Two-factor Convergence Model of Cox-Ingersoll-Ross Type

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Abstract

Corzo and Schwartz [Convergence within the European Union: Evidence from Interest Rates, *Economic Notes* **29**, 2000, pp. 243–268] proposed a short rate model for a country before adopting Euro currency, which is based on the Vasicek model. The evolution of the European short rate is given by the one-factor Vasicek model. The domestic short rate is a similar process, but the drift depends on the current level of the European rate. We study an analogous model based on the Cox-Ingersoll-Ross (CIR) model. Bond prices are solutions to a partial differential equation. We show that their easy calculation by separation of variables (possible in the one-factor Vasicek a CIR models as well as in the convergence model by Corzo and Schwartz) can be done only in the case of uncorrelated increments of Wiener processes in the stochastic differential equations for the European and the domestic rates. Therefore, we study a possibility of an analytical approximation of bond prices in the correlated case and the order of accuracy. Finally, we present some numerical examples based on Slovak interest rates before the Slovak Republic adopted the Euro currency.

Keywords: two-factor • term structure • short-rate • convergence model • Vasicek • CIR • zero-coupon • bond • approximation • order of accuracy.

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1 Introduction

Term structure models describe a functional dependence between the time to maturity of a discount bond and its present price. The relation between the price $P(t,T)$ of a zero-coupon bond at the time t with maturity at T and the interest rate $R(t,T)$ is given by

$$P(t,T) = \mathrm{e}^{-R(t,T)(T-t)}, \text{ i.e. } R(t,T) = -\frac{1}{T-t}\ln[P(t,T)],$$

cf. Section 7.1.2 in Kwok (1998). Continuous interest rate models are often formulated in terms of stochastic differential equations (SDEs) for the instantaneous interest rate (short-rate) as well as SDEs for other relevant quantities.

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In two-factor models there are two sources of uncertainty yielding different term structures for the same short-rate as they may depend on the value of the other factor. Moreover, two-factor models have a richer variety of possible shapes of term structures. Compared with one-factor models, which are governed only by one SDE, two-factor models have a richer variety of possible shapes of term structures.

There are several forms of incorporating the second factor into a model. One approach is to take a parameter from a one factor model and consider its stochastic character. In this way, we can obtain models with stochastic volatility (see, e.g., Fong and Vasicek (1991), Anderson and Lund (1996) or Fouque, Papanicolaou and Sircar (2000)), or models with stochastic limit of the short-rate (see, e.g., Balduzzi, Das and Foresi (1998)) are obtained. Another approach is to take a quantity that is assumed to influence the short rate. The models by Schaefer and Schwartz (1984), Brennan and Schwartz (1982), and Christiansen (2005) are based on the consol rate. Corzo and Schwartz (2000) and Corzo and Gómez Biscarri (2005) proposed a model of the domestic rate which is being influenced by an European interest rate.

In general, a two factor model is given by the system of SDEs

$$\begin{aligned} \mathrm{d}r &= \mu_{\mathrm{r}}(r,x,t)\mathrm{d}t + \sigma_{\mathrm{r}}(r,x,t)\mathrm{d}W_1, \\ \mathrm{d}x &= \mu_{\mathrm{x}}(r,x,t)\mathrm{d}t + \sigma_{\mathrm{x}}(r,x,t)\mathrm{d}W_2, \\ \mathrm{Cov}[\mathrm{d}W_1, \mathrm{d}W_2] &= \rho \mathrm{d}t. \end{aligned}$$

where $\rho \in [-1,1]$ is a correlation between the increments of Wiener processes. After the specification of the so-called market prices of risk λ_{r} and λ_{x} , the price of a discount bond $P(r,x,t)$ is a solution to the partial differential equation (PDE)

$$\frac{\partial P}{\partial t} + (\mu_{\mathrm{r}} - \lambda_{\mathrm{r}}\sigma_{\mathrm{r}})\frac{\partial P}{\partial r} + (\mu_{\mathrm{x}} - \lambda_{\mathrm{x}}\sigma_{\mathrm{x}})\frac{\partial P}{\partial x} + \frac{\sigma_{\mathrm{r}}^2}{2}\cdot\frac{\partial^2 P}{\partial r^2} + \frac{\sigma_{\mathrm{x}}^2}{2}\cdot\frac{\partial^2 P}{\partial x^2} + \rho\sigma_{\mathrm{r}}\sigma_{\mathrm{x}}\frac{\partial^2 P}{\partial r \partial x} - rP = 0, \quad (1)$$

with the terminal condition $P(r,x,T) = 1$.

The reader is referred to Kwok (1998) and Brigo and Mercurio (2006) for detailed discussion on interest rate modelling.

In this paper we deal with a convergence model, where the domestic short-rate converges to the European short-rate, while both rates have stochastic behaviour. We motive this model by market data. A recent example is depicted in Figure 1, where we show the Slovak (BRIBOR) and European (EONIA) overnight interest rates in the period before Slovakia adopted the Euro currency on 1st January 2009.

The paper is organised as follows: Section 2 is focused on the first convergence model proposed by Corzo and Schwartz (2000), where we provide a correct solution and derive the limit of the domestic term structure of interest rates for this model. In Section 3 we formulate a Cox-Ingersoll-Ross modification to the original model. In the next section, Section 4, we try to find the price of the discount bond for this model. Firstly, we assume that the increments of the Wiener processes for the domestic and European rates are uncorrelated (we call this "the case of a zero correlation"), then we solve the bond-pricing PDE and, eventually, state and prove some of its properties. We also show that there is no "separable" solution to the bond-pricing PDE in the nonzero correlation case.

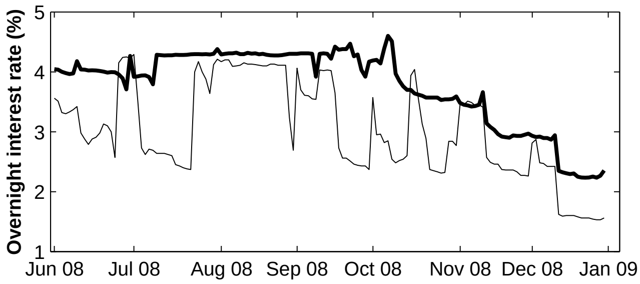

Figure 1: *The Slovak (thin line) and European (thick line) overnight interest rate before Slovakia adopted the Euro currency. Source: http://www.nbs.sk, http://www.euribor.org.*

Consequently, in Section 5, we try to approximate the solution in the case of a nonzero correlation by the solution in the case of a zero correlation in both original and modified models. An empirical example is provided in Section 6.

2 Convergence model of Vasicek type

In the pioneering paper by Corzo and Schwartz (2000) the first convergence model was formulated. The authors assumed that the domestic short-rate r_{d} and the European short-rate r_{e} were linked in the following way:

$$\begin{aligned} \mathrm{d}r_{\mathrm{d}} &= [a + b(r_{\mathrm{e}} - r_{\mathrm{d}})]\mathrm{d}t + \sigma_{\mathrm{d}}\mathrm{d}W_{\mathrm{d}}, \\ \mathrm{d}r_{\mathrm{e}} &= c(d - r_{\mathrm{e}})\mathrm{d}t + \sigma_{\mathrm{e}}\mathrm{d}W_{\mathrm{e}}, \\ \mathrm{Cov}[\mathrm{d}W_{\mathrm{d}}, \mathrm{d}W_{\mathrm{e}}] &= \rho\mathrm{d}t, \end{aligned} \tag{2}$$

where the constants b, c, σ_{d}, σ_{e} were assumed to be positive and d to be non-negative. The process for the European short-rate r_{e} is governed by the Vasicek (1977) model, and the process for the domestic rate r_{d} is a constant volatility process that converges to r_{e} with a possible minor divergence given by a. We will refer this model as *the convergence model of Vasicek type.*

The corresponding bond-pricing PDE (using the transformation $\tau = T - t$) is

$$\begin{aligned} -\frac{\partial P}{\partial \tau} + [a + b(r_{\mathrm{e}} - r_{\mathrm{d}}) - \lambda_{\mathrm{d}}\sigma_{\mathrm{d}}]\frac{\partial P}{\partial r_{\mathrm{d}}} + [c(d - r_{\mathrm{e}}) - \lambda_{\mathrm{e}}\sigma_{\mathrm{e}}]\frac{\partial P}{\partial r_{\mathrm{e}}} + & \\ \frac{1}{2}\sigma_{\mathrm{d}}^2\frac{\partial^2 P}{\partial r_{\mathrm{d}}^2} + \frac{1}{2}\sigma_{\mathrm{e}}^2\frac{\partial^2 P}{\partial r_{\mathrm{e}}^2} + \rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}\frac{\partial^2 P}{\partial r_{\mathrm{d}}\partial r_{\mathrm{e}}} - r_{\mathrm{d}}P &= 0, \\ P(r_{\mathrm{d}}, r_{\mathrm{e}}, 0) &= 1. \end{aligned} \tag{3}$$

Setting a solution of the form

$$P(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau) = \mathrm{e}^{A(\tau) - D(\tau)r_{\mathrm{d}} - U(\tau)r_{\mathrm{e}}}, \tag{4}$$

to the equation (3) and collecting the coefficients at r_{d}, r_{e} and the constant term yields

$$\begin{aligned} \dot{D} &= 1 - bD, \\ \dot{U} &= bD - cU, \\ \dot{A} &= (-a + \lambda_{\mathrm{d}}\sigma_{\mathrm{d}})D + (-cd + \lambda_{\mathrm{e}}\sigma_{\mathrm{e}})U + \frac{1}{2}\sigma_{\mathrm{d}}^2 D^2 + \frac{1}{2}\sigma_{\mathrm{e}}^2 U^2 + \rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}DU, \end{aligned} \tag{5}$$

with initial condition $A(0) = D(0) = U(0) = 0$. Using standard methods for solving ordinary differential equations (ODE) we obtain that

$$D(\tau) = \frac{1 - \mathrm{e}^{-b\tau}}{b}, \tag{6}$$

$$U(\tau) = \begin{cases} \frac{b}{c-b}\left(D(\tau) - \Xi(\tau)\right), & \text{if } b \neq c \\ \Xi(\tau) - \tau \mathrm{e}^{-c\tau}, & \text{if } b = c \end{cases}, \tag{7}$$

$$\text{where } \Xi(\tau) = \frac{1 - \mathrm{e}^{-c\tau}}{c},$$

and $A(\tau)$ can be obtained by integrating the equation (5). It follows from the fact that $\lim_{\tau\to\infty} D(\tau)/\tau = \lim_{\tau\to\infty} U(\tau)/\tau = 0$ that $\lim_{\tau\to\infty} R(r_\mathrm{d}, r_\mathrm{e}, \tau) = \lim_{\tau\to\infty} -A(\tau)/\tau$, and after long, but straightforward computations, we obtain the following

Proposition 1. *The limit of the domestic term structure of interest rates in the convergence model of Vasicek type is*

$$\lim_{\tau\to\infty} R(r_\mathrm{d}, r_\mathrm{e}, \tau) = \frac{a}{b} + d - \frac{c^2\sigma_\mathrm{d} + b^2\sigma_\mathrm{e}(2c\lambda_\mathrm{e} + \sigma_\mathrm{e}) + 2bc\sigma_\mathrm{d}(c\lambda_\mathrm{d} + \rho\sigma_\mathrm{e})}{2b^2c^2}.$$

3 Convergence model of Cox-Ingersoll-Ross type

In this section we formulate the Cox, Ingersoll, Jr. and Ross (1985) (CIR) counterpart to the model (2), i.e., the constant volatilities σ_d and σ_e are replaced by their $\sqrt{r_\mathrm{d}}$ and $\sqrt{r_\mathrm{e}}$ multiples, respectively:

$$\begin{aligned} \mathrm{d}r_\mathrm{d} &= [a + b(r_\mathrm{e} - r_\mathrm{d})]\mathrm{d}t + \sigma_\mathrm{d}\sqrt{r_\mathrm{d}}\mathrm{d}W_\mathrm{d}, \\ \mathrm{d}r_\mathrm{e} &= c(d - r_\mathrm{e})\mathrm{d}t + \sigma_\mathrm{e}\sqrt{r_\mathrm{e}}\mathrm{d}W_\mathrm{e}. \end{aligned} \tag{8}$$

In general, the correlation between $\mathrm{d}W_\mathrm{d}$ and $\mathrm{d}W_\mathrm{e}$ is equal to ρ. In this model, the European short-rate is governed by the CIR model. The meaning of the parameters is the same as in the model (2). We note that the domestic and European interest rates cannot reach negative values in the model (8), unlike in the Vasicek case. We will refer this model as *the convergence model of CIR type*. A simulation of such a process is depicted in Figure 2.

In what follows we assume that $a \geq 0$. Note that because of the square root in the stochastic differential equation for r_d, the process cannot exist for $r_\mathrm{d} < 0$. However, suppose that it attains zero. Then the volatility is zero and hence its behaviour on infinitesimal time interval is nonstochastic and determined by the drift. If $a < 0$ and the European rate r_e is sufficiently close to zero, the drift is negative and the process collapses. We will also see that the condition $a \geq 0$ ensures that the bond price is from the interval $(0, 1)$ and hence the whole term structure is positive.

4 A solution to the bond-pricing PDE

Let the European market price of risk be equal to $\nu_\mathrm{e}\sqrt{r_\mathrm{e}}$, where ν_e is a constant. Then the price of the European discount bond is given by the CIR bond-pricing

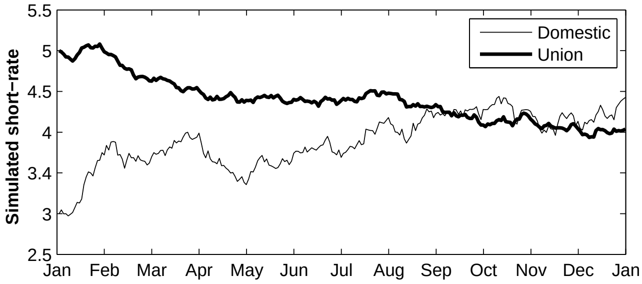

Figure 2: *A simulation of the process* (8) *with parameters* $a = 0.01$, $b = 3.67$, $c = 0.21$, $d = 0.03$, $\sigma_{\mathrm{d}} = 0.05$, $\sigma_{\mathrm{e}} = 0.02$ *a* $\rho = 0.22$. *The initial values are 3 percent for the domestic and 5 percent for the European short-rate.*

formula (see, Cox et al. (1985)). By setting the corresponding drifts and volatilities to the PDE (1), we obtain that the price of the domestic discount bond $P(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau)$ is a solution to

$$\begin{aligned} -\tfrac{\partial P}{\partial \tau} + [a + b(r_{\mathrm{e}} - r_{\mathrm{d}}) - \lambda_{\mathrm{d}}(r_{\mathrm{d}}, r_{\mathrm{e}})\sigma_{\mathrm{d}}\sqrt{r_{\mathrm{d}}}]\tfrac{\partial P}{\partial r_{\mathrm{d}}} + [c(d - r_{\mathrm{e}}) - \nu_{\mathrm{e}}\sigma_{\mathrm{e}} r_{\mathrm{e}}]\tfrac{\partial P}{\partial r_{\mathrm{e}}} \\ + \tfrac{\sigma_{\mathrm{d}}^2 r_{\mathrm{d}}}{2} \cdot \tfrac{\partial^2 P}{\partial r_{\mathrm{d}}^2} + \tfrac{\sigma_{\mathrm{e}}^2 r_{\mathrm{e}}}{2} \cdot \tfrac{\partial^2 P}{\partial r_{\mathrm{e}}^2} + \rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}\sqrt{r_{\mathrm{d}} r_{\mathrm{e}}}\tfrac{\partial^2 P}{\partial r_{\mathrm{d}}\partial r_{\mathrm{e}}} - r_{\mathrm{d}} P = 0, \end{aligned} \tag{9}$$

where $\tau = T - t$ and $\lambda_{\mathrm{d}}(r_{\mathrm{d}}, r_{\mathrm{e}})$ is the domestic market price of risk.

4.1 The case of a zero correlation

Let be the domestic market price of risk taken to be $\lambda_{\mathrm{d}}(r_{\mathrm{d}}, r_{\mathrm{e}}) = \nu_{\mathrm{d}}\sqrt{r_{\mathrm{d}}}$, where ν_{d} is a constant. Then PDE (9) yields

$$\begin{aligned} -\tfrac{\partial P}{\partial \tau} + [a + b(r_{\mathrm{e}} - r_{\mathrm{d}}) - \nu_{\mathrm{d}}(r_{\mathrm{d}}, r_{\mathrm{e}})\sigma_{\mathrm{d}} r_{\mathrm{d}}]\tfrac{\partial P}{\partial r_{\mathrm{d}}} + [c(d - r_{\mathrm{e}}) - \nu_{\mathrm{e}}\sigma_{\mathrm{e}} r_{\mathrm{e}}]\tfrac{\partial P}{\partial r_{\mathrm{e}}} \\ + \tfrac{\sigma_{\mathrm{d}}^2 r_{\mathrm{d}}}{2} \cdot \tfrac{\partial^2 P}{\partial r_{\mathrm{d}}^2} + \tfrac{\sigma_{\mathrm{e}}^2 r_{\mathrm{e}}}{2} \cdot \tfrac{\partial^2 P}{\partial r_{\mathrm{e}}^2} + \rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}\sqrt{r_{\mathrm{d}} r_{\mathrm{e}}}\tfrac{\partial^2 P}{\partial r_{\mathrm{d}}\partial r_{\mathrm{e}}} - r_{\mathrm{d}} P = 0. \end{aligned} \tag{10}$$

Let us assume that the solution to (10) has the form (4) and ρ equal to 0. By setting the solution (4) to the PDE (10) and collecting the coefficients at r_{d}, r_{e} and the constant term, we obtain that

$$\dot{D} = 1 - (b + \nu_{\mathrm{d}}\sigma_{\mathrm{d}})D - \frac{\sigma_{\mathrm{d}}^2}{2} D^2, \tag{11}$$

$$\dot{U} = bD - (c + \nu_{\mathrm{e}}\sigma_{\mathrm{e}})U - \frac{\sigma_{\mathrm{e}}^2}{2} U^2, \tag{12}$$

$$\dot{A} = -aD - cdU, \tag{13}$$

with initial conditions $A(0) = D(0) = U(0) = 0$, which follow from the initial condition $P(r_{\mathrm{d}}, r_{\mathrm{e}}, 0) = 1$. Using standard methods for solving ODEs we obtain that the solution to equation (11) is

$$D(\tau) = \frac{D_{\oplus}(1 - \mathrm{e}^{k\tau})}{1 - \frac{D_{\oplus}}{D_{\ominus}}\mathrm{e}^{k\tau}}, \tag{14}$$

where

$$k = \sqrt{(b + \nu_{\mathrm{d}}\sigma_{\mathrm{d}})^2 + 2\sigma_{\mathrm{d}}^2},$$

and

$$D_{\oplus} = -\frac{b+\nu_{\mathrm{d}}\sigma_{\mathrm{d}}+k}{\sigma_{\mathrm{d}}^2} < 0,\ D_{\ominus} = -\frac{b+\nu_{\mathrm{d}}\sigma_{\mathrm{d}}-k}{\sigma_{\mathrm{d}}^2} > 0.$$

We were not able to find an explicit solution to ODE (12); nevertheless, it is easy to solve it numerically and obtain values of $U(\tau)$. The function $A(\tau)$ can then be obtained by a numerical integration of equation (13).

In the following statements we formulate some properties of functions A, D and U.

Lemma 2. *Let $A(\tau)$, $D(\tau)$ and $U(\tau)$ be solutions to the system of ODEs* (11)–(13). *Then:* **i)** *$D(\tau) > 0$ is strictly increasing, and $\lim_{\tau\to\infty} D(\tau) = D_{\ominus}$,* **ii)** *$U(\tau) > 0$ is strictly increasing and bounded, and* **iii)** *if $a \geq 0$ then $A(\tau) < 0$; for all $\tau > 0$.*

Proof. **i)** The monotonicity of D follows directly from the derivative of solution (14) with respect to τ, which is positive:

$$\dot{D}(\tau) = \frac{-D_{\oplus}\left(1-\frac{D_{\oplus}}{D_{\ominus}}\right)k\mathrm{e}^{k\tau}}{\left(1-\frac{D_{\oplus}}{D_{\ominus}}\mathrm{e}^{k\tau}\right)^2} > 0,$$

since $k > 0$, $D_{\oplus} < 0$, and $D_{\ominus} > 0$. The fact that $D(0) = 0$ and $\dot{D}(\tau) > 0$ for τ greater than 0 implies the positivity of D. For $\tau \to \infty$ we obtain:

$$\begin{aligned}\lim_{\tau\to\infty} D(\tau) &= \lim_{\tau\to\infty}\left[\frac{D_{\oplus}}{1-\frac{D_{\oplus}}{D_{\ominus}}\mathrm{e}^{k\tau}} - \frac{D_{\oplus}\mathrm{e}^{k\tau}}{1-\frac{D_{\oplus}}{D_{\ominus}}\mathrm{e}^{k\tau}}\right]\\ &= \lim_{\tau\to\infty}\left[\frac{-D_{\oplus}}{\mathrm{e}^{-k\tau}-D_{\oplus}/D_{\ominus}}\right] = D_{\ominus}.\end{aligned}$$

ii) The initial condition $U(0) = 0$ and equation (12) imply that $\dot{U}(0) = 0$ and $\ddot{U}(0) = b > 0$. Therefore, U is positive in some neighbourhood of $\tau = 0$. To prove the positivity of U for all τ greater than 0, it is sufficient to show that $\dot{U}(\tau^*) > 0$ whenever $U(\tau^*) = 0$. This holds since if $U(\tau^*) = 0$, then, due to equation (12), we obtain $\dot{U}(\tau^*) = bD(\tau^*) > 0$. To prove that U is monotonous and increasing, we have to show that $\dot{U}$ is positive. To do this we show that if $\dot{U}(\tau^*) = 0$, then $\ddot{U}(\tau^*) = b\dot{D}(\tau^*) - (c+\nu_{\mathrm{e}}\sigma_{\mathrm{e}})\dot{U}(\tau^*) - \frac{\sigma_{\mathrm{e}}^2}{2}U(\tau^*)\dot{U}(\tau^*) = b\dot{D}(\tau^*) > 0$. To prove that U is bounded it is sufficient to show that there exists M such that if $U(\tau^*) = M > 0$, then $\dot{U}(\tau^*) \leq 0$: $\dot{U}(\tau^*) = bD(\tau^*) - (c+\nu_{\mathrm{e}}\sigma_{\mathrm{e}})M - \frac{\sigma_{\mathrm{e}}^2}{2}M^2 \leq 0$. **iii)** Since $a \geq 0$, $D(\tau) > 0$ and $U(\tau) > 0$ for all $\tau > 0$, equation (13) implies that $\dot{A}(\tau) < 0$ for all $\tau > 0$, i.e., $A(\tau)$ is strictly decreasing with an origin in 0, which proves the third part. □

The previous lemma implies

Corollary 3. *The limit of $U(\tau)$ for $\tau \to \infty$ is*

$$\widehat{U} = \lim_{\tau\to\infty} U(\tau) = \frac{(c+\sigma_{\mathrm{e}}\nu_{\mathrm{e}}) - \sqrt{(c+\sigma_{\mathrm{e}}\nu_{\mathrm{e}})^2 + 2b\sigma_{\mathrm{e}}^2 D_{\ominus}}}{-\sigma_{\mathrm{e}}^2}.$$

Proof. The boundedness and monotonicity of function U gives the existence of $\lim_{\tau\to\infty} U(\tau)$, and, consequently, $\lim_{\tau\to\infty} \dot{U}(\tau)$ (cf. equation (12)). Since U is increasing, $\lim_{\tau\to\infty} \dot{U}(\tau) \geq 0$. Suppose that $\lim_{\tau\to\infty} \dot{U}(\tau) = L > 0$. Then, from the definition of a limit, there exists K such that for all $\tau \in (K,\infty)$ we have $\dot{U}(\tau) \geq L/2$. Lagrange's mean value theorem yields that for any $s,t \in (K,\infty)$, we have $U(s)-U(t) = \dot{U}(\tau)(s-t) \geq \frac{L}{2}(s-t)$. Therefore, $U(s) \geq U(t)+\frac{L}{2}(s-t)$, which implies a contradiction for $s \to \infty$. Consequently, $\lim_{\tau\to\infty} \dot{U}(\tau) = 0$ and by denoting the limit $\lim_{\tau\to\infty} U(\tau)$ by $\widehat{U}$ we obtain that

$$\lim_{\tau\to\infty} \dot{U}(\tau) = 0 = bD_{\ominus} - (c+\sigma_{\mathrm{e}}\nu_{\mathrm{e}})\widehat{U} - \frac{\sigma_{\mathrm{e}}^2}{2}\widehat{U}^2.$$

The positive solution of the previous equation is the limit of U. □

It follows that if $a \geq 0$, then the price of the discount bond lies between 0 and 1 for all $\tau > 0$; hence, the term structures starting from the positive short-rate are always positive. Note that this is not necessarily true in two-factor models; Stehlíková and Ševčovič (2005) showed that a certain constraint on the market price of risk has to be imposed to ensure the positivity of the interest rates in the Fong-Vasicek model.

In the following proposition we state the limit of the domestic term structure of interest rates in the convergence model of CIR type.

Proposition 4. *The limit of the domestic term structure of interest rates in the convergence model of CIR type is*

$$\lim_{\tau\to\infty} R(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau) = aD_{\ominus} + cd\frac{(c+\nu_{\mathrm{e}}\sigma_{\mathrm{e}}) - \sqrt{(c+\nu_{\mathrm{e}}\sigma_{\mathrm{e}})^2 + 2bD_{\ominus}\sigma_{\mathrm{e}}^2}}{-\sigma_{\mathrm{e}}^2}.$$

Proof. Lemma 2 and Corollary 3 imply that

$$\lim_{\tau\to\infty} \frac{D(\tau)}{\tau} = 0, \text{ and } \lim_{\tau\to\infty} \frac{U(\tau)}{\tau} = 0.$$

Using l'Hospital's rule and ODE (13), the limit of the term structure is

$$\lim_{\tau\to\infty} R(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau) = -\lim_{\tau\to\infty} A(\tau)/\tau = -\lim_{\tau\to\infty} \dot{A}(\tau) = a\lim_{\tau\to\infty} D(\tau) + cd\lim_{\tau\to\infty} U(\tau),$$

which completes the proof. □

4.2 The case of nonzero correlation

In the case of a nonzero correlation the term $\rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}\sqrt{r_{\mathrm{d}}r_{\mathrm{e}}}\frac{\partial^2 P}{\partial r_{\mathrm{d}}\partial r_{\mathrm{e}}}$ in equation (9) is not eliminated. The only acceptable domestic market price of risk is of the form $\lambda_{\mathrm{d}} = \nu_{\mathrm{d}}\sqrt{r_{\mathrm{d}}} + \overline{\nu_{\mathrm{e}}}\sqrt{r_{\mathrm{e}}}$, where ν_{d} and $\overline{\nu_{\mathrm{e}}}$ are constants (this approach enables us to obtain one more term with $\sqrt{r_{\mathrm{d}}r_{\mathrm{e}}}$; the other choice would lead to a single term that we would not be able to eliminate). If we assume the solution of the form (4), the only change is that the system of ODEs (11)–(13) is extended by the equation

$$0 = \overline{\nu_{\mathrm{e}}}\sigma_{\mathrm{d}}D + \rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}DU \ \forall\tau. \tag{15}$$

However, Lemma 2 implies that in the solution of the form (4) the function D is positive. It is obvious that U is not a constant function; therefore, equation (15) is not satisfied for $\rho \neq 0$.

Maturity	Difference
1/4	4.5×10^{-5}
1/2	2.2×10^{-4}
3/4	4.8×10^{-4}
1	7.8×10^{-4}
5	5.0×10^{-3}
10	8.1×10^{-3}
20	1.1×10^{-2}

Table 1: *Differences (in percentual points) in interest rates between the convergence model of Vasicek type with and without a correlation. The parameters of the model were taken from Corzo and Schwartz (2000).*

5 Approximation of a solution in the case of nonzero correlation and its accuracy

Although we proved that there is no separable solution of the form (4), we can try to approximate a solution in the nonzero correlation case by a solution in the zero correlation case. In this section we also investigate how much these solutions differ. We demonstrate our motivation on the convergence model of Vasicek type.

Let us consider the two-factor convergence model of Vasicek type; for the sake of simplicity assume that $c \neq d$. Let $P_{\mathrm{Vas}}(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho)$ be the price of the domestic bond, where the dependence on the correlation ρ is explicitly marked. Analogously, let $R(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho)$ be the corresponding term structure of interest rates and $A(\tau; \rho)$ be the function in (4). By expanding the explicit solution into the Taylor series with respect to τ we obtain:

Proposition 5. *Let $P_{\mathrm{Vas}}(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho)$ be a solution to the bond-pricing PDE* (3) *of the convergence model of Vasicek type. Then*

$$\ln[P_{\mathrm{Vas}}(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; 0)] - \ln[P_{\mathrm{Vas}}(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho)] = -\frac{1}{8} b\rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}\, \tau^4 + \mathrm{o}(\tau^4).$$

Since we know the bond price explicitly, we are able not only to derive the order of the difference as in the previous proposition, but also to compute its concrete values and to find its dependence on time to maturity.

In Table 1 we exhibit the difference between interest rates in the convergence model of Vasicek type with parameters taken from Corzo and Schwartz (2000) with the same model with a zero correlation and the other parameters remaining. The market data are quoted with two decimal places; therefore, the differences in Table 1 are observable only for long-time maturities. However, even in the case of a twenty-year maturity, the difference is only 0.01 percent (for the given parameters).

Proposition 6. *In the convergence model of Vasicek type, the difference between the term structures of interest rates $|R(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; 0) - R(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho)|$ in the case of the zero and nonzero correlation is an increasing function of time to maturity τ and is less than or equal to $|\rho|\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}/(bc)$.*

Proof. Let us denote the difference between the term structures of interest rates with and without a correlation

$$R(\tau) = R(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; 0) - R(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho) = -\frac{A(\tau; 0) - A(\tau; \rho)}{\tau}.$$

First we show that $R(\tau)$ is monotonous. By differentiating $R(\tau)$ and applying ODE (5) we get

$$\dot{R}(\tau) = -\frac{[\dot{A}(\tau; 0) - \dot{A}(\tau; \rho)]\tau - [A(\tau; 0) - A(\tau; \rho)]}{\tau^2} = -\frac{R(\tau)}{\tau} + \frac{\rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}D(\tau)U(\tau)}{\tau}. \tag{16}$$

Since $R(0) = 0$, the solution is given by

$$R(\tau) = \rho\sigma_{\mathrm{d}}\sigma_{\mathrm{e}}F(\tau),$$

where

$$F(\tau) = \frac{1}{\tau}\int_0^{\tau} D(s)U(s)\mathrm{d}s.$$

To prove that R is monotonous it is sufficient to show that $\dot{F}$ is positive for $\tau > 0$. The derivative of F is

$$\dot{F}(\tau) = \frac{D(\tau)U(\tau)\tau - \int_0^{\tau} D(s)U(s)\mathrm{d}s}{\tau^2}.$$

Since D and U are both increasing and positive, then the product DU is increasing and positive, too. Therefore,

$$\int_0^{\tau} D(s)U(s)\mathrm{d}s < \int_0^{\tau} \max_{t \in \langle 0, \tau \rangle} D(t)U(t)\mathrm{d}s = \int_0^{\tau} D(\tau)U(\tau)\mathrm{d}s = D(\tau)U(\tau)\tau,$$

which implies that $\dot{F} > 0$ for τ greater than 0. Thus $|R|$ is bounded by

$$\begin{aligned} \sup_{\tau \in (0,\infty)} |R(\tau)| &= |\rho|\sigma_{\mathrm{d}}\sigma_{\mathrm{e}} \sup_{\tau \in (0,\infty)} F(\tau) \\ &= |\rho|\sigma_{\mathrm{d}}\sigma_{\mathrm{e}} \lim_{\tau \to \infty} F(\tau) = |\rho|\sigma_{\mathrm{d}}\sigma_{\mathrm{e}} \lim_{\tau \to \infty} D(\tau)U(\tau). \end{aligned}$$

To complete the proof we set the limits $\lim_{\tau\to\infty} D(\tau) = 1/b$ (cf. equation (6)), and $\lim_{\tau\to\infty} U(\tau) = 1/c$ (cf. equation (7)). □

This approach motivates us to determine the difference between the logarithm of the price of the discount bond in the case of a zero correlation and the logarithm of the price of the discount bond in the case of a nonzero correlation in the two-factor convergence model of CIR type. However, we do not know any exact solution to the price of the discount bond in the convergence model of CIR type; therefore, we are only able to derive the order of an approximation. The result is formulated in the following

Theorem 7. *Let $P_{\mathrm{CIR}}(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho)$ be a solution to the bond-pricing PDE* (9) *of the convergence model of CIR type. Then*

$$\ln[P_{\mathrm{CIR}}(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; 0)] - \ln[P_{\mathrm{CIR}}(r_{\mathrm{d}}, r_{\mathrm{e}}, \tau; \rho)] = c_3(r_{\mathrm{d}}, r_{\mathrm{e}}; \rho)\tau^3 + \mathrm{o}(\tau^3),$$

where the coefficient c_3 is not identically equal to zero.

Proof. Let $f = \ln(P)$ be the logarithm of the domestic bond-price, and let $K_{\rm d} = [a + b(r_{\rm e} - r_{\rm d}) - \nu_{\rm d}\sigma_{\rm d} r_{\rm d}]$, $L_{\rm d} = \sigma_{\rm d}^2 r_{\rm d}/2$, $K_{\rm e} = [c(d - r_{\rm e}) - \nu_{\rm e}\sigma_{\rm e} r_{\rm e}]$, $L_{\rm e} = \sigma_{\rm e}^2 r_{\rm e}/2$. Then f satisfies the following PDE:

$$\begin{aligned}-\frac{\partial f}{\partial \tau} + K_{\rm d}\frac{\partial f}{\partial r_{\rm d}} + K_{\rm e}\frac{\partial f}{\partial r_{\rm e}} + L_{\rm d}\left[\left(\frac{\partial f}{\partial r_{\rm d}}\right)^2 + \frac{\partial^2 f}{\partial r_{\rm d}^2}\right] + L_{\rm e}\left[\left(\frac{\partial f}{\partial r_{\rm e}}\right)^2 + \frac{\partial^2 f}{\partial r_{\rm e}^2}\right]&\\ +2\rho\sqrt{L_{\rm d}L_{\rm e}}\left(\frac{\partial f}{\partial r_{\rm d}}\frac{\partial f}{\partial r_{\rm e}} + \frac{\partial^2 f}{\partial r_{\rm d}\partial r_{\rm e}}\right) - r_{\rm d} &= 0,\end{aligned} \tag{17}$$

which follows from (9). For our purposes, we denote by $P_{\rm ex}$ an exact solution to equation (9) for $\rho > 0$; we denote by $P_{\rm ap}$ our solution to equation (9) with $\rho = 0$ by which we want to approximate $P_{\rm ex}$, and $f = \ln(P_{\rm ex})$ and $f_0 = \ln(P_{\rm ap})$. Let us see what PDE $g = f_0 - f$ satisfies. Using $(\frac{\partial g}{\partial r})^2 = (\frac{\partial f_0}{\partial r})^2 - (\frac{\partial f}{\partial r})^2 - 2\frac{\partial f}{\partial r}\frac{\partial g}{\partial r}$, for $r = r_{\rm d}, r_{\rm e}$, we obtain

$$\begin{aligned}&-\frac{\partial g}{\partial \tau} + K_{\rm d}\frac{\partial g}{\partial r_{\rm d}} + K_{\rm e}\frac{\partial g}{\partial r_{\rm e}} + L_{\rm d}\left[\left(\frac{\partial g}{\partial r_{\rm d}}\right)^2 + \frac{\partial^2 g}{\partial r_{\rm d}^2}\right] + L_{\rm e}\left[\left(\frac{\partial g}{\partial r_{\rm e}}\right)^2 + \frac{\partial^2 g}{\partial r_{\rm e}^2}\right]\\
&\quad +2\rho\sqrt{L_{\rm d}L_{\rm e}}\left(\frac{\partial g}{\partial r_{\rm d}}\frac{\partial g}{\partial r_{\rm e}} + \frac{\partial^2 g}{\partial r_{\rm d}\partial r_{\rm e}}\right) =\\
&-\frac{\partial f_0}{\partial \tau} + K_{\rm d}\frac{\partial f_0}{\partial r_{\rm d}} + K_{\rm e}\frac{\partial f_0}{\partial r_{\rm e}} + L_{\rm d}\left[\left(\frac{\partial f_0}{\partial r_{\rm d}}\right)^2 + \frac{\partial^2 f_0}{\partial r_{\rm d}^2}\right] + L_{\rm e}\left[\left(\frac{\partial f_0}{\partial r_{\rm e}}\right)^2 + \frac{\partial^2 f_0}{\partial r_{\rm e}^2}\right] - r_{\rm d}\\
&\quad -\Bigg(-\frac{\partial f}{\partial \tau} + K_{\rm d}\frac{\partial f}{\partial r_{\rm d}} + K_{\rm e}\frac{\partial f}{\partial r_{\rm e}} + L_{\rm d}\left[\left(\frac{\partial f}{\partial r_{\rm d}}\right)^2 + \frac{\partial^2 f}{\partial r_{\rm d}^2}\right]\\
&\quad +L_{\rm e}\left[\left(\frac{\partial f}{\partial r_{\rm e}}\right)^2 + \frac{\partial^2 f}{\partial r_{\rm e}^2}\right] + 2\rho\sqrt{L_{\rm d}L_{\rm e}}\left(\frac{\partial f}{\partial r_{\rm d}}\frac{\partial f}{\partial r_{\rm e}} + \frac{\partial^2 f}{\partial r_{\rm d}\partial r_{\rm e}}\right) - r_{\rm d}\Bigg)\\
&\quad -2L_{\rm d}\frac{\partial f}{\partial r_{\rm d}}\frac{\partial g}{\partial r_{\rm d}} - 2L_{\rm e}\frac{\partial f}{\partial r_{\rm e}}\frac{\partial g}{\partial r_{\rm e}} + 2\rho\sqrt{L_{\rm d}L_{\rm e}}\left(\frac{\partial f}{\partial r_{\rm d}}\frac{\partial f}{\partial r_{\rm e}} + \frac{\partial^2 f}{\partial r_{\rm d}\partial r_{\rm e}}\right)\\
&\quad +2\rho\sqrt{L_{\rm d}L_{\rm e}}\left(\frac{\partial g}{\partial r_{\rm d}}\frac{\partial g}{\partial r_{\rm e}} + \frac{\partial^2 g}{\partial r_{\rm d}\partial r_{\rm e}}\right)\\
&= 4\rho\sqrt{L_{\rm d}L_{\rm e}}DU + 2L_{\rm d}\left[\left(\frac{\partial f}{\partial r_{\rm d}}\right)^2 + D\frac{\partial f}{\partial r_{\rm d}}\right] + 2L_{\rm e}\left[\left(\frac{\partial f}{\partial r_{\rm e}}\right)^2 + U\frac{\partial f}{\partial r_{\rm e}}\right]\\
&\quad +2\rho\sqrt{L_{\rm d}L_{\rm e}}\left(2\frac{\partial f}{\partial r_{\rm d}}\frac{\partial f}{\partial r_{\rm e}} + D\frac{\partial f}{\partial r_{\rm e}} + U\frac{\partial f}{\partial r_{\rm d}}\right).\end{aligned} \tag{18}$$

Now, we expand g into the Taylor series, i.e., $g(r_{\rm d}, r_{\rm e}, \tau) = \sum_{k=\omega}^{\infty} c_k(r_{\rm d}, r_{\rm e})\tau^k$; that is, we expect the first $\omega - 1$ terms to be zero. Therefore, $\frac{\partial g}{\partial \tau} = \omega c_\omega \tau^{\omega-1} + {\rm o}(\tau^{\omega-1})$. The rest of the terms on the left-hand side of (18) are of the order τ^ω (because the rest are derivatives of g with respect to $r_{\rm d}$ and $r_{\rm e}$); hence, the left-hand side is of the order $\tau^{\omega-1}$. Let us analyse the right-hand side of the equation (18). Note that f is of the order τ, since its value for $\tau = 0$ is the logarithm of the bond price at maturity, i.e., zero. It follows that the derivatives $\frac{\partial f}{\partial r_{\rm d}}$ and $\partial f r_{\rm e}$ are of the order τ as well. Equation (11) and the initial condition $D(0) = 0$ give $\dot{D}(0) = 1$ and $\ddot{D}(0) = -(b + \nu_{\rm d}\sigma_{\rm d})$. Analogously, equation (12) and $U(0) = 0$ yield that $\dot{U}(0) = 0$ and $\ddot{U}(0) = b$. Therefore, we obtain the expansion $D(\tau)U(\tau) = \frac{1}{2}b\tau^3 + {\rm o}(\tau^3)$. Consequently, we get that the right-hand side of equation (18) is of the order at least τ^2. Therefore, ω is at least 3. An order higher than 3 would be attained if the coefficient at τ^2 in the expansion of the right-hand side of (18) was eliminated. In the following we show that that is not the case. Since $U = b\tau^2 + {\rm o}(\tau^2)$, $b > 0$, and $\frac{\partial g}{\partial r_{\rm e}} = -U - \frac{\partial f}{\partial r_{\rm e}}$, we have an extra information that $\frac{\partial f}{\partial r_{\rm e}} = k_2\tau^2 + {\rm o}(\tau^2)$. Repeating the previous analysis of the right-hand side of (18) with this additional information, we obtain that the only ${\rm O}(\tau^2)$ term is (up to a multiplicative constant independent of τ) equal to $\frac{\partial f}{\partial r_{\mathrm{d}}}\left(\frac{\partial f}{\partial r_{\mathrm{d}}}+D\right)$. We prove that this term is not constantly equal to zero. To derive a contradiction, assume that

$$\frac{\partial f}{\partial r_{\mathrm{d}}}\left(\frac{\partial f}{\partial r_{\mathrm{d}}}+D\right)=0, \tag{19}$$

for all $\tau, r_{\mathrm{d}}, r_{\mathrm{e}}$. It follows from the continuity of $\frac{\partial f}{\partial r_{\mathrm{d}}}$ and the behaviour of D that we only have two options: either $\frac{\partial f}{\partial r_{\mathrm{d}}}=0$ for all $\tau, r_{\mathrm{d}}, r_{\mathrm{e}}$, or $\frac{\partial f}{\partial r_{\mathrm{d}}}+D=0$ for all $\tau, r_{\mathrm{d}}, r_{\mathrm{e}}$. If $\frac{\partial f}{\partial r_{\mathrm{d}}}=0$, then $f=f(r_{\mathrm{e}},\tau)$ and $\frac{\partial^2 f}{\partial r_{\mathrm{d}}^2}=0$. Equation (17) reduces to

$$-\frac{\partial f}{\partial \tau}+K_{\mathrm{e}}\frac{\partial f}{\partial r_{\mathrm{e}}}+L_{\mathrm{e}}\left[\left(\frac{\partial f}{\partial r_{\mathrm{e}}}\right)^2+\frac{\partial^2 f}{\partial r_{\mathrm{e}}^2}\right]-r_{\mathrm{d}}=0,$$

and if we differentiate the previous equation with respect to r_{d}, we obtain that $-1=0$, which is a contradiction. In the other case, integrating $\frac{\partial f}{\partial r_{\mathrm{d}}}=-D$ with respect to r_{d} yields the form of the solution f as $f=-D(\tau)r_{\mathrm{d}}+w(\tau,r_{\mathrm{e}})$ for some function $w(\tau,r_{\mathrm{e}})$. By setting such a solution to PDE (17), we obtain that $\frac{\partial w}{\partial r_{\mathrm{e}}}=0$, i.e. $\frac{\partial f}{\partial r_{\mathrm{e}}}=0$, which leads to a contradiction in the same way as in the previous case. Therefore, the term in (19) is not constantly zero, which completes the proof. $\square$

6 Empirical implementation for the case of Slovak interest rates

6.1 The data

We use Slovak and European data. In particular, the data consist of 62 daily (that is, $\Delta=1/252$) observations from 1st October 2008 to 31st December 2008. The reason why we use data for such a short period is the influence of the economic crisis. Figure 3 depicts the evolution of the Slovak and European overnight, 1 week, 2 week, 1 month, 2 month, 3 month, 6 month, 9 month and 1 year interest rates from 2nd June 2008 to 31st December 2008. In the last quarter of 2008 (from 1st October 2008) we can see the influence of the upcoming economic crisis that highlights the strong dependence between both interest rates (specially interest rates for bonds with a long maturity) immediately before the Slovak Republic adopted the Euro currency. We note that there is a structural breakpoint, that is, a change in the settings of the economy (and, therefore, in the parameters). The European market data, EONIA and EURIBOR, are available at *http://www.euribor.org*. The Slovak market data BRIBOR are taken as the middle between an offer and a bid, which is available at the National Bank of Slovakia website, *http://www.nbs.sk*. We use the overnight interest rates as the short-rates.

Tables 2 and 3 provide some descriptive statistics for the short-rates.

6.2 Methodology

Consider an equidistant discrete sample $\{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n\}$ with time difference Δ of a multivariate $\boldsymbol{\theta}$-parametrized Itō process $\boldsymbol{X}_t$ with values in $\mathbb{R}^n$ and an initial

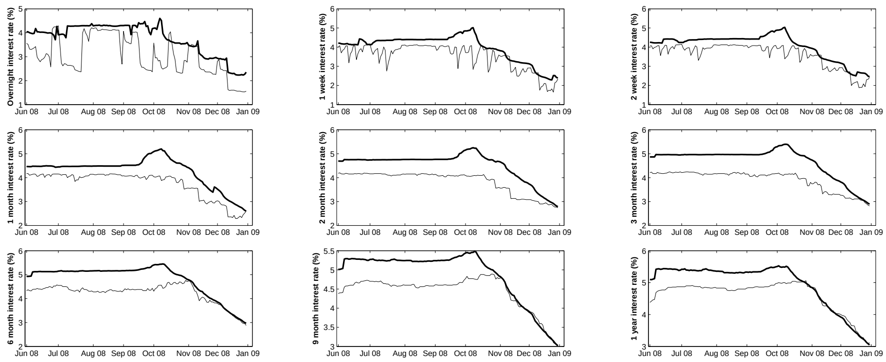

Figure 3: *The evolution of BRIBOR (thin line) and EURIBOR (thick line) from* 2^{nd} *June* 2008 *to* 31^{st} *December* 2008.

	$\boldsymbol{r}_{\mathbf{svk}}$	$\mathbf{d}\boldsymbol{r}_{\mathbf{svk}}$	$\boldsymbol{r}_{\mathbf{emu}}$	$\mathbf{d}\boldsymbol{r}_{\mathbf{emu}}$	$\boldsymbol{r}_{\mathbf{emu}}-\boldsymbol{r}_{\mathbf{svk}}$
Mean	0.025258	−0.000330	0.031916	−0.000301	0.006658
Std. dev.	0.006589	0.003047	0.006322	0.001359	0.004772
Skewness	0.196581	1.069584	0.098701	−2.523980	0.283366
Kurtosis	2.478430	9.940486	2.316502	11.911318	3.252789
Maximum	0.040400	0.013400	0.046000	0.002500	0.019700
Minimum	0.015300	−0.008400	0.022340	−0.005940	−0.004000

Table 2: *Individual descriptive statistics for the selected variables.*

	$\mathbf{d}\boldsymbol{r}_{\mathbf{svk}}$	$\boldsymbol{r}_{\mathbf{emu}}$	$\mathbf{d}\boldsymbol{r}_{\mathbf{emu}}$	$\boldsymbol{r}_{\mathbf{emu}}-\boldsymbol{r}_{\mathbf{svk}}$
$\boldsymbol{r}_{\mathbf{svk}}$	0.227530	0.715676	0.101294	−0.423370
$\mathbf{d}\boldsymbol{r}_{\mathbf{svk}}$		0.001662	0.318626	−0.305394
$\boldsymbol{r}_{\mathbf{emu}}$			0.079707	0.329754
$\mathbf{d}\boldsymbol{r}_{\mathbf{emu}}$				−0.033528

Table 3: *Correlation between the selected variables.*

value $\boldsymbol{X}_0$ that follow SDE

$$\mathrm{d}\boldsymbol{X} = \boldsymbol{\mu}(\boldsymbol{X}, t, \boldsymbol{\theta})\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}, t, \boldsymbol{\theta})\mathrm{d}\boldsymbol{W}, \tag{20}$$

and let $\boldsymbol{\gamma}(\boldsymbol{x}, \boldsymbol{\theta})$ be an invertible function such that

$$\mathbf{J}_{\boldsymbol{x}}\gamma(\boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{\sigma}^{-1}(\boldsymbol{x}, \boldsymbol{\theta}), \tag{21}$$

where $\mathbf{J}_{\boldsymbol{x}}\gamma(\boldsymbol{x}, \boldsymbol{\theta})$ is the Jacobian matrix of $\boldsymbol{\gamma}(\boldsymbol{x}, \boldsymbol{\theta})$ and $\boldsymbol{\sigma}^{-1}(\boldsymbol{x}, \boldsymbol{\theta})$ is the matrix inverse of $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\theta})$. By $\boldsymbol{\gamma}^{\mathrm{inv}}(., \boldsymbol{\theta})$ we denote the inverse transformation for $\boldsymbol{\gamma}(., \boldsymbol{\theta})$. In this case we say that the diffusion $\boldsymbol{X}_t$ is reducible to a unit diffusion (as we will see later, mentioned convergence models are reducible diffusions). Itō's lemma implies that the process $\boldsymbol{Y}_t = \boldsymbol{\gamma}(\boldsymbol{X}_t, \boldsymbol{\theta})$ follows SDE

$$\mathrm{d}\boldsymbol{Y} = \boldsymbol{\mu}_{\mathbf{Y}}(\boldsymbol{Y})\mathrm{d}t + \mathrm{d}\boldsymbol{W},$$

where

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{Y}}(\boldsymbol{y})\mathrm{d}t &= \boldsymbol{\sigma}^{-1}[\boldsymbol{\gamma}^{\mathrm{inv}}(\boldsymbol{y}),\boldsymbol{\theta}]\boldsymbol{\mu}[\boldsymbol{\gamma}^{\mathrm{inv}}(\boldsymbol{y}),\boldsymbol{\theta}]\mathrm{d}t \\ &+ \left.\begin{pmatrix} (\mathrm{d}\boldsymbol{X})'(\boldsymbol{\nabla}^2_{\boldsymbol{x}}\gamma_1(\boldsymbol{x},\boldsymbol{\theta}))\mathrm{d}\boldsymbol{X} \\ \vdots \\ (\mathrm{d}\boldsymbol{X})'(\boldsymbol{\nabla}^2_{\boldsymbol{x}}\gamma_n(\boldsymbol{x},\boldsymbol{\theta}))\mathrm{d}\boldsymbol{X} \end{pmatrix}\right|_{\boldsymbol{x}=\boldsymbol{\gamma}^{\mathrm{inv}}(\boldsymbol{y},\boldsymbol{\theta})}, \end{aligned} \tag{22}$$

$\gamma_i(\boldsymbol{x},\boldsymbol{\theta})$ is the ith component of $\boldsymbol{\gamma}(\boldsymbol{x},\boldsymbol{\theta})$, $(\mathrm{d}W_i)^2 = \mathrm{d}t$ and $(\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}W_i = 0$. Clearly, $\boldsymbol{Y}_0 = \boldsymbol{\gamma}(\boldsymbol{X}_0,\boldsymbol{\theta})$. Now, we perform another transformation, namely $\boldsymbol{Z}_t = \Delta^{-1/2}(\boldsymbol{Y}_t - \boldsymbol{Y}_0)$, and we recall that Δ is the time difference between two values in the given sample. Again, Itō's lemma yields

$$\mathrm{d}\boldsymbol{Z} = \Delta^{-1/2}\mathrm{d}\boldsymbol{Y} = \Delta^{-1/2}\boldsymbol{\mu}_{\mathbf{Y}}(\boldsymbol{Y},\boldsymbol{\theta})\mathrm{d}t + \Delta^{-1/2}\mathrm{d}\boldsymbol{W},$$

and $\boldsymbol{Z}_0 = \mathbf{0}$. According to Aït-Sahalia (2002, 2008), $\boldsymbol{Z}$ is close enough to a normal distribution. If we assume that Δ is small enough (for daily data $\Delta = 1/252$, which is small enough), we substitute Δ for $\mathrm{d}t$. Consequently, we can write $\boldsymbol{\varepsilon} = \Delta^{-1/2}\mathrm{d}\boldsymbol{W}$ approximately has the $\mathcal{N}(\mathbf{0},\mathbf{I})$ distribution. It follows from $\boldsymbol{Y}_t = \Delta^{1/2}\boldsymbol{Z}_t + \boldsymbol{Y}_0$ that

$$\mathrm{d}\boldsymbol{Z}_t \approx \Delta^{1/2}\boldsymbol{\mu}_{\mathbf{Y}}(\Delta^{1/2}\boldsymbol{Z}_t + \boldsymbol{Y}_0,\boldsymbol{\theta}) + \boldsymbol{\varepsilon}.$$

We are interested in the distribution of $\boldsymbol{Z}_\Delta$, depending on $\boldsymbol{Z}_0$ for Δ small. We perform another approximation. The fact that $\boldsymbol{Z}_0 = \mathbf{0}$ implies $\boldsymbol{Z}_\Delta = \boldsymbol{Z}_\Delta - \boldsymbol{Z}_0 \approx \mathrm{d}\boldsymbol{Z}_0$, which results in the final approximation

$$\boldsymbol{Z}_\Delta \mid \boldsymbol{Y}_0 \sim \mathcal{N}(\Delta^{1/2}\boldsymbol{\mu}_{\mathbf{Y}}(\boldsymbol{Y}_0,\boldsymbol{\theta}),\mathbf{I}).$$

After a little degree of effort we obtain that for Δ small the density function of $\boldsymbol{X}_\Delta$ conditioned on $\boldsymbol{X}_0$ is approximately

$$\begin{aligned} f_{\boldsymbol{X}}(\boldsymbol{x},\Delta,\boldsymbol{\theta} \mid \boldsymbol{X}_0) = {} & \frac{\Delta^{-1/2}}{(2\pi)^{n/2}}\det[\boldsymbol{\sigma}^{-1}(\boldsymbol{x},\boldsymbol{\theta})] \\ & \times \exp\left\{-\frac{1}{2\Delta}\left\|\boldsymbol{\gamma}(\boldsymbol{x},\boldsymbol{\theta}) - \boldsymbol{\gamma}(\boldsymbol{X}_0,\boldsymbol{\theta}) - \Delta\boldsymbol{\mu}_{\mathbf{Y}}[\gamma(\boldsymbol{X}_0),\boldsymbol{\theta}]\right\|^2\right\}. \end{aligned}$$

Note also that

$$\boldsymbol{\mu}_{\mathbf{Y}}[\gamma(\boldsymbol{X}_0),\boldsymbol{\theta}]\mathrm{d}t = \boldsymbol{\sigma}^{-1}(\boldsymbol{X}_0,\boldsymbol{\theta})\boldsymbol{\mu}(\boldsymbol{X}_0,\boldsymbol{\theta})\mathrm{d}t + \left.\begin{pmatrix} (\mathrm{d}\boldsymbol{X})'(\boldsymbol{\nabla}^2_{\boldsymbol{x}}\gamma_1(\boldsymbol{x},\boldsymbol{\theta}))\mathrm{d}\boldsymbol{X} \\ \vdots \\ (\mathrm{d}\boldsymbol{X})'(\boldsymbol{\nabla}^2_{\boldsymbol{x}}\gamma_n(\boldsymbol{x},\boldsymbol{\theta}))\mathrm{d}\boldsymbol{X} \end{pmatrix}\right|_{\boldsymbol{x}=\boldsymbol{X}_0},$$

where $(\mathrm{d}W_i)^2 = \mathrm{d}t$ and $(\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}W_i = 0$. Consequently, the density is invariant with respect to the addition of a constant vector to $\boldsymbol{\gamma}(\boldsymbol{x})$.

It is a reasonable economic assumption that the evolution of the European interest rate is not influenced by the domestic interest rate. Therefore, the parameters c, d, σ_{e}, ν_{e} are estimated using only the EURIBOR data. Firstly, we estimate parameters c, d, σ_{e} using the maximum likelihood method and the approximation of the density by Aït-Sahalia (1999). Afterwards, the constant

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{\sigma_{\mathrm{d}}}$	$\boldsymbol{\rho}$	$\boldsymbol{\nu_{\mathrm{d}}}$
0.0	6.96169	0.285294	0.382321	−2.03
$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{\sigma_{\mathrm{e}}}$	$\boldsymbol{\nu_{\mathrm{e}}}$	**MSE**
7.75816	0.022182	0.117200	−38.91972	2.278×10^{-5}

Table 4: *Calibration summary*

ν_{e} from the market price of risk is obtained by minimizing the mean square error

$$\mathrm{MSE}(\nu_{\mathrm{e}}) = \frac{1}{MN}\sum_{i=1}^{N}\sum_{j=1}^{M}[R(r_{\mathrm{e},i},\tau_j;\nu_{\mathrm{e}}) - R_{ij}]^2, \tag{23}$$

where $R(r_{\mathrm{e},i},\tau_j;\nu_{\mathrm{e}})$ is the interest rate with maturity τ_j observed on ith day computed from the model (we have marked its dependence on ν_{e}) and R_{ij} is the corresponding interest rate from the marked data. Note that for the European interest rates we assume the one-factor interest rate model. Hence, $R(r_{\mathrm{e}},\tau;\nu_{\mathrm{e}})$ is known in the closed form (cf. Cox et al. (1985)).

In the second step we estimate the correlation ρ and the parameters b, σ_{d} ν_{d}. We set the parameter a to zero, since the data which we use come from the time period directly before the adoption of the Euro currency. Hence we assume that there is no divergence and the domestic rate is pushed towards the European rate. We insert the already estimated parameters into the approximation of the likelihood function derived above and maximize the likelihood function with respect to the parameters ρ, b, σ_{d}. Then we estimate the parameter ν_{d} by minimizing the mean square error

$$\mathrm{MSE}(\nu_{\mathrm{d}}) = \frac{1}{MN}\sum_{i=1}^{N}\sum_{j=1}^{M}[R(r_{\mathrm{d},i},r_{\mathrm{e},i},\tau_j;\nu_{\mathrm{d}}) - R_{ij}]^2, \tag{24}$$

where $R(r_{\mathrm{d},i},r_{\mathrm{e},i},\tau_j;\nu_{\mathrm{d}})$ is the interest rate with maturity τ_j observed on ith day computed from the model by numerically solving the bond pricing equation in the uncorrelated case and R_{ij} is the corresponding interest rate from the marked data.

6.3 Results

Table 4 provides the estimates of the parameters in the proposed model. Figure 4 compares theoretical term structures with the market term structures from 1st October 2008 with period 10 days.

7 Conclusion

In this thesis we study two-factor convergence term structure models of interest rates, which describe the evolution of the interest rate of a country before adopting the Euro currency.

In the first part we focus our attention on the two-factor convergence model of Vasicek type proposed by Corzo and Schwartz (2000). We figure out a solution to the bond-pricing partial differential equation and compute the limit of the term structure of interest rates for the maturity going to infinity.

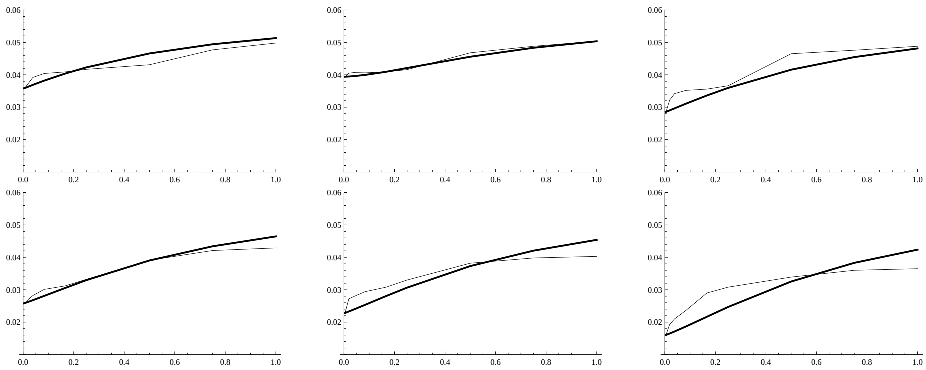

Figure 4: *Comparison of the real (solid curves) and estimated (bold curves) term structures*

The second part deals with the proposed two-factor convergence model of Cox-Ingersoll-Ross (CIR) type, where the domestic and European overnight rates are governed by the Bessel square root process. We show that a separable solution exists only if the correlation between the increments of Wiener processes is zero; we derive its properties and compute the limit of the term structure of interest rates for the maturity going to infinity. In the other case we demonstrate that the separable solution for a zero correlation is a good approximation of the bond price in the case of nonzero correlation: we derive the order of approximation for the difference in logarithms of the bond prices with and without a correlation between the increments of Wiener processes.

In the last part we give an empirical example using the Slovak data before the Slovak Republic adopted the Euro currency.

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