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MODELING VOLATILITY CLUSTERS WITH APPLICATION TO TWO–FACTOR INTEREST RATE MODELS

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This paper is motived by the study of two-factor interest rate model for valuing term structures. The instantaneous interest rate (short rate) follows a stochastic differential equation, in which the volatility itself is a solution of another stochastic differential equation. We are interested in properties of a volatility corresponding to a prescribed asymptotic form of the distribution. Moreover, we construct a drift function such that the long-time behavior of the distribution of the stochastic volatility has two humps corresponding to two clusters of volatility. The constructed asymptotic distribution is a convex combination of distributions corresponding to two mean-reversion processes with different mean levels. Such a model can explain volatility clustering observed in real interest rate data.

Keywords: two-factor interest rate model, limiting volatility distribution, volatility clustering

2000 *Mathematics Subject Classification*: 91B28

1 INTRODUCTION

The instantaneous interest rate (short rate) is fundamental for pricing of derivatives traded in the market. Therefore, a lot of work has been done in modeling interest rates.

It is an accepted and documented fact that the volatility of the short rate is not constant. There are many models which assuming the level-effect, i.e. higher volatility for higher interest rates. A comprehensive comparison of a class of such models can be found in [5].

This is not the only source of changes in the volatility. It is possible to observe the periods with stable high interest rates and periods with unstable low interest rates (see [4]). One possible way to model such a behaviour is to use GARCH models (see for example [6]). However, the interest rate level does not affect the volatility.

It is convenient to consider models including a stochastic volatility depending also on the interest rate level. In continuous setting, they are often generalizations of level-models by considering the volatility to follow a stochastic differential equation. They are therefore referred to as two-factor interest rate models. Examples of this type are models by Fong and Vasicek [7], Anderson and Lund [2], Brenner [4] is an example of a discrete model. Many comparisons of such models have been done (see for example [3]). In [9], the method for computing average values and confidence intervals for bond prices and interest rates in the presence of unobservable stochastic volatility was proposed. It was applied to Fong-Vasicek model.

The main goal of this paper is to propose and analyze a stochastic differential equation for interest rate volatility, which can be used in continuous two-factor models of the type

$$dr = \kappa_r(\theta_r - r)dt + \sqrt{y}r^\gamma dw_1 , \qquad (1)$$

$$dy = a(y)dt + v\sqrt{y}dw_2 . \qquad (2)$$

We are looking for a process capable of describing the volatility clustering. In this case, the long-term distribution of volatility has two humps. We describe one class of such distributions, for which we are able to derive the corresponding stochastic differential equation.

Simulation of such a process is shown in Fig. 1.

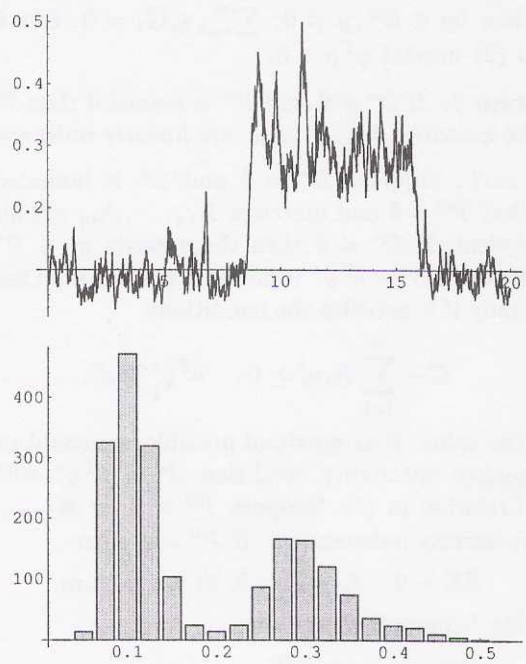

Fig. 1. Simulation of the process and the histogram of its values.

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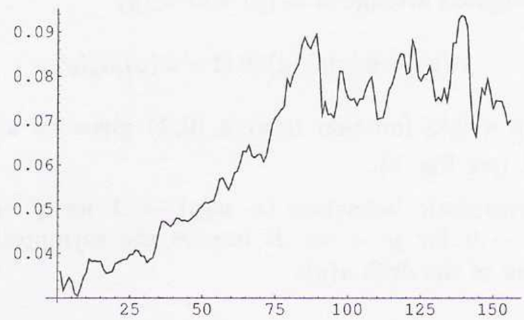

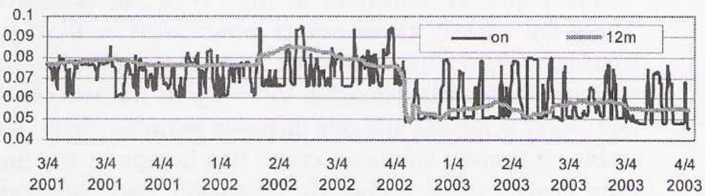

Fig. 3. BRIBOR, 2001-2003.

A real interest rate data example illustrating the change in volatility is shown in Fig. 2. It shows weekly observations of 3-month US Treasury Bills rates in years 1972-1974. It can be seen that the volatility is higher in the second half of this time period irrespective of higher level of interest rate.

It was documented in [10] and [11] that one-factor models are not sufficient to capture the behaviour of the short rate in western economies as well as in central European economies. See Fig. 3 from [11] which shows daily data of BRIBOR in years 2001-2003. It suggests that there are periods with low and periods with high volatility.

2 DISTRIBUTION OF VOLATILITY

Let the process $y(t)$ satisfy the stochastic differential equation

$$dy = a(t,y)dt + b(t,y)dw$$

subject to the initial condition $y(0) = y_0$. According to [8], the density function $f(t,y)$ of $y(t)$ is a solution to the Fokker-Planck equation

$$-\frac{\partial f}{\partial t} - \frac{\partial (af)}{\partial y} + \frac{\partial^2}{\partial y^2}\Big(\frac{b^2 f}{2}\Big) = 0\,, \quad t > 0 \tag{3}$$

subject to the initial condition $f(0,y) = \delta_0(y - y_0)$ where δ_0 is the Dirac function. The density $f(t,y)$ can be expressed in the closed form only in few cases. But if we assume that the process evolves a long enough time, we may approximate its distribution by a limiting distribution. If the limit $\lim_{t\to\infty} f(t,y)$ exists and it is a density function, we will call it a limiting distribution of the process $y(t)$ and denote $g(y) = \lim_{t\to\infty} f(t,y)$.

Let the process $y(t)$ be non-negative, so it can model a volatility of the interest rate. In what follows, we will consider a process with the volatility $b(y) = v\sqrt{y}$. We restrict ourselves to limiting densities only for this case. Suppose that the process has a limiting density $g(y)$. From (3) it follows that $g(y)$ satisfies a stationary Fokker-Planck equation

$$-\frac{d(a(y)g(y))}{dy} + \frac{d^2}{dy^2}\Big(\frac{v^2 y g(y)}{2}\Big) = 0\,, \quad y > 0 \tag{4}$$

and the normalization condition

$$\int_0^\infty g(y)dy = 1 \tag{5}$$

because $g(y) = 0$ for $y \le 0$. Integrating equation (4) and assuming that $\lim_{y\to 0+} g(y) = 0$ and $g'(y)$ is bounded on the neighborhood of zero, we get

$$-a(y)g(y) + \frac{d}{dy}\Big(\frac{v^2 y g(y)}{2}\Big) = 0\,. \tag{6}$$

The general solution of (6) is

$$g(y) = \frac{c}{y}\exp\Big(\int_{c_0}^{y} \frac{2a(s)}{v^2 s}ds\Big) \tag{7}$$

where $c_0 \in (0,\infty)$ is arbitrary and c is a positive constant. It is chosen in such a way that the normalization condition (5) is satisfied.

3 MEAN–REVERTING VOLATILITY

In the case $a(y) = \kappa(\theta - y)$, the process $y(t)$ satisfying (2) is called Bessel square root process. Then a solution of (6), satisfying the normalization condition (5), is given by

$$g(y) = \frac{\alpha^{\alpha\theta}}{\Gamma(\alpha\theta)} y^{\alpha\theta - 1} e^{-\alpha y} \tag{8}$$

where $\alpha = \frac{2\kappa}{v^2}$. It is a density function of the Gamma distribution with parameters $(\alpha, \alpha\theta)$. The restrictions imposed on the behavior of the limiting density in the neighborhood of zero (i.e. $g(0+) = 0$, g' is bounded) are satisfied for $\alpha\theta \ge 2$.

4 VOLATILITY CLUSTERING

Now we consider two Bessel processes with drifts $a_1(y) = \kappa(\theta_1 - y)$ and $a_2(y) = \kappa(\theta_2 - y)$ where $\theta_1 < \theta_2$, and the same volatility $b(y) = v\sqrt{y}$. Denote $\alpha = \frac{2\kappa}{v^2}$ and suppose that $\alpha\theta_1 \ge 2$, $\alpha\theta_2 \ge 2$. If we denote g_1 and g_2 the limiting distributions of these two processes then their convex combination

$$g(y) = kg_1(y) + (1-k)g_2(y), \;\; k \in (0,1)\,, \tag{9}$$

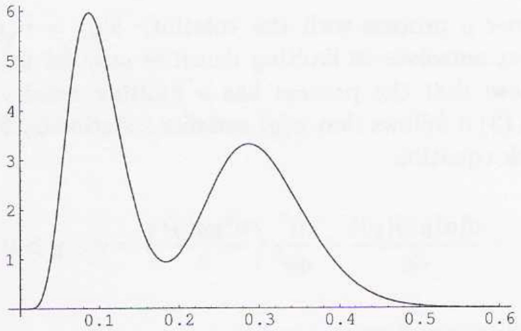

Fig. 4. Example of the convex combination of two Gamma distributions.

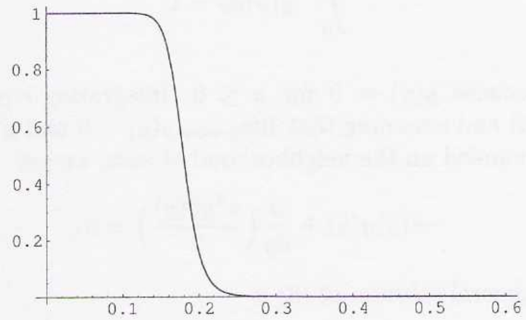

Fig. 5. A weight function $w(y)$.

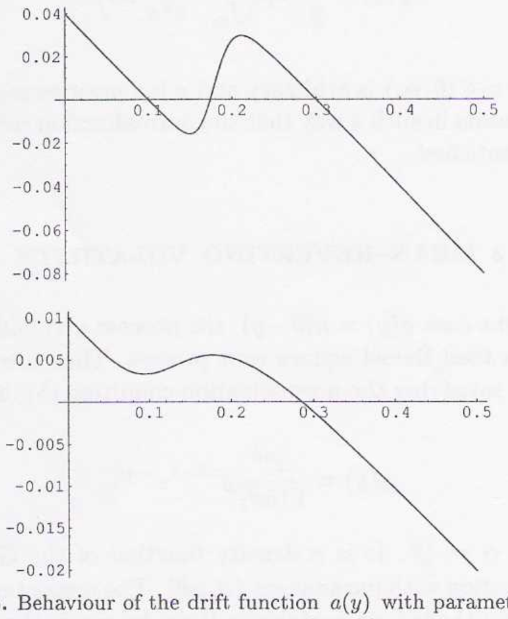

Fig. 6. Behaviour of the drift function $a(y)$ with parameters $\kappa = 0.4$, $\theta_1 = 0.1$, $\theta_2 = 0.3$, $v = 0.1$ (above), $\kappa = 0.1$, $\theta_1 = 0.1$, $\theta_2 = 0.3$, $v = 0.1$ (below).

is a distribution with two humps which can be used in order to model the volatility clustering. Example of such a distribution is shown in Fig. 4.

In the following part, we find a process having a limiting distribution of the form (9). It will have the same volatility $b(y)$ as the two compounded processes. From (4) it follows that $a(y) = \frac{1}{g(y)} \frac{d}{dy}\left(\frac{b^2(y)g(y)}{2}\right)$. Using a similar relationship between $a_1(y)$ and $g_1(y)$ and between $a_2(y)$ and $g_2(y)$, we can write $a(y) = \frac{kg_1(y)}{g(y)} a_1(y) + \frac{(1-k)g_2(y)}{g(y)} a_2(y)$. Hence the drift $a(y)$ can be written as the weighted average of $a_1(y)$ and $a_2(y)$:

$$a(y) = w(y)a_1(y) + (1 - w(y))a_2(y)$$

with a weight function $w(y) \in (0,1)$ given by $w(y) = \frac{kg_1(y)}{g(y)}$ (see Fig. 5).

Its asymptotic behaviour is: $w(y) \to 1$ for $y \to 0+$, $w(y) \to 0$ for $y \to \infty$. It implies the asymptotic behaviour of the drift $a(y)$:

$$a(y) \sim a_1(y) \text{ for } y \to 0+,\ a(y) \sim a_2(y) \text{ for } y \to \infty .$$

The expected behaviour of $a(y)$ is in Fig. 6. above. There are exactly three points where $a(y) = 0$. If we consider only the deterministic part of the process, the ordinary differential equation $dy = a(y)dt$ has two stable stationary solutions and one unstable between them. The stable stationary solutions cause two humps in the limiting density after adding the stochastic part. However, on the graph bellow we see that it is not always the case for a particular choice of parameters. In what follows, we show that if the value of α is large enough (with fixed k, θ_1 and θ_2) then the behaviour of $a(y)$ is the one in Fig. 6 above.

To simplify the expressions we denote

$$c = \frac{1-k}{k} \frac{\Gamma(\alpha\theta_1)}{\Gamma(\alpha\theta_2)} \alpha^{\alpha(\theta_2-\theta_1)}, \quad q = \alpha(\theta_2 - \theta_1). \tag{10}$$

Then the weight $w(y)$ can be written as $\frac{1}{1+cy^q}$. From the expression for the drift we have

$$a(y) = \frac{\kappa}{1+cy^q}\left((\theta_1 - y) + cy^q(\theta_2 - y)\right). \tag{11}$$

It follows that $a(y) > 0$ for $0 \le y \le \theta_1$ and $a(y) < 0$ for $y \ge \theta_2$. Hence all points for which $a(y) = 0$ are between θ_1 and θ_2.

From (11) we also see that $a(y) = 0$, iff $f(y) = 0$ where $f(y) = (\theta_1 - y) + cy^q(\theta_2 - y)$. For this function there are at most three points y in (θ_1, θ_2) where $f(y) = 0$. If there are k points, by Rolle's theorem there are $k-1$ points where $f'(y) = 0$ and $k-2$ points where $f''(y) = 0$ and all these points are also in (θ_1, θ_2). By computing the second derivative of f, we get that it equals zero only for $y = 0$ and $y = \frac{q-1}{q+1}\theta_2$, so there is at most one point in (θ_1, θ_2) where $f''(y) = 0$. Hence $k \le 3$.

The equation $f(y) = 0$ on (θ_1, θ_2) can be equivalently written as $f_1(y) = f_2(y)$ where

$$f_1(y) = cy^q, \quad f_2(y) = \frac{y - \theta_1}{\theta_2 - y}.$$

Since $f_1(\theta_1) > f_2(\theta_1)$ and $f_1(\theta_2)$ is finite, whereas $f_2(y)$ approaches infinity for $y \to \theta_2-$, it suffices to find $y_1 < y_2$ such that

$$f_1(y_1) < f_2(y_1),\ f_1(y_2) > f_2(y_2). \tag{12}$$

Then we will have shown the existence of at least three points, for which $f_1(y) = f_2(y)$. As we already know, the existence of more then three such points is not possible.

To show the existence of such points y_1, y_2, we will compute the limit of $f_1(y)$ as α approaches infinity. By substituting the expressions for c and q, we obtain

$$f_1(y) = \frac{1-k}{k}\frac{\Gamma(\alpha\theta_1)}{\Gamma(\alpha\theta_2)}(\alpha y)^{\alpha(\theta_2-\theta_1)}. \qquad (13)$$

Recall that the Stirling formula holds for $x > 0$ (see [1]):

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}\xi} \qquad (14)$$

for some $\xi = \xi(x) \in (0,1)$. Using it for $x = \alpha\theta_1 - 1$ and $x = \alpha\theta_2 - 1$ (by assumptions on the parameters, they are both greater than or equal to 1), we write (13) as the product of ρ_1 and ρ_2 where

$$\rho_1 = \frac{1-k}{k}\Big(\frac{\alpha\theta_1 - 1}{\alpha\theta_2 - 1}\Big)^{-\frac{1}{2}}\frac{\big(1-\frac{1}{\alpha\theta_1}\big)^{\alpha\theta_1}}{\big(1-\frac{1}{\alpha\theta_2}\big)^{\alpha\theta_2}}\frac{e^{\frac{1}{12(\alpha\theta_1-1)}\xi_1(\alpha)}}{e^{\frac{1}{12(\alpha\theta_2-1)}\xi_2(\alpha)}},$$

$$\rho_2 = \Big(e^{\theta_1\log\frac{\theta_1}{y}+\theta_2\log\frac{y}{\theta_2}+\theta_2-\theta_1}\Big)^{\alpha}.$$

Factor ρ_1 converges to a positive number $\frac{1-k}{k}\big(\frac{\theta_1}{\theta_2}\big)^{-\frac{1}{2}}$ for $\alpha \to \infty$. From the sign of the exponent of e in ρ_2 it follows that ρ_2 converges to zero on some right neighborhood of θ_1 and it converges to infinity on some left neighborhood of θ_2. Hence also $f_1(y)$ has the same limit and we are able to choose $y_1 < y_2$ from (θ_1, θ_2) such that $f_1(y_1) \to 0$ and $f_2(y_2) \to \infty$ for $\alpha \to \infty$. There exists α_0 such that for $\alpha > \alpha_0$ we have $f_1(y_1) < f_2(y_1)$ and $f_1(y_2) > f_2(y_2)$, so the conditions in (12) are satisfied. Hence for $\alpha > \alpha_0$ there are exactly three points where $a(y) = 0$.

5 CONCLUSION

We described a class of distributions which can be used to describe the volatility clustering. For distributions from this class we derived a process with this limiting distribution and discussed the properties of its drift function. The future work will be the study of bond prices and term structures in interest rate models with this kind of stochastic volatility.

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