AVERAGED BOND PRICES IN GENERALIZED COX-INGERSOLL-ROSS MODEL OF INTEREST RATES

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Abstract

In short rate interest rate models, the behaviour of the short rate is given by a stochastic differential equation (in one-factor models) or a system of stochastic differential equations (in multi-factor models). Interest rates with different maturities are determined by bond prices, which are solutions of the parabolic partial differential equation. We consider the generalized Cox-Ingersoll-Ross model, where the short rate is a sum of two Bessel square root processes, which evolve independently. The bond price is a function of maturity and the level of each of the components of the short rate. We do not observe all values necessary to obtain a bond price. Therefore, we propose the averaging of the bond prices. We consider the limiting distribution of the short rate components, conditioned to have the sum equal to the observable short rate level. In this way, we obtain the averaged bond prices, which depend only on maturity and short rate. We prove that there is no one-factor model yielding the same bond prices as are the averaged values described above.

1. GENERALIZED COX-INGERSOLL-ROSS MODEL OF INTEREST RATES

Term structure models describe the dependance between the time to maturity of a discount bond and its present price which implies the interest rate. Continuous short rate models are formulated in terms of a stochastic differential equation, or a sysstem of them, for the instanteneous interest rate r (short rate). The bond prices, and hence the term structures of the itnerest rates, are then obtain by solving the partial differential equation.

In one-factor models, the process describing the short rate, is given by

$$dr = \alpha(t,r)dt + \beta(t,r)dw, \tag{1}$$

where $\alpha(t,r)$ and $\beta(t,r)$ are non-stochastic functions. If $\alpha(t,r) = \kappa(\theta - r)$, $\kappa > 0$, the process has the property of mean-reversion to the level θ. A popular class of models is obtained by taking

$\sigma(t,r) = \sigma r^\gamma$. It includes Vasicek model with $\gamma = 0$ (Vasicek (1977)), Cox-Ingersoll-Ross (CIR hereafter) model with $\gamma = \frac{1}{2}$ (Cox et al. (1985)); an important article on comparison of the models with different γ is Chan et al. (1992).

If the short rate evolves according to 1, then the discount bond with maturity T has the price $P(t,r)$ which depends on the time t and the current level of the short rate r. It is given by the following partial differential equation:

$$-\frac{\partial P}{\partial t} + (\alpha - \lambda\beta)\frac{\partial P}{\partial r} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial r^2} - rP = 0, \quad t \in (0,T) \tag{2}$$

$$P(T,r) = 1, \tag{3}$$

$$\tag{4}$$

where $\lambda = \lambda(t,r)$ is the market price of risk. The interest rates are then obtained from the bond prices by $R(t,r) = -\frac{\log P(t,r)}{T-t}$. (See Kwok (1998).)

For the specific choices of the market price of risk in Vasicek and CIR models, it is known that the bond price can be written in the closed form. If $\lambda(t,r) = \lambda\sqrt{r}$ in CIR model then the price of bond with time to maturity $\tau = T - t$ has the form

$$P(\tau,r) = A(\tau)e^{-B(\tau)r}.$$

The functions $A(\tau)$ and $B(\tau)$ satisfy the following system of ordinary differential equations

$$\begin{aligned} \dot{A}(\tau) &= \kappa\theta A(\tau)B(\tau) \\ \dot{B}(\tau) &= -(\kappa + \lambda\sigma)B - \frac{1}{2}\sigma^2 B(\tau)^2 + 1 \end{aligned} \tag{5}$$

with initial conditions $A(0) = 1$, $B(0) = 0$. It can be solved analytically.

There are several possibilities of generalizing one-factor models, which lead to multifactor models. They include making a parameter of 1-factor model stochastic (e.g. stochastic volatility models Anderson and Lund (1996), Fong and Vasicek (1991)), adding another relevant quantity (consol rate in Brennan and Schwartz (1982), European interest rate in Corzo and Schwartz (2000), Santamaria and Biscarri (2005)), composition of short rate from more components (generalized CIR model in Cox et al. (1985), consol rate and the spread between the short rate and consol rate in Schaefer and Schwartz (1984), Christiansen (2002)).

In generalized CIR model, the short rate r is the sum of two independent Bessel square root processes:

$$\begin{aligned} r &= r_1 + r_2, \\ dr_1 &= \kappa_1(\theta_1 - r_1)dt + \sigma_1\sqrt{r_1}dw_1, \\ dr_2 &= \kappa_2(\theta_2 - r_2)dt + \sigma_2\sqrt{r_2}dw_2, \end{aligned} \tag{6}$$

where the Wiener processes w_1 and w_2 are independent. If the market prices of risk corresponding to r_1 and r_2 are taken to be $\lambda_1\sqrt{r_1}$ and $\lambda_2\sqrt{r_2}$, then the bond price $P(\tau, r_1, r_2)$ has the form

$$\pi(\tau, r_1, r_2) = A(\tau)e^{-B_1(\tau)r_1 - B_2(\tau)r_2}, \tag{7}$$

where $A(\tau) = A_1(\tau)A_2(\tau)$ and $A_1(\tau)$, $A_2(\tau)$, $B_1(\tau)$, $B_2(\tau)$ are the solutions of the systems of ordinary differential equations 5 arising in 1-factor model, with the appropriate index.

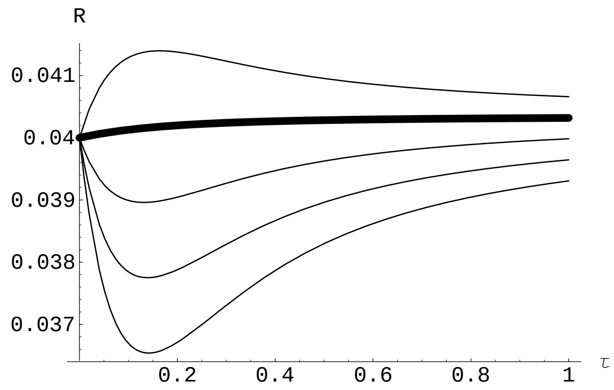

Figure 1: Examples of term structures correspondening to different pairs of r_1 and r_2 such that $r_1 + r_2 = 0.04$. The averaged term structure is in bold.

2. AVERAGING IN TWO-FACTOR MODELS

Since the components of the short rate r_1 and r_2 are not observable and the observable variable is only their sum r, the interesting questions are the properties of the averaging of the bond prices conditioned to the given sum of r_1 and r_2. This is motived by papers Fouque et al. (2003) about averaging in stochastic volatility models of stock prices and Cotton et al. (2004) about averaging in stochastic volatility models of ond prices (where the unobservable random quantity is the volatility), which are used in the series expansion of the prices. The asymptotic distribution of the hidden process is used. It can be justified if the processes have been evolving for a sufficiently long time.

In the same way, we consider the limit distributions in generalized CIR model. It is well known that the limit distribution of a Bessel square root process is a gamma distribution. Hence the limit distribution of each of r_i $(i = 1, 2)$ in (7) are given by

$$f_i(r_i) = \frac{a_i^{b_i}}{\Gamma(b_i)} e^{-a_i r_i} r_i^{b_i - 1}$$

where $a_i = \frac{2\kappa_i}{\sigma_i^2}$, $b_i = \frac{2\kappa_i\theta_i}{\sigma_i^2}$ for $r_i > 0$ and zero otherwise, and the limit density of r_1 conditioned to $r_1 + r_2 = r$ is

$$f(r_1, r) = \frac{f_1(r_1) f_2(r - r_1)}{\int_0^r f_1(s) f_2(r - s) ds} = \frac{f_1(r_1) f_2(r - r_1)}{M(r)}, \tag{8}$$

where we denoted the numerator of the fraction by $M(r)$ to simplify the notation of the following computations. The bond price (7) can be written in terms of τ, r, r_1 and the averaged value is computed as

$$P(\tau, r) = \int_0^r \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1. \tag{9}$$

In the same way, the averaged term structure is given by

$$P(\tau, r) = \int_0^r \left[-\frac{\log \pi(\tau, r_1, r - r_1)}{\tau} \right] f(r_1, r) dr_1. \tag{10}$$

In Fig. 1 we give an example. It shows the term structures obtained by the generalized CIR model and the averaged term structure computed in the way described above.

3. THE MAIN RESULT

In this paper, we study the following problem: We ask, whether there are such functions α and β that the bond prices are same as the averaged prices from 2-factor CIR model. We restrict ourselves to certain processes. Drift and volatility of the process, as well as the market price of risk are time-independent. For zero level of short rate, we require the volatility to be zero,. This consition is needed to ensure the nonnegativity of short rate. We also assume that the volatility parameters σ are different for the two processes forming the short rate in 2-factor CIR model.

Theorem 3.1 *Suppose that*

1. *functions α, β, λ depend only on r (and not on τ),*
2. *functions α, β, λ are continuous in $r = 0$,*
3. $\beta(0) = 0$,
4. $\sigma_1 \neq \sigma_2$.

Then

1. $P(\tau, r) \to A(\tau)$ *as* $r \to 0$,
2. $\frac{\partial P}{\partial \tau}(\tau, r) \to \dot{A}(\tau)$ *as* $r \to 0$,
3. $\frac{\partial P}{\partial r}(\tau, r) \to -A(\tau)\left(\frac{b_1}{b_1+b_2}B_1(\tau) + \frac{b_2}{b_1+b_2}B_2(\tau)\right)$ *as* $r \to 0$,
4. $\frac{\partial^2 P}{\partial r^2}(\tau, r)$ *is bounded on the neighbourhood of* $r = 0$.

Now, we state some properties of the Kummer cofluent hypergeometric functions ${}_1F_1$ in the following lemma, which will be used in the subsequent proof of the theorem 3.1. They can be found in Abramovitz and Stegun (1972).

Lemma 3.2 1. *The following equality holds:*

$$\int_0^r e^{-ax} x^{b-1} (r-x)^c dx = r^{b+c} \frac{\Gamma(b)\Gamma(1+c)}{\Gamma(1+b+c)} {}_1F_1(b, 1+b+c, -ar)$$

2. *The series expansion of* ${}_1F_1(a, b, z)$ *is:*

$$ {}_1F_1(a, b, z) = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} z^2 + \dots .$$

Proof of the theorem 3.1: Firstly, we write the term $M(r)$ appearing in the density $f(r_1, r)$ and the density itself in the form which will be useful later.

$$\begin{aligned} M(r) &= \int_0^r f_1(r_1) f_2(r - r_1) dr_1 = \\ &= \frac{a_1^{b_1} a_2^{b_2}}{\Gamma(b_1 + b_2)} e^{-a_2 r} r^{b_1 + b_2 - 1} {}_1F_1(b_1, b_1 + b_2, -(a_1 - a_2) r). \end{aligned}$$

Substituting into the density yields

$$\begin{aligned} f(r_1, r) &= \frac{1}{M(r)} f_1(r_1) f_2(r - r_1) = \\ &= \frac{1}{{}_1F_1(b_1, b_1 + b_2, -(a_1 - a_2) r)} \frac{\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} \frac{1}{r^{b_1 + b_2 - 1}} \left[e^{-(a_1 - a_2) r_1} r_1^{b_1 - 1} (r - r_1)^{b_2 - 1} \right]. \end{aligned} \tag{11}$$

Now, we proceed to prove the assertions of the theorem:

1. Substituting (11) into the expression for the averaged bond price gives

$$\begin{aligned} P(\tau, r) &= \int_0^r \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1 = \\ &= A e^{-Br} \frac{{}_1F_1(b_1, b_1 + b_2, -((B_1 - B_2) + (a_1 - a_2) r))}{{}_1F_1(b_1, b_1 + b_2, -(a_1 - a_2) r)}. \end{aligned} \tag{12}$$

   Since both denominator and numerator of the fraction in (12) converge to unity as $r \to 0$, we have

$$\lim_{r \to 0} P(\tau, r) = A(\tau).$$

2. We compute the derivative of P with respect to τ:

$$\begin{aligned} \frac{\partial P}{\partial \tau} &= \int_0^r \frac{\partial \pi}{\partial \tau}(\tau, r_1, r - r_1) f(r_1, r) dr_1 = \\ &= P(\tau, r) \left[\left(\frac{\dot{A}}{A} - \dot{B}_2 r \right) - (\dot{B}_1 - \dot{B}_2) \frac{\int_0^r r_1 \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1}{\int_0^r \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1} \right] \end{aligned} \tag{13}$$

   The numerator of the fraction in (13) is positive for all $r > 0$ and can be bounded from above by $r \int_0^r \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1$. Hence the fraction is positive and bounded from above by r, which implies that it converges to zero as $r \to 0$. Since we already know that $P(\tau, r) \to A(\tau)$ for $r \to 0$, we obtain from (13) that

$$\lim_{r \to} \frac{\partial P}{\partial \tau}(\tau, r) = \dot{A}(\tau).$$

3. In the computation of the derivative $\frac{\partial P}{\partial r}$

$$\frac{\partial P}{\partial r} = \int_0^r \frac{\partial \pi}{\partial r}(\tau, r_1, r - r_1) f(r_1, r) + \pi(\tau, r_1, r - r_1) \frac{\partial f}{\partial r}(r_1, r) dr_1 \tag{14}$$

there are two derivatives which need to be computed: $\frac{\partial \pi}{\partial r}$ and $\frac{\partial f}{\partial r}$. Now, we evaluate these expressions. Firstly,

$$\frac{\partial \pi}{\partial r}(\tau, r_1, r - r_1) = -B_2(\tau)\pi(\tau, r_1, r - r_1). \tag{15}$$

Secondly,

$$\begin{aligned} \frac{\partial f}{\partial r}(r_1, r) &= \frac{f_1(r_1) f_2'(r - r_1)}{M(r)} - \frac{f_1(r_1) f_2(r - r_1)}{M^2(r)} M'(r) \\ &= f(r_1, r) \left[\frac{f_2'(r - r_1)}{f_2(r - r_1)} - \frac{\int_0^r f_1(s) f_2'(r - s) ds}{\int_0^r f_1(s) f_2(r - s) ds} \right] \end{aligned} \tag{16}$$

Noting that

$$\frac{f_2'(x)}{f_2(x)} = -a_2 + (b_2 - 1)\frac{1}{x}$$

and using it in (16) gives

$$\frac{\partial f}{\partial r}(r_1, r) = f(r_1, r)(b_2 - 1) \left[\frac{1}{r - r_1} - \frac{\int_0^r \frac{1}{r-s} f_1(s) f_2(r - s) ds}{\int_0^r f_1(s) f_2(r - s) ds} \right]. \tag{17}$$

Substituting (15) and (17) into (14) yields after the rearrangement

$$\begin{aligned} \frac{\partial P}{\partial r} =& P \left[-B_2 + (b_2 - 1) \left(\frac{\int_0^r \frac{1}{r-r_1} \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1}{\int_0^r \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1} \right.\right. \\ & \left.\left. - \frac{\int_0^r \frac{1}{r-r_1} f_1(r_1) f_2(r - r_1) dr_1}{\int_0^r f_1(r_1) f_2(r - r_1) dr_1} \right) \right]. \end{aligned} \tag{18}$$

Let us denote

$$X_1 = \frac{\int_0^r \frac{1}{r-r_1} \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1}{\int_0^r \pi(\tau, r_1, r - r_1) f(r_1, r) dr_1}, \; X_2 = \frac{\int_0^r \frac{1}{r-r_1} f_1(r_1) f_2(r - r_1) dr_1}{\int_0^r f_1(r_1) f_2(r - r_1) dr_1}.$$

In this notation,

$$\frac{\partial P}{\partial r} = P(\tau, r) \left[-B_2 + (b_2 - 1)\left(X_1 - X_2 \right) \right] \tag{19}$$

We write each of the expressions X_1 and X_2 in terms of functions ${}_1F_1$:

$$X_1 = \frac{1}{r} \frac{b_1 + b_2 - 1}{b_2 - 1} \frac{{}_1F_1(b_1, b_1 + b_2 - 1, -((B_1 - B_2) + (a_1 - a_2)r))}{{}_1F_1(b_1, b_1 + b_2, -((B_1 - B_2) + (a_1 - a_2)r))} \tag{20}$$

and in a similar way

$$X_2 = \frac{1}{r} \frac{b_1 + b_2 - 1}{b_2 - 1} \frac{{}_1F_1(b_1, b_1 + b_2 - 1, -(a_1 - a_2)r)}{{}_1F_1(b_1, b_1 + b_2, -(a_1 - a_2)r)}. \tag{21}$$

Hence

$$X_1 - X_2 = \frac{\partial P}{\partial r} = \frac{1}{r}\frac{b_1 + b_2 - 1}{b_2 - 1}\left[\frac{G_1}{G_2} - \frac{G_3}{G_4}\right],$$

where we denoted

$$\begin{aligned}
G_1 &= {}_1F_1(b_1, b_1 + b_2 - 1, -((B_1 - B_2) + (a_1 - a_2))r),\\
G_2 &= {}_1F_1(b_1, b_1 + b_2, -((B_1 - B_2) + (a_1 - a_2))r),\\
G_3 &= {}_1F_1(b_1, b_1 + b_2 - 1, -(a_1 - a_2)r),\\
G_4 &= {}_1F_1(b_1, b_1 + b_2, -(a_1 - a_2)r). \qquad (22)
\end{aligned}$$

Because $G_2G_4 \to 1$ as $r \to 0$, we need to compute $G_1G_4 - G_2G_3$ to be able to compute the limit of (18) Since

$$\begin{aligned}
G_1 &= 1 - \frac{b_1}{b_1 + b_2 - 1}((B_1 - B_2) + (a_1 - a_2))r + o(r),\\
G_2 &= 1 - \frac{b_1}{b_1 + b_2}((B_1 - B_2) + (a_1 - a_2))r + o(r),\\
G_3 &= 1 - \frac{b_1}{b_1 + b_2 - 1}(a_1 - a_2)r + o(r),\\
G_4 &= 1 - \frac{b_1}{b_1 + b_2}(a_1 - a_2)r + o(r), \qquad (23)
\end{aligned}$$

we have

$$G_1G_4 - G_2G_3 = r\left(-\frac{b_1}{b_1 + b_2 - 1} + \frac{b_1}{b_1 + b_2}\right) + o(r). \qquad (24)$$

Hence

$$X_1 - X_2 = \frac{b_1 + b_2 - 1}{b_2 - 1}\frac{1}{G_2G_4}\left[(B_1 - B_2)\left(-\frac{b_1}{b_1 + b_2 - 1} + \frac{b_1}{b_1 + b_2}\right) + \frac{o(r)}{r}\right] =$$

and

$$\lim_{r\to 0} X_1 - X_2 = \frac{b_1 + b_2 - 1}{b_2 - 1}(B_1 - B_2)\left(-\frac{b_1}{b_1 + b_2 - 1} + \frac{b_1}{b_1 + b_2}\right)$$

Finally, we can compute the limit of (18)

$$\begin{aligned}
\lim_{r\to 0}\frac{\partial P}{\partial r}(\tau, r) &= \lim_{r\to 0} P(\tau, r)\left[-B_2 + (b_2 - 1)(X_1 - X_2)\right] =\\
&= A\left[-B_2 + (b_1 + b_2 - 1)(B_1 - B_2)\left(-\frac{b_1}{b_1 + b_2 - 1} + \frac{b_1}{b_1 + b_2}\right)\right] =\\
&= -A\left[\frac{b_1}{b_1 + b_2}B_1 + \frac{b_2}{b_1 + b_2}B_2\right].
\end{aligned}$$

4. We show that there is a finite limit of $\frac{\partial^2 P}{\partial r^2}(\tau, r)$ as $r \to 0$, from which the boundedness follows.

From (18) we have

$$\frac{\partial^2 P}{\partial r^2} = \frac{\partial P}{\partial r}\left[-B_2 + (b_2-1)(X_1 - X_2)\right] + P\frac{\partial\left[-B_2 + (b_2-1)(X_1-X_2)\right]}{\partial r}$$

From the definition of X_1 and X_2 and already computed limits it follows, that it suffices to show the existence of the finite limit of $\frac{\partial}{\partial r}\left(\frac{1}{r}F(r)\right)$ for $r \to 0+$, where

$$F(r) = \frac{G_1(r)}{G_2(r)} - \frac{G_3(r)}{G_4(r)}. \tag{25}$$

Assuming $F(r)$ has the series expansion $F(r) = \sum_{k=0}^{\infty} a_k r^k$., the condition $a_0 = 0$ is sufficient for boundedness of the term $\frac{\partial}{\partial r}\left(\frac{1}{r}F(r)\right)$ in the neighbourhood of $r = 0$, which holds for (25).

Theorem 3.3 *Under the hypotheses of the theorem 3.1, there is no one-factor interest rate model, for which the averaged bond prices satisfy the PDE up to the boundary* $r = 0$.

Proof: By limit $r \to 0$ in the PDE (2) we obtain, using the results from the previous theorem, that for all $\tau > 0$.

$$-\dot{A}(\tau) + \alpha(r=0)(-A(\tau))\left(\frac{b_1}{b_1+b_2}B_1(\tau) + \frac{b_2}{b_1+b_2}B_2(\tau)\right) = 0$$

From this we calculate the value of the function α for $r = 0$:

$$\alpha(r=0) = -\frac{\dot{A}(\tau)}{A(\tau)}\frac{1}{\frac{b_1B_1(\tau)}{b_1+b_2} + \frac{b_2B_2(\tau)}{b_1+b_2}} = -\frac{\dot{A}(\tau)}{A(\tau)}\frac{b_1+b_2}{b_1B_1(\tau)+b_2B_2(\tau)}.$$

It follows that

$$-\frac{\dot{A}(\tau)}{A(\tau)}\frac{b_1+b_2}{b_1B_1(\tau)+b_2B_2(\tau)} = K_1, \tag{26}$$

where K_1 is a constant (independent of τ).

Now we recall that the the function $A(\tau)$ from the 2-factor CIR model can be writen as $A(\tau) = A_1(\tau)A_2(\tau)$, where $A_1(\tau)$ and $A_2(\tau)$ are functions appearing in the original CIR model, correspondening to each of the equations for r_1 and r_2. Hence they satisfy

$$\dot{A}_i(\tau) = \kappa_i\theta_i A_i(\tau)B_i(\tau) \quad (i = 1, 2)$$

and so we get

$$\frac{\dot{A}(\tau)}{A(\tau)} = \frac{\dot{A_1}(\tau)A_2(\tau) + A_1(\tau)\dot{A}_2(\tau)}{A_1(\tau)A_2(\tau)} = \frac{\dot{A}_1(\tau)}{A_1(\tau)} + \frac{\dot{A_2}(\tau)}{A_2(\tau)} = \kappa_1\theta_1B_1(\tau) + \kappa_2\theta_2B_2(\tau).$$

So the expression in (26) is

$$K_1 = -\frac{\dot{A}(\tau)}{A(\tau)}\frac{b_1+b_2}{b_1B_1(\tau)+b_2B_2(\tau)} = -(\kappa_1\theta_1B_1(\tau) + \kappa_2\theta_2B_2(\tau))\frac{b_1+b_2}{b_1B_1(\tau)+b_2B_2(\tau)}.$$

Since $b_1 + b_2$ is constant, the important part is the following fraction, which has to be equal to some constant K:

$$\frac{\kappa_1\theta_1 B_1(\tau) + \kappa_2\theta_2 B_2(\tau)}{b_1 B_1(\tau) + b_2 B_2(\tau)} = K.$$

It implies that

$$\kappa_1\theta_1 B_1(\tau) + \kappa_2\theta_2 B_2(\tau) = K(b_1 B_1(\tau) + b_2 B_2(\tau))$$

and so

$$(\kappa_1\theta_1 - Kb_1)B_1(\tau) = (Kb_2 - \kappa_2\theta_2)B_2(\tau)$$

for each $\tau > 0$. It is possible in two ways:

1. $\kappa_1\theta_1 - Kb_1 = 0$, $Kb_2 - \kappa_2\theta_2 = 0$,
2. $B_1(\tau) = cB_2(\tau)$, where c is a constant.

Now we look at each of these possibilities:

1. The same constant K appears in both equalities. From the first one (i.e. $\kappa_1\theta_1 - Kb_1 = 0$), we get $K = \frac{\kappa_1\theta_1}{b_1}$ and by substituting the value of $b_1 = \frac{2\kappa_1\theta_1}{\sigma_1^2}$, we obtain $K = \frac{\sigma_1^2}{2}$. In the same way, from the second equality (i.e. $Kb_2 - \kappa_2\theta_2 = 0$), we obtain $K = \frac{\sigma_2^2}{2}$. But by the hypothesis, $\sigma_1^2 \neq \sigma_2^2$, which is a contradiction.

2. We recall the equation for B_1 from CIR model:

$$-\dot{B}_1(\tau) = (\kappa_1 + \lambda_1\sigma_1)B_1(\tau) + \frac{1}{2}\sigma_1^2 B_1(\tau)^2 - 1. \tag{27}$$

   From the similar equation for $B_2(\tau)$

$$-\dot{B}_2(\tau) = (\kappa_2 + \lambda_2\sigma_2)B_2(\tau) + \frac{1}{2}\sigma_2^2 B_2(\tau)^2 - 1, \tag{28}$$

   together with $B_1(\tau) = cB_2(\tau)$, we obtain another expression for B_1:

$$-\dot{B}_1(\tau) = c\left[(\kappa_2 + \lambda_2\sigma_2)B_2(\tau) + \frac{1}{2}\sigma^2 B_2(\tau)^2 - 1\right] \tag{29}$$

   The right-hand sides of (27) and (29) have to be equal:

$$c\left[(\kappa_2 + \lambda_2\sigma_2)B_2(\tau) + \frac{1}{2}\sigma^2 B_2(\tau)^2 - 1\right] = (\kappa_1 + \lambda_1\sigma_1)B_1(\tau) + \frac{1}{2}\sigma_1^2 B_1(\tau)^2 - 1$$

   for all $\tau > 0$. By continuity, the equality holds also in the limit $\tau = 0+$. From this, we get $c = 1$ and hence the functions $B_1(\tau)$ and $B_2(\tau)$ coincide. We denote this function by $B(\tau)$. By subtracting equations (27) and (28) we obtain:

$$\left[-(\kappa_1 + \lambda_1\sigma_1) + (\kappa_2 + \lambda_1\sigma_1)\right]B(\tau) + \left[-\frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right]B^2(\tau) = 0$$

   and, dividing by $B(\tau)$ (which is nonzero)

$$\left[-(\kappa_1 + \lambda_1\sigma_1) + (\kappa_2 + \lambda_1\sigma_1)\right] - \frac{1}{2}\left[\sigma_2^2 - \sigma_1^2\right]B(\tau) = 0.$$

   Since $\sigma_1 \neq \sigma_2$, it implies that $B(\tau)$ is a constant function, which is a contradiction.

Since both possibilities lead to a contradiction, the theorem is proved.

4. CONCLUSION

We considered the 2-factor Cox-Ingersoll-Ross model of interest rates and the averaged bond prices with respect to the asymptotic distribution of the processes forming the short rate, conditioned on the observable short rate level. Such averaged values are functions of the maturity and the short rate. Solutions of one factor models are the fuctions of the same variables. Hence we studied the question, whether there is a one factor model yielding the same bond prices as those obtained by averaging in the 2-factor Cox-Ingersoll-Ross model. We proved that the answer is negative. In the future, we plan to study this question also for another two-factor models.

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