_n^1, s_n^2, s_n^3]$ appreciators for $[t_{\xi(n)-1}, t_{\xi(n)}]$ from $n_-$ to $n$

How to simulate $s_n^j$ **?**

$s_n^j = \frac{1+r_n^j}{(1+\rho_n^{avg})^{l_n}}$ where $\rho_n^{avg} =$ average in the period, known

$r_n^j =$ for the overall period

Calculate $r_n^j$ from stock and bond returns:

$$r_n^1 = 0.8 r_n^{(s)} + 0.2 r_n^{(b)}$$
$$r_n^2 = 0.5 r_n^{(s)} + 0.5 r_n^{(b)}$$
$$r_n^3 = r_n^{(b)}$$

Need to simulate $r_n^{(s)}, r_n^{(b)}$.

# Numerical implementation - scenarios



$r_n(s), r_n(b)$ independent
$\forall n$ : 3 scenarios for both $r_n(s), r_n(b)$, i.e. +,-,0

$$dS_t = \nu S_t dt + \sigma S_t dW_t$$

$S_{t+l} = S_t \exp((\nu - 0.5\sigma^2)l + \sigma(W_{t+l} - W_t))$ for interval of length $l$

$\Rightarrow 1 + r^{(s)}_{\{n\}+} = \exp((\nu - 0.5\sigma^2)l_{\xi(n)+1} + \sigma\sqrt{l_{\xi(n)+1}} Z_{\xi(n)+1})$
where $Z_{\xi(n)+1} \sim N(0,1)$, independent, for $(t_{\xi(n)}, t_{\xi(n)+1})$

3-point discretization of $N(0,1)$:
point masses $= (-\sqrt{2}, 0, \sqrt{2})$, probabilities $= (1/4, 1/2, 1/4)$

$\longrightarrow K_s = 3$ scenarios for stocks, $K_b = 3$ for bonds,
$\Rightarrow K_s * K_b = 9$ scenarios for $r^j_{\{n\}+}$, resp. $s^j_{\{n\}+}$, i.e. $9$ successors for each $n \in \mathcal{N} \setminus \mathcal{T}$. $\quad (N \sim 6 * 10^5)$

# Results



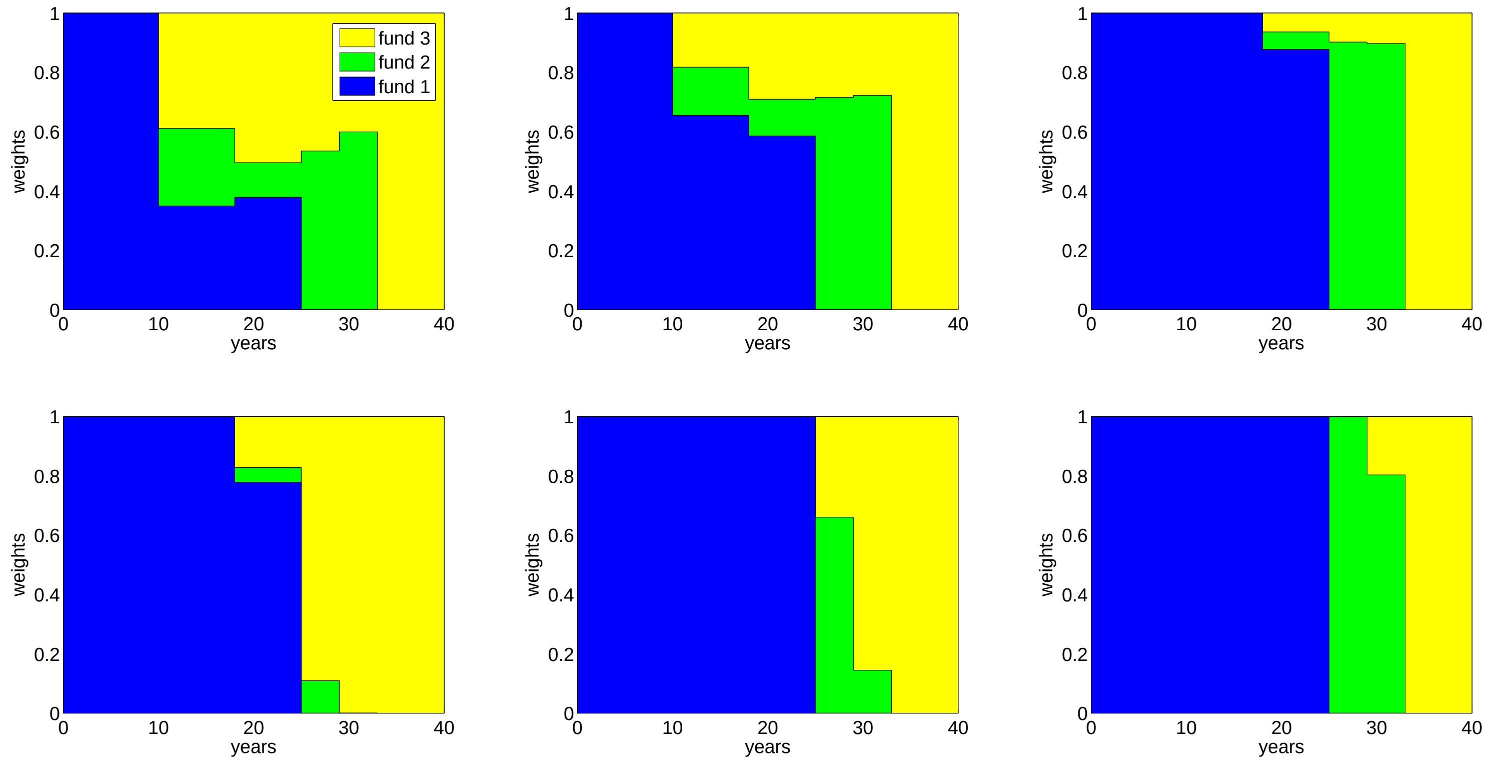
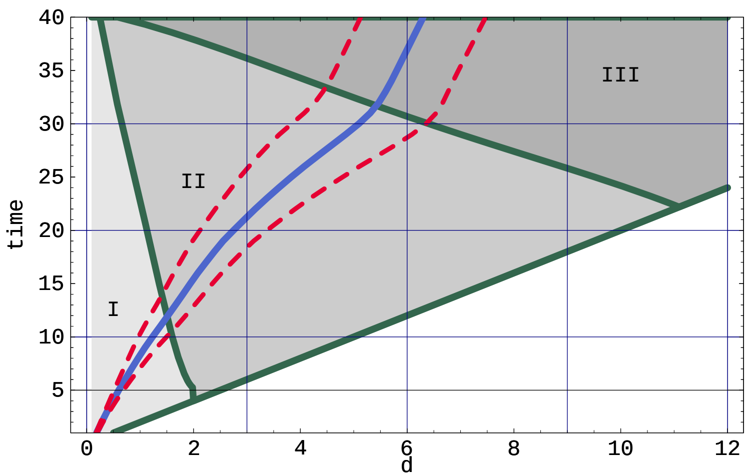



Figure 2: Averaged optimal weights of funds $1, 2, 3$ for the one-period AVaRD (top) and the multi-period AVaRD (bottom), and for the target wealth (after $33$ years) $\mu = 4$ (left), $\mu = 4.5$ (middle) and $\mu = 5$ (right).


# Comparison



utility approach




$$I \longrightarrow II \longrightarrow III$$