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A comparison of asymptotic analytical formulae with finite-difference approximations for pricing zero coupon bond

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Abstract In this paper we solve numerically a *degenerate* parabolic equation with *dynamical* boundary condition for pricing zero coupon bond and compare numerical solution with asymptotic analytical solution. First, we discuss an approximate analytical solution of the model and its order of accuracy. Then, starting from the divergent form of the equation we implement the finite-volume method of Song Wang (IMA J Numer Anal 24:699–720, 2004) to discretize the differential problem. We show that the system matrix of the discretization scheme is a M-matrix, so that the discretization is *monotone*. This provides the non-negativity of the price with respect to time if the initial distribution is nonnegative. Numerical experiments demonstrate second order of convergence for difference scheme when the node is not too close to the point of degeneration.

Keywords Degenerate parabolic equation · Bond pricing · Finite volume · Difference scheme · M-matrix

1 Introduction

Term structure models give the dependence of time to maturity of a discount bond and its present price. One-factor models are often formulated in terms

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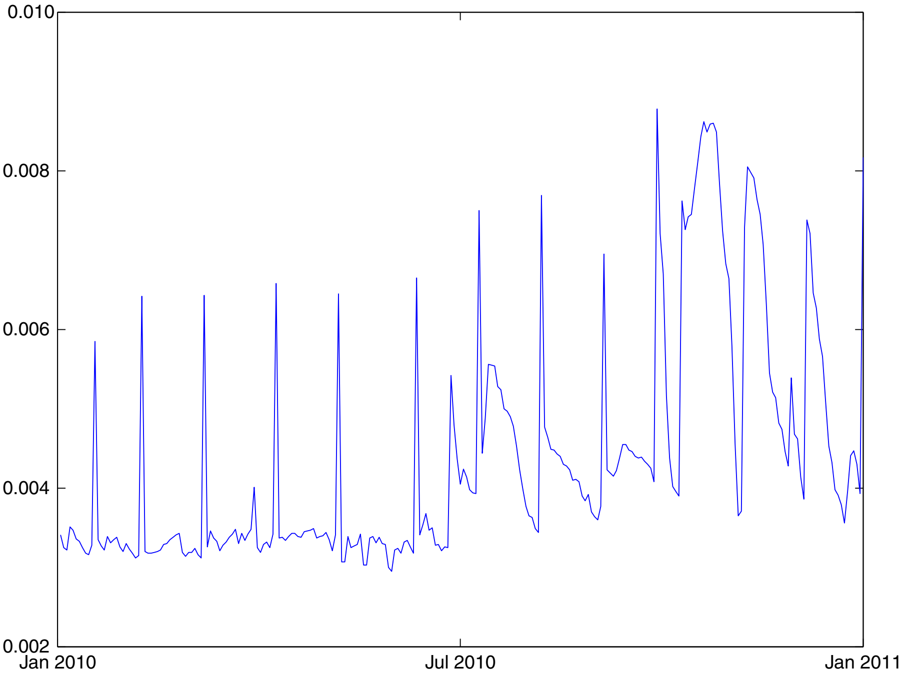


Fig. 1 Eonia overnight interest rate in 2010

of a stochastic differential equation for the instantaneous interest rate (short rate).

As a concrete example, let us consider the interest rates in European monetray union. Eonia (Euro OverNight Index Average) is an effective overnight rate computed as a weighted average of all overnight unsecured lending transactions in the interbank market, initiated within the euro area by the contributing panel banks. Euribor (Euro Interbank Offered Rate) is the rate at which euro interbank term deposits are being offered by one prime bank to another within the EMU zone (http://www.euribor-ebf.eu). We show the evolution of the overnight Eonia rate in 2010 in Fig. 1. Selected term structures consisting of Euribor rates from this time period are shown in Fig. 2.

In non-arbitrage term structure models the prices of financial derivatives, in particular bond prices (yielding the interest rates),[1] are given by a solution to a parabolic partial differential equation.

It is often assumed that the short rate $r = r(t)$ evolves according to the stochastic differential equation

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dw^{\mathbb{P}}, \tag{1}$$

[1] If P is the price of bond with maturity τ years, the interest rate R is given by $R = -\frac{\ln P}{\tau}$. To obtain interest rates in percentages, this value has to be multiplied by 100. Unless stating otherwise, we will not do it and we will consider interest rates as decimal numbers.

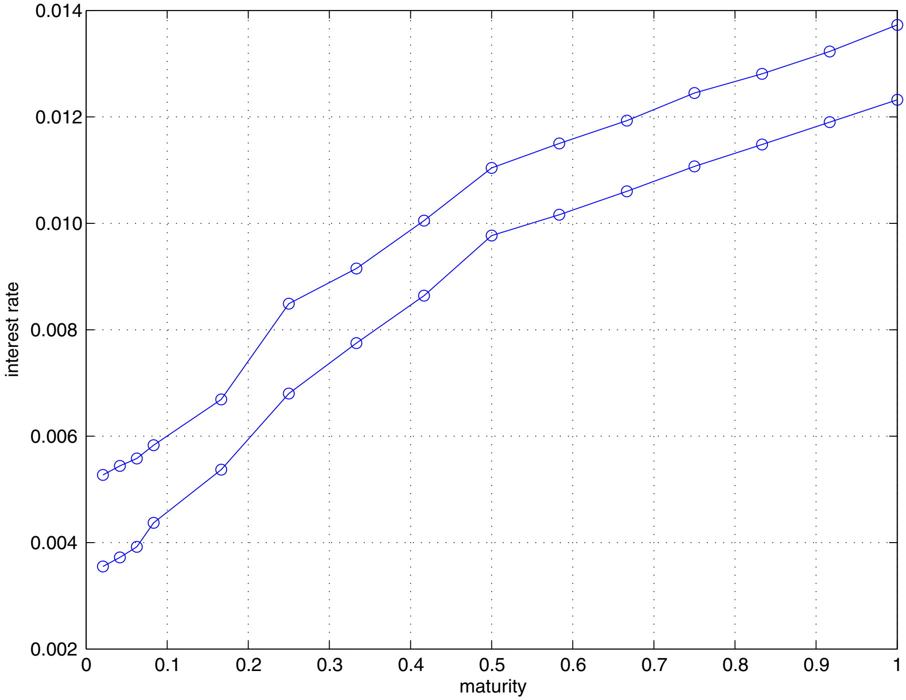


Fig. 2 Monthly averages of Euribor term structures in January 2010 (*bottom curve*) and July 2010 (*top curve*)

where $w^{\mathbb{P}}$ is a Wiener process in the probability measure $\mathbb{P}$ and the parameters satisfy $\alpha > 0, \beta < 0, \sigma > 0, \gamma \geq 0$. It means that the short rate r follows a mean-reverting process with the limit $(-\alpha/\beta)$ to which its value reverts (hence the name mean-reverting) and its volatility is given by σr^{γ}. Hence the parameter γ describes the dependence of the volatility on the current level of the short rate. To price the derivatives, one needs to specify so called market price of risk $\lambda = \lambda(r, t)$ which, with opposite sign, gives the increase in excepted instantaneous return of a bond for one unit of risk (cf. [7]).

This specification (1) includes the Vasicek model [13] with $\gamma = 0, \lambda(r, t) = \lambda$ and Cox–Ingersoll–Ross (CIR) model [5] with $\gamma = 1/2, \lambda(r, t) = \lambda\sqrt{r}$ in which the explicit solutions to bond pricing partial differential equations are known. With the exception of these two models, such an explicit solution is not available. However, estimating the general model (1) from the time series of short rates suggests that other choices of γ are more suitable—see the pioneering paper [2]; for an evidence from a wider range of data see, e.g., [12], where the data from eight countries were considered. In these cases no closed form expression for bond prices is available.

An alternative to specifying the model in the real probability measure $\mathbb{P}$ (i.e., where the probabilities refer to the observed ones) and by market price of risk, is specifying it in risk neutral measure $\mathbb{Q}$. It is an equivalent measure in which the derivatives prices can be computed by means of expected value. Volatilities of the process in both measures $\mathbb{P}$ and $\mathbb{Q}$ are the same; the drifts have the relation *(risk neutral drift)* = *(real drift)* − *(market price of risk)* × *(volatility)*, see [7].

For more details about interest rate models see, e.g., [1] and [7].

The paper is organized as follows. In the second section we present the partial differential equation for bond price in the given one-factor model, its approximate analytical solution and the order of accuracy. In the third section we present the numerical scheme for solving this equation and in the fourth section we present the results of a numerical experiment. In the fifth section we summarize our results.

2 Bond pricing PDE in one-factor model and approximate analytical formula for its solution

Suppose that the evolution of the short rate r in the risk neutral measure is given by

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dw^{\mathbb{Q}}, \tag{2}$$

with $\sigma > 0$, $\gamma \geq 0$. Note that for specific choices of market price of risk it leads to a mean reverting process (1) in the real probability measure. As an example, let us consider the process (2) with parameters $\alpha = 0.02$, $\beta = -1$, $\sigma = 0.35$, $\gamma = 1$. Simulations of the process for several choices of market prices of risk, together with depicted the limit value, are shown in Fig. 3. The simulations were performed using Euler–Maruyama discretization of the stochastic differential equation.

It is known that the price $P = P(\tau, r)$ of the zero-coupon with maturity τ, when the current level of the short rate is r, is a solution to the partial differential equation

$$-\frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2 r^{2\gamma} \frac{\partial^2 P}{\partial r^2} + (\alpha + \beta r)\frac{\partial P}{\partial r} - rP = 0, \;\; r > 0, \;\; \tau \in (0, T), \tag{3}$$

satisfying the initial condition $P(0, r) = 1$ for all $r > 0$, see, e.g., [1, 7].

In [4] the following approximate analytical solution was suggested. Its derivation is based on approximating the integral in the probabilistic representation of the solution.

Theorem 1 [4, Theorem 2] *The approximate analytical solution* P^{ap} *is given by*

$$\begin{aligned}\ln P^{\mathrm{ap}}(\tau, r) = -rB + \frac{\alpha}{\beta}(\tau - B) + \left(r^{2\gamma} + q\tau\right)\frac{\sigma^2}{4\beta}\left[B^2 + \frac{2}{\beta}(\tau - B)\right] \\ - q\frac{\sigma^2}{8\beta^2}\left[B^2(2\beta\tau - 1) - 2B\left(2\tau - \frac{3}{\beta}\right) + 2\tau^2 - \frac{6\tau}{\beta}\right]\end{aligned} \tag{4}$$

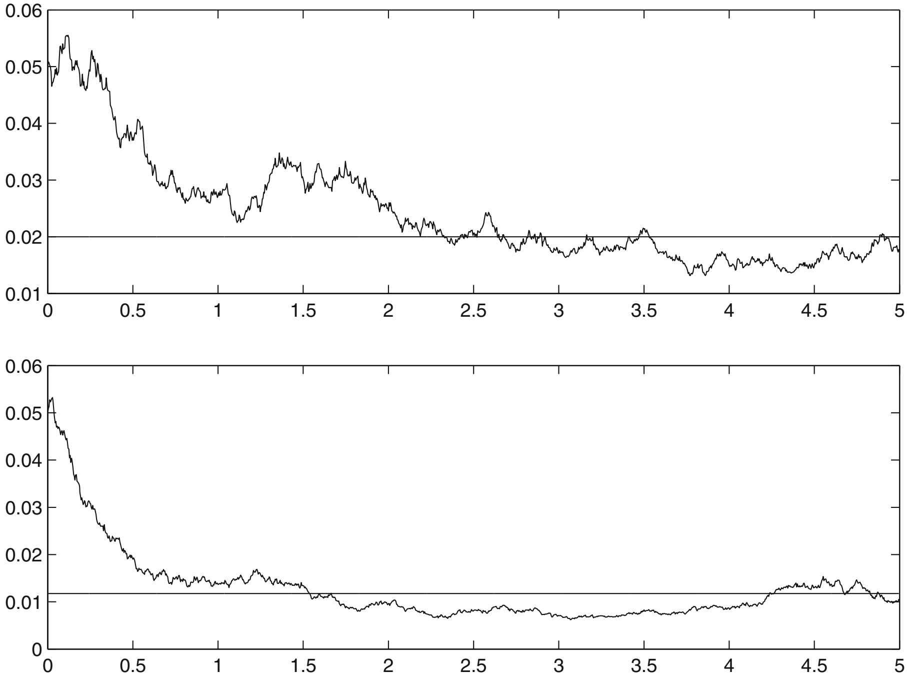


Fig. 3 Simulation of the short rate evolution with depicted limit value. Risk neutral parameters: $\alpha = 0.02$, $\beta = -1$, $\sigma = 0.35$, $\gamma = 1$; market price of risk: $\lambda = 0$ (*above*), $\lambda = -2$ (*below*)

where

$$q(r) = \gamma\,(2\gamma - 1)\,\sigma^2 r^{2(2\gamma-1)} + 2\gamma r^{2\gamma-1}(\alpha + \beta r) \tag{5}$$

and

$$B(\tau) = \left(e^{\beta\tau} - 1\right)/\beta. \tag{6}$$

The order of accuracy for this approximation has been derived in [10].

Theorem 2 [10, Theorem 3] *Let P^{ap} be the approximative solution given by* (4) *and P^{ex} be the exact bond price given as a unique complete solution*[2] *to* (3). *Then*

$$\ln P^{\text{ap}}(\tau, r) - \ln P^{\text{ex}}(\tau, r) = c_5(r)\tau^5 + O\left(\tau^6\right)$$

[2]The uniqueness of the solution (under some growth conditions) has been proved in [10] for $1/2 < \gamma < 3/2$ and for $\gamma = 1/2$ if the condition $2\alpha \geq \sigma^2$ is satisfied. If $\gamma = 1/2$ and $2\alpha < \sigma^2$, solution is not unique, as it has been shown in [6].

as $\tau \to 0^+$ *where*

$$c_5(r) = -\frac{1}{120}\gamma r^{2(\gamma-2)}\sigma^2 \left[2\alpha^2(-1+2\gamma)r^2 + 4\beta^2\gamma r^4 - 8r^{3+2\gamma}\sigma^2 + 2\beta\left(1-5\gamma+6\gamma^2\right)r^{2(1+\gamma)}\sigma^2 + \sigma^4 r^{4\gamma}(2\gamma-1)^2(4\gamma-3) + 2\alpha r\left(\beta(-1+4\gamma)r^2 + (2\gamma-1)(3\gamma-2)r^{2\gamma}\sigma^2\right)\right]. \quad (7)$$

Convergence is uniform w. r. to r on compact subintervals $[r_1, r_2] \subset (0, \infty)$.

Remark 1 The function $c_5(r)$ remains bounded as $r \to 0^+$ for the case of the CIR model in which $\gamma = 1/2$ or for the case when $\gamma \geq 1$. On the other hand, if $1/2 < \gamma < 1$, then $c_5(r)$ becomes singular, $c_5(r) = O(r^{2(\gamma-1)})$ as $r \to 0^+$.

Term structures, obtained from P^{ap} for the parameters $\alpha = 0.02$, $\beta = -1$, $\sigma = 0.35$, $\gamma = 1$, which were used in the previous simulations of the short rate process, are shown in Fig. 4.

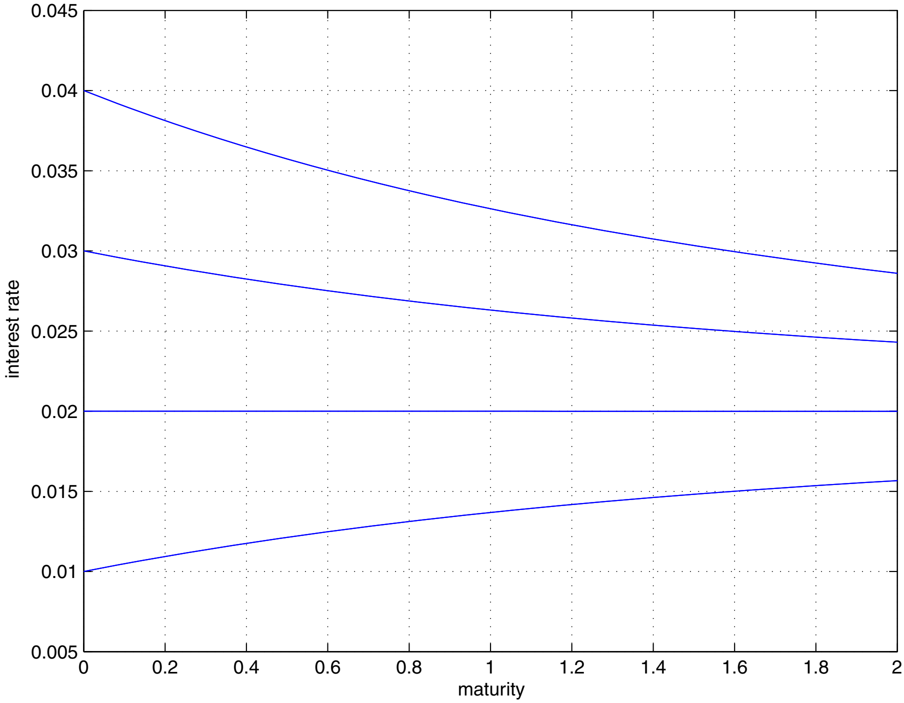


Fig. 4 Term structures of interest rates. Risk neutral parameters: $\alpha = 0.02$, $\beta = -1$, $\sigma = 0.35$, $\gamma = 1$

3 Numerical scheme

3.1 The differential problem

Recall that the bond pricing equation reads as

$$\frac{\partial P}{\partial \tau} = \frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 P}{\partial r^2} + (\alpha + \beta r)\frac{\partial P}{\partial r} - rP, \tag{8}$$

where $P = P(\tau, r)$, $\tau \in (0, T]$, $r \in [0, R)$, i. e., we cut the short rate interval to a finite one. We also assume, that $\gamma \geq 0.5$. We will explain a bit more about the restriction on γ.

Being different from conventional parabolic equation in which the coefficient of the 2nd order term is assumed to be bounded below by a positive constant, (8) with $\gamma > 0$ belongs to the second order differential equations with non-negative characteristic form. The main character of such kind of equations is *degeneracy*. By the well known Fichera's theory (see [8]) for degenerate parabolic equation (8) we have that at the degenerate boundary $r = 0$ the boundary condition should not be given. But if one attempts to obtain the numerical solution of (8) the *dynamic* boundary condition

$$\frac{\partial P}{\partial \tau}(\tau, 0) = \alpha \frac{\partial P}{\partial r}(\tau, 0), \quad \tau \in [0, T] \tag{9}$$

should be considered [11].

At $r = R$ we consider the Dirichlet boundary condition

$$P(\tau, R) = P_R(\tau), \quad \tau \in [0, T], \tag{10}$$

where $P_R(\tau)$ is a given function. The initial condition is given by

$$P(0, r) = 1, \quad r \in [0, R]. \tag{11}$$

Before discussing the discretization method, we first transform (8) in the self-adjoint form. The resulting equation can be written as

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial r}\left(k(r)\frac{\partial P}{\partial r} + m(r)P\right) - p(r)P, \tag{12}$$

where

$$k(r) = 0.5\sigma^2 r^{2\gamma}, \quad m(r) = \alpha + \beta r - \sigma^2\gamma r^{2\gamma-1},$$
$$p(r) = \beta - \sigma^2\gamma\,(2\gamma - 1)\, r^{2\gamma-2} + r.$$

Further in the numerical scheme we need to calculate the coefficient m at $r = 0$. That is why we consider the case $\gamma \geq 0.5$.

3.2 Short rate discretization

Let the interval $(0, R)$ be divided into N sub-intervals $I_i = (r_i, r_{i+1})$, $i = 0, 1, 2, \ldots, N-1$ with $0 = r_0 < r_1 < r_2 < \ldots < r_N = R$. Let $h_i = r_{i+1} - r_i$ for $i = 0, 1, 2, \ldots, N-1$ and $h = \max_{0 \leq i \leq N-1} h_i$. We also let $r_{i-1/2} = 0.5(r_{i-1} + r_i)$,

$r_{i+1/2} = 0.5(r_i + r_{i+1})$ for each $i = 1, 2, \ldots, N-1$. These mid-points form a second partition of the interval $(0, R)$ if we define $r_{-1/2} = r_0 = 0$ and $r_{N+1/2} = r_N = R$.

For interest rate discretization of the problem we will use Song Wang's method, described in [14] for the Black-Scholes equation. Let us note that earlier we have applied this method to a similar problem and our investigations shoved that this method gives better results near the points of degeneration of the equation in comparison to classical approximations [3].

The Song Wang's method is based on a finite volume formulation of the problem coupled with a fitted local approximation to the solution. The local approximation is determined by a set of two-point boundary value problem defined on the element edges.

Let us rewrite (12) in the form

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial r}\left(a(r)r^2\frac{\partial P}{\partial r} + b(r)rP\right) - p(r)P, \tag{13}$$

where

$$a(r) = \frac{k(r)}{r^2} = \frac{\sigma^2 r^{2\gamma-2}}{2}, \quad b(r) = \frac{m(r)}{r} = \frac{\alpha}{r} + \beta - \sigma^2\gamma r^{2\gamma-2}.$$

Integrating (13) over the interval $(r_{i-1/2},\ r_{i+1/2})$, $i = 1, 2, \ldots, N-1$ we get

$$\int_{r_{i-1/2}}^{r_{i+1/2}} \frac{\partial P}{\partial \tau}\, dr = \left[r\left(ar\frac{\partial P}{\partial r} + b\,P\right)\right]\Bigg|_{r_{i-1/2}}^{r_{i+1/2}} - \int_{r_{i-1/2}}^{r_{i+1/2}} pPdr.$$

Applying the mid-point quadrature rule to the integrals we obtain from the above

$$\left.\frac{\partial P}{\partial \tau}\right|_{r_i} \hbar_i = \left[r_{i+1/2}\ \rho(P)|_{r_{i+1/2}} - r_{i-1/2}\ \rho(P)|_{r_{i-1/2}}\right] - p_i P_i \hbar_i, \tag{14}$$

where $\hbar_i = r_{i+1/2} - r_{i-1/2}$, $p_i = p(r_i)$, P_i is the nodal approximation to $P(\tau, r_i)$ to be determined and $\rho(P)$ is the flux associated with P and defined by

$$\rho(P) = ar\frac{\partial P}{\partial r} + b\,P. \tag{15}$$

To derive an approximation to the flux at the end-points $r_{i+1/2}$ and $r_{i-1/2}$ we consider the following two-point boundary problem

$$\left(a_{i+1/2}rv' + b_{i+1/2}v\right)' = 0, \quad r \in I_i, \tag{16}$$

$$v(r_i) = P_i, \quad v(r_{i+1}) = P_{i+1}, \tag{17}$$

where $a_{i+1/2} = a\left(r_{i+1/2}\right)$, $\quad b_{i+1/2} = b\left(r_{i+1/2}\right)$. Integrating (16) yields the first order linear differential equation

$$\rho_i(v) = a_{i+1/2}rv' + b_{i+1/2}v = C_1,$$

where C_1 denotes an additive constant (depending on the time τ). The analytic solution of this equation is

$$v = C_2 r^{-\alpha_i} + \frac{C_1}{b_{i+1/2}}, \tag{18}$$

where C_2 is also an additive constant and $\alpha_i = \frac{b_{i+1/2}}{a_{i+1/2}}$. Applying the boundary conditions in (17) to the (18) and solving the corresponding linear system we get

$$\rho_i(P) = C_1 = b_{i+1/2} \frac{r_{i+1}^{\alpha_i} P_{i+1} - r_i^{\alpha_i} P_i}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}}, \tag{19}$$

which gives a representation for the flux $\rho(P)$ defined in (15) at the point $r_{i+1/2}$. Similarly, one can obtain the approximation of the flux at the point $r_{i-1/2}$ for $i = 2, 3, \ldots, N$:

$$\rho_{i-1}(P) = C_1 = b_{i-1/2} \frac{r_i^{\alpha_{i-1}} P_i - r_{i-1}^{\alpha_{i-1}} P_{i-1}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}}. \tag{20}$$

Let us note that the above analysis does not apply to the approximation of the flux on the interval $(0, r_1)$, because (13) is degenerate (see [14]). For the interval $(0, r_1)$ instead of problem (16), (17) we consider the problem

$$\left(a_{1/2} r v' + b_{1/2} v\right)' = C_1, \quad r \in I_0, \tag{21}$$

$$v(0) = P_0, \quad v(r_1) = P_1, \tag{22}$$

where C_1 is an unknown to be determined. One integration of (21) leads to

$$a_{1/2} r v' + b_{1/2} v = C_1 r + C_2.$$

Using first of the conditions (22) we get $b_{1/2} P_0 = C_2$ and then the above equation becomes

$$\rho_0(v) = a_{1/2} r v' + b_{1/2} v = C_1 r + b_{1/2} P_0. \tag{23}$$

Solving this equation analytically gives

$$v = \begin{cases} P_0 + \dfrac{C_1 r}{a_{1/2} + b_{1/2}} + C_3 r^{-\alpha_0}, & \alpha_0 \neq -1, \\ P_0 + \dfrac{C_1}{a_{1/2}} r \ln r + C_3 r, & \alpha_0 = -1, \end{cases}$$

where C_3 is an additive constant, depending on τ and $\alpha_0 = \frac{b_{1/2}}{a_{1/2}}$. Next consider the case $\alpha_0 \neq -1$ and using boundary conditions (22) we obtain

$$C_3 = 0, \quad C_1 = \frac{1}{r_1} \left(a_{1/2} + b_{1/2}\right) \left(P_1 - P_0\right).$$

When $\alpha_0 = -1$, boundary conditions (22) lead to

$$C_1 = 0, \quad C_3 = \frac{P_1 - P_0}{r_1}.$$

Therefore from (23) we get that in both cases—$\alpha_0 \neq -1$ and $\alpha_0 = -1$

$$v = P_0 + \frac{P_1 - P_0}{r_1} r, \quad r \in I_0,$$

and then for $\rho_0(P)$ we have

$$\rho_0(P) = \frac{1}{2}\left[\left(a_{1/2} + b_{1/2}\right) P_1 - \left(a_{1/2} - b_{1/2}\right) P_0\right]. \tag{24}$$

Let us note, that the problem under consideration in [14] has Dirichlet boundary conditions. In our problem the boundary condition at $r = 0$ is dynamic. To obtain an approximation to the dynamic boundary condition (9) let us integrate the (13) over the interval $(r_{-1/2},\ r_{1/2}) \equiv (r_0,\ r_{1/2}) \equiv (0,\ r_{1/2})$:

$$\int_0^{r_{1/2}} \frac{\partial P}{\partial \tau}\, dr = \left[r\left(ar\frac{\partial P}{\partial r} + b\, P \right)\right]\Bigg|_0^{r_{1/2}} - \int_0^{r_{1/2}} pP\, dr.$$

Then

$$\left.\frac{\partial P}{\partial \tau}\right|_{r=0} \frac{h_0}{2} = \left[r_{1/2}\ \rho(P)|_{r_{1/2}} - (mP)|_{r=0}\right] - p_0 P_0 \frac{h_0}{2}. \tag{25}$$

Using $r_{1/2} = 0.5h_0$ and $m|_{r=0} = \alpha$ (note that we suppose $\gamma \geq 0.5$) from (25) we get

$$\left.\frac{\partial P}{\partial t}\right|_{r=0} = \rho(P)|_{r_{1/2}} - \left(p_0 + \frac{2\alpha}{h_0}\right) P_0 + f_0. \tag{26}$$

Now, substituting the expressions (19), (20) and (24) in (14) and (26) we obtain

$$\begin{aligned}
\left.\frac{\partial P}{\partial \tau}\right|_{r=r_0=0} &= \frac{1}{2}\left[\left(a_{1/2} + b_{1/2}\right) P_1 - \left(a_{1/2} - b_{1/2}\right) P_0\right] - \left(p_0 + \frac{2\alpha}{h_0}\right) P_0,\\
\left.\frac{\partial P}{\partial \tau}\right|_{r=r_1} \hbar_1 &= \left[r_{3/2} b_{3/2} \frac{r_2^{\alpha_1} P_2 - r_1^{\alpha_1} P_1}{r_2^{\alpha_1} - r_1^{\alpha_1}} - \frac{h_0}{4}\left[\left(a_{1/2} + b_{1/2}\right) P_1 - \left(a_{1/2} - b_{1/2}\right) P_0\right]\right]\\
&\quad - \hbar_1 p_1 P_1,\\
\left.\frac{\partial P}{\partial \tau}\right|_{r=r_i} \hbar_i &= \left[r_{i+1/2} b_{i+1/2} \frac{r_{i+1}^{\alpha_i} P_{i+1} - r_i^{\alpha_i} P_i}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}} - r_{i-1/2} b_{i-1/2} \frac{r_i^{\alpha_{i-1}} P_i - r_{i-1}^{\alpha_{i-1}} P_{i-1}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}}\right]\\
&\quad - \hbar_i p_i P_i, \quad i = 2, 3, \ldots, N-1,\\
P|_{r=r_N=R} &= P_R(\tau, R).
\end{aligned} \tag{27}$$

The scheme (27) is first order accurate in the interest rate grid size h [14].

3.3 Time discretization

The system (27) is a first order linear ODE system. To discretize this system, we let τ_k, $(k = 0, 1, \ldots, K)$ be a set of partition points in the interval $[0, T]$, $0 = \tau_0 < \tau_1 < \ldots < \tau_K = T$, $\Delta\tau_k = \tau_{k+1} - \tau_k$, $\Delta\tau = \max\limits_{0 \le k \le K-1} \tau_k$. Then, we apply the two-level time-stepping method with splitting parameter $\theta \in [0, 1]$ to yield

$$\begin{aligned}\frac{\hat{P}_0 - P_0}{\Delta\tau_k} &= \frac{\theta}{2}\left[\left(a_{1/2} + b_{1/2}\right)\hat{P}_1 - \left(a_{1/2} - b_{1/2}\right)\hat{P}_0\right] \\ &\quad + \frac{1-\theta}{2}\left[\left(a_{1/2} + b_{1/2}\right)P_1 - \left(a_{1/2} - b_{1/2}\right)P_0\right] \\ &\quad - \theta\left(p_0 + \frac{2\alpha}{h_0}\right)\hat{P}_0 - (1-\theta)\left(p_0 + \frac{2\alpha}{h_0}\right)P_0, \end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\hat{P}_1 - P_1}{\Delta\tau_k}\hbar_1 &= \theta\left[r_{3/2}b_{3/2}\frac{r_2^{\alpha_1}\hat{P}_2 - r_1^{\alpha_1}\hat{P}_1}{r_2^{\alpha_1} - r_1^{\alpha_1}} - \frac{h_0}{4}\left[\left(a_{1/2}+b_{1/2}\right)\hat{P}_1 - \left(a_{1/2}-b_{1/2}\right)\hat{P}_0\right]\right] \\ &\quad + (1-\theta)\left[r_{3/2}b_{3/2}\frac{r_2^{\alpha_1}P_2 - r_1^{\alpha_1}P_1}{r_2^{\alpha_1} - r_1^{\alpha_1}} - \frac{h_0}{4}\right. \\ &\quad \left.\times\left[\left(a_{1/2} + b_{1/2}\right)P_1 - \left(a_{1/2} - b_{1/2}\right)P_0\right]\right] \\ &\quad - \theta\hbar_1 p_1\hat{P}_1 - (1-\theta)\hbar_1 p_1 P_1, \end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\hat{P}_i - P_i}{\Delta\tau_k}\hbar_i &= \theta\left[r_{i+1/2}b_{i+1/2}\frac{r_{i+1}^{\alpha_i}\hat{P}_{i+1} - r_i^{\alpha_i}\hat{P}_i}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}} - r_{i-1/2}b_{i-1/2}\frac{r_i^{\alpha_{i-1}}\hat{P}_i - r_{i-1}^{\alpha_{i-1}}\hat{P}_{i-1}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}}\right] \\ &\quad + (1-\theta)\left[r_{i+1/2}b_{i+1/2}\frac{r_{i+1}^{\alpha_i}P_{i+1} - r_i^{\alpha_i}P_i}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}}\right. \\ &\quad \left. - r_{i-1/2}b_{i-1/2}\frac{r_i^{\alpha_{i-1}}P_i - r_{i-1}^{\alpha_{i-1}}P_{i-1}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}}\right] \\ &\quad - \theta\hbar_i p_i\hat{P}_i - (1-\theta)\hbar_i p_i P_i, \quad i = 2, 3, \ldots, N-1, \end{aligned} \tag{30}$$

$$\hat{P}_N = P_R\left(\tau_{k+1}, R\right), \tag{31}$$

where $\hat{P}_i$ denotes the approximation of P at the point (τ_{k+1}, r_i) and P_i—the approximation of P at the point (τ_k, r_i), $i = 0, 1, \ldots, N$, $k = 0, 1, \ldots, K-1$. When $\theta = 0$ the scheme is explicit, when $\theta = 0.5$ the scheme is Crank–Nicolson and when $\theta = 1$ it is implicit scheme. The latter two schemes are absolutely stable and they are second and first order accuracy, respectively [14].

At every time layer τ_{k+1} system (28–31) has three diagonal matrix and we have solved it by Thomas procedure [9].

Theorem 3 *For any given $k = 1, 2, \ldots, K-1$ if $\Delta\tau_k$ is sufficiently small, the system matrix of (28–31) can be reduced to an M-matrix.*

Proof Let us rewrite the system (28–31) in the form

$$\left| \begin{array}{l} C_0 \hat{P}_0 + B_0 \hat{P}_1 \ = F_0, \\ A_i \hat{P}_{i-1} + C_i \hat{P}_i + B_i \hat{P}_{i+1} = F_i, \quad i = 1, 2, \ldots N-1, \\ A_N \hat{P}_{N-1} + C_N \hat{P}_N \qquad = F_N, \end{array} \right.$$

where

$$C_0 = \frac{1}{\Delta\tau_k} + \frac{\theta}{2}\left(a_{1/2} - b_{1/2}\right) + \theta\left(p_0 + \frac{2\alpha}{h_0}\right), \quad B_0 = -\frac{\theta}{2}\left(a_{1/2} + b_{1/2}\right),$$

$$F_0 = \left[\frac{1}{\Delta\tau_k} - \frac{1-\theta}{2}\left(a_{1/2} - b_{1/2}\right) - (1-\theta)\left(p_0 + \frac{2\alpha}{h_0}\right)\right] P_0 + \frac{1-\theta}{2}\left(a_{1/2} + b_{1/2}\right) P_1,$$

$$A_1 = -\frac{\theta h_0}{4}\left(a_{1/2} - b_{1/2}\right), \quad B_1 = -\theta r_{3/2} b_{3/2} \frac{r_2^{\alpha_1}}{r_2^{\alpha_1} - r_1^{\alpha_1}},$$

$$C_1 = \frac{\hbar_1}{\Delta\tau_k} + \theta r_{3/2} b_{3/2} \frac{r_1^{\alpha_1}}{r_2^{\alpha_1} - r_1^{\alpha_1}} + \frac{\theta h_0}{4}\left(a_{1/2} + b_{1/2}\right) + \theta \hbar_1 p_1,$$

$$F_1 = \left[\frac{\hbar_1}{\Delta\tau_k} - (1-\theta)\left(r_{3/2} b_{3/2} \frac{r_1^{\alpha_1}}{r_2^{\alpha_1} - r_1^{\alpha_1}} + \frac{h_0}{4}\left(a_{1/2} + b_{1/2}\right) + \hbar_1 p_1\right)\right] P_1 + (1-\theta)\left(\frac{h_0}{4}\left(a_{1/2} - b_{1/2}\right) P_0 + r_{3/2} b_{3/2} \frac{r_2^{\alpha_1}}{r_2^{\alpha_1} - r_1^{\alpha_1}} P_2\right),$$

$$A_i = -\theta r_{i-1/2} b_{i-1/2} \frac{r_{i-1}^{\alpha_{i-1}}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}}, \quad B_i = -\theta r_{i+1/2} b_{i+1/2} \frac{r_{i+1}^{\alpha_i}}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}},$$

$$C_i = \frac{\hbar_i}{\Delta\tau_k} + \theta r_{i+1/2} b_{i+1/2} \frac{r_i^{\alpha_i}}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}} + \theta r_{i-1/2} b_{i-1/2} \frac{r_i^{\alpha_{i-1}}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}} + \theta p_i \hbar_i,$$

$$F_i = \left[\frac{\hbar_i}{\Delta\tau_k} - (1-\theta)\left(\frac{r_{i+1/2} b_{i+1/2} r_i^{\alpha_i}}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}} + \frac{r_{i-1/2} b_{i-1/2} r_i^{\alpha_{i-1}}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}} + \hbar_i p_i \right) \right] P_i,$$

$$+ (1-\theta)\left(\frac{r_{i-1/2} b_{i-1/2} r_{i-1}^{\alpha_{i-1}}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}} P_{i-1} + \frac{r_{i+1/2} b_{i+1/2} r_{i+1}^{\alpha_i}}{r_{i+1}^{\alpha_i} - r_i^{\alpha_i}} P_{i+1} \right),$$

$$i = 2, 3 \ldots N-1,$$

$$A_N = 0, \quad C_N = 1, \quad F_N = P_R(\tau_{k+1}, R).$$

Let us first investigate the off-diagonal elements of the system matrix A_i and B_i, $i = 2, 3, \ldots, N-1$. From the formula for A_i we have $A_i < 0$. That is because

$$A_i = -\theta r_{i-1/2} b_{i-1/2} \frac{r_{i-1}^{\alpha_{i-1}}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}} = -\theta r_{i-1/2} a_{i-1/2} \alpha_{i-1} \frac{r_{i-1}^{\alpha_{i-1}}}{r_i^{\alpha_{i-1}} - r_{i-1}^{\alpha_{i-1}}}$$

$$= -\theta r_{i-1/2} a_{i-1/2} \frac{\alpha_{i-1}}{\bar{r}_{i-1} - 1} < 0, \quad 0 < \bar{r}_{i-1} = \frac{r_i^{\alpha_{i-1}}}{r_{i-1}^{\alpha_{i-1}}} < 1$$

for each $i = 2, 3, \ldots, N-1$.We have used that $\theta r_{i-1/2} a_{i-1/2} > 0$ and $\bar{r}_i - 1$ has just the sign of α_{i-1}.

At the same way one can prove that $B_i < 0$. We should also note that C_i is always positive since $\Delta\tau_k$ is sufficiently small. For $i = N$ $A_N = 0$, $C_N = 1$. More over

$$C_i + A_i + B_i = \frac{\hbar_i}{\Delta\tau_k} - \theta\left(r_{i+1/2} b_{i+1/2} - r_{i-1/2} b_{i-1/2} \right) + \theta p_i \hbar_i \geq 0$$

if $\Delta\tau_k$ is sufficiently small and $C_N + A_N = 1 > 0$.

Different is situation for $i = 0, 1$. For the first three equations we find

$$\tilde{C}_2 = C_2 - \frac{A_2 B_1}{E}, \quad \tilde{F}_2 = F_2 + \frac{A_2 D}{E},$$

$$\hat{P}_0 = \frac{F_0}{C_0} - \frac{B_0}{C_0} \hat{P}_1, \quad \hat{P}_1 = \frac{D}{E} - \frac{B_1}{E} \hat{P}_2,$$

$$E = C_1 - \frac{A_1 B_0}{C_0}, \quad D = F_1 - \frac{A_1 F_0}{C_0},$$

$$\tilde{C}_2 \hat{P}_2 + B_2 \hat{P}_3 = \tilde{F}_2,$$

$$\tilde{C}_2 = C_2 - \frac{A_2 B_1}{E}, \quad \tilde{F}_2 = F_2 - \frac{A_2 D}{E}.$$

When $\Delta\tau_k$ is sufficiently small $E > 0$, $E = O\left(\frac{1}{\Delta\tau_k}\right)$, $C_2 = O\left(\frac{1}{\Delta\tau_k}\right)$. Therefore for $\Delta\tau_k$ sufficiently small we have $\tilde{C}_2 > 0$, $\tilde{C}_2 = O\left(\frac{1}{\Delta\tau_k}\right)$ and $\tilde{C}_2 + B_2 \geq 0$.

As a result we obtain a system of linear algebraic equation with unknowns $\hat{P}_2, \hat{P}_3, \ldots, \hat{P}_N$ which matrix is a M-matrix. □

While $F_3, \ldots, F_N$ are non-negative, we have to prove that $\tilde{F}_2$ is also non-negative. From the formulae for $\tilde{F}_2$ it follows that when $\Delta\tau_k$ is small $\tilde{F}_2$ is non-negative since $F_2 = O\left(\frac{1}{\Delta\tau_k}\right)$ and D, E are of the same order with respect to $\Delta\tau_k$.

Since the load vector $(\tilde{F}_2, F_3, \ldots F_N)$ is non-negative and corresponding matrix is a M-matrix we can conclude that $\hat{P}_2, \ldots \hat{P}_N$ are non-negative. Finally, using the formulas for $\hat{P}_0$ and $\hat{P}_1$ one can easily check that they are non-negative too if $\Delta\tau_k$ is small.

Remark 2 Theorem 3 shows that the fully discretized system (28–31) satisfies the discrete maximum principle and because of that fact the above discretization is monotone. This guarantees the following: for non-negative initial function $P(0, r)$ the numerical solution $\hat{P}_i$, obtained via this method, is also non-negative as expected, because the price of the bond is a positive number.

4 Numerical experiments

We perform numerical experiments for the following values of the coefficients in (8): $\sigma = 0.35,\ \gamma = 1,\ \alpha = 0.02,\ \beta = -1$. Note that these values were used to show examples of interest rates behaviour in Section 2. We take $R = 0.2$ (as a sufficiently high level of the short rate; it corresponds to 20%) and $T = 1$ (the approximate analytical solution, which we use at the right boundary condition, as well as for the comparison, has the accuracy derived for $\tau \to 0^+$; hence T cannot be too large).

In order to investigate numerically the convergence and the accuracy of the constructed schemes, we approximately solve the model problem with known

Table 1 Norms of the error

N	$C-$norm of the error	L_2-norm of the error
8	4.168 E-3	5.171 E-4
16	1.623 E-3	1.567 E-4
32	1.086 E-3	6.379 E-5
64	7.753 E-3	3.600 E-5
128	4.441 E-4	1.825 E-5

analytical solution $P(\tau, r) = P^{\text{ap}}(\tau, r)$ described in Section 2:

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial r}\left(k(r)\frac{\partial P}{\partial r} + m(r)P\right) - p(r)P + f(\tau, r), \quad \tau \in (0, T], \quad r \in [0, R),$$

$$\frac{\partial P}{\partial \tau}(\tau, 0) = \alpha\frac{\partial P}{\partial r}(\tau, 0) + f(\tau, 0), \quad \tau \in [0, T],$$

$$P(\tau, R) = P_R(\tau), \quad \tau \in [0, T],$$

$$P(0, r) = 1, \quad r \in [0, R].$$

Let us note that in the model problem in equation and in the boundary condition at $r = 0$ in the right hand site appears the function

$$f = \frac{\partial P^{\text{ap}}}{\partial \tau} - k(r)\frac{\partial^2 P^{\text{ap}}}{\partial r^2} - (\alpha + \beta r)\frac{\partial P^{\text{ap}}}{\partial r} + rP^{\text{ap}}.$$

and $P_R(\tau) = P^{\text{ap}}(\tau, R)$.

Below we present some results from computational experiments for Crank–Nicolson scheme.

Table 1 contains the computed errors of $z = P - P^{\text{ap}}$ in C and L_2 grid norms, where

$$\|z\|_C = \max_{\substack{0 \le i \le N,\\ 0 \le k \le K}} \left|P_i^k - P_i^{\text{ap},k}\right|, \quad \|z\|_{L_2} = \sqrt{\sum_{i=0}^{N-1}\sum_{k=0}^{K-1} h_i \Delta\tau_k \left(P_i^k - P_i^{\text{ap},k}\right)^2}.$$

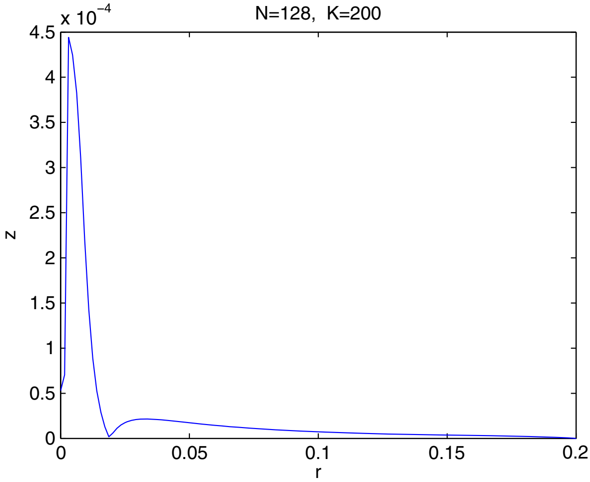
Everywhere the calculations are performed with constant time step $\Delta\tau_k = \Delta\tau = 0.005$. It can be seen from the Table 1 that the Crank–Nicolson scheme converges in both grid norms.

We use Runge method for practical estimation of the *rate of convergence s* of the scheme with respect to the space variable r at fixed value of $\tau = T$. In the case when the exact solution $P^{\text{ap}}(r, \tau)$ of the model problem is known, the

Table 2 Rate of convergence

1.84	0.49	1.03	1.55	1.95	1.97	1.98	1.99	1.98	1.98	1.97
1.98	1.98	1.98	1.98	1.98	1.98	1.99	1.99	1.99	1.99	1.99
1.99	1.99	1.99	1.99	1.99	2.00	2.00	2.00	2.00	2.00	1.99
1.99	1.99	1.99	1.99	1.99	1.99	1.99	1.99	1.99	1.99	1.99
1.99	1.98	1.98	1.98	1.98	1.98	1.98	1.97	1.97	1.97	1.97
1.97	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96		

Fig. 5 $z = |P_{\mathrm{num}}(T, r) - P^{\mathrm{ap}}(T, r)|$

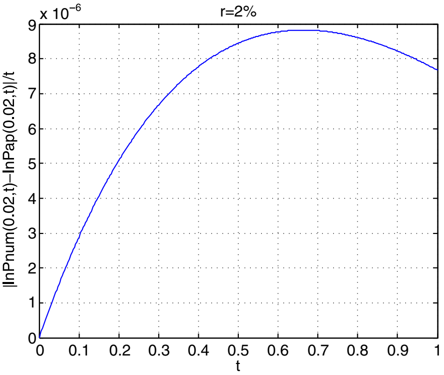


formula for s is

$$s = \ln \left| \frac{P^{\mathrm{ap}}(r, T) - P_h(r, T)}{P^{\mathrm{ap}}(r, T) - P_{h/2}(r, T)} \right| / \ln 2,$$

where $P_h(r, T)$ and $P_{h/2}(r, T)$ are computed values of the numerical solution at point r on the grid with step h and $h/2$ respectively. We use two inserted grids with 64 and 128 nodes in the interval $[0, R]$ and the analytical solution P^{ap} of the model problem. Results from calculations are presented in Table 2. It can be seen from Table 2 that the order of convergence rate is about two, when the node is not too close to the point of degeneration.

In Fig. 5 we present the difference between the approximate analytical and numerical solutions $z = |P - P^{\mathrm{ap}}|$ over the interval $[0, R]$ at the terminal time



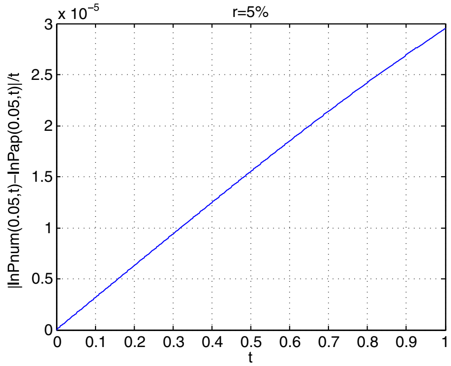


Fig. 6 The difference $|R_{\mathrm{num}}(\tau, r) - R^{\mathrm{ap}}(\tau, r)|$. The short rate r equals 0.02 (*left*) and 0.05 (*right*)

Table 3 Interest rates in percentages rounded to market precision, computed from approximate analytical solution P^{ap} and numerical solution P_{num}

Maturity	1M	2M	3M	4M	5M	6M
Analytical	3.919	3.842	3.769	3.701	3.635	3.573
Numerical	3.919	3.843	3.770	3.702	3.637	3.575
	7M	8M	9M	10M	11M	12M
Analytical	3.515	3.459	3.406	3.356	3.308	3.263
Numerical	3.517	3.461	3.409	3.359	3.312	3.266

The short rate r is 5.000%

$\tau = T$. It can be seen from Fig. 5 that this difference is the largest near to the point of degeneration of the (8).

In Fig. 6 we present the error in interest rates, i.e., $|R_{num} - R^{ap}|$ where $R_{num} = -\frac{\ln P_{num}(\tau,r)}{\tau}$ and $R^{ap} = -\frac{\ln P^{ap}(\tau,r)}{\tau}$. This error can be related to the precision of the market data. Let us consider, for example, Euribor. These rates are quoted in percentages rounded to three decimal points. Hence the difference in the last decimal place corresponds to 10^{-5} order of difference in R. A numerical example is shown in Table 3.

5 Conclusion

We presented a numerical scheme for solving bond pricing partial differential equation in one-factor short rate model. This scheme is constructed using the Song Wang's method, which is based on a finite volume formulation of the problem coupled with a fitted local approximation to the solution. We show that the system matrix of the discretization scheme is a M-matrix and performed numerical experiments for a meaningful set of parameters. We computed the norms of the error of the numerical solution with respect to the approximate analytical solution and the experimental rate of convergence. Finally, we considered the error in interest rates relative to the precision of the market data.

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