$$B(t,T) = E_Q^t\left[e^{-\int_t^T r(u)\,du}\right]. \tag{7.2.2}$$

Once the dynamics of the short rate $r(t)$ is specified, we are able to compute the bond price. This is why most earlier interest rate models are based on the characterization of the dynamics of the short rate.

7.2.1 Short Rate Models and Bond Prices

Assume that the short rate r_t follows the Ito process as described by the following stochastic differential equation

$$dr_t = \mu(r_t,t)\,dt + \rho(r_t,t)\,dZ_t, \tag{7.2.3}$$

where dZ_t is the differential of the standard Brownian process, $\mu(r_t,t)$ and $\rho(r_t,t)^2$ are the instantaneous drift and variance. We would like to derive the governing differential equation for the bond price using the no arbitrage argument. Since the short rate is not a traded security, the differential equation is expected to involve the market price of risk of r_t. The prices of bonds with varying maturities are shown to satisfy certain consistency relations in order to ensure absence of arbitrage opportunities. We express the bond price in terms of the expectation under the physical measure, and from which we deduce the Radon–Nikodym derivative for the change of measure from the physical measure to the risk neutral measure.

Throughout this section, we assume the bond price to be dependent on r_t only, independent of default risk, liquidity and other factors. If we write the bond price as $B(r,t)$ (suppressing T when there is no ambiguity and dropping the time index t in stochastic processes of r_t, Z_t, etc.), then the use of Ito's lemma gives the dynamics of the bond price as

$$dB = \left(\frac{\partial B}{\partial t} + \mu\frac{\partial B}{\partial r} + \frac{\rho^2}{2}\frac{\partial^2 B}{\partial r^2}\right)dt + \rho\frac{\partial B}{\partial r}dZ. \tag{7.2.4}$$

When the above dynamics of $B(r,t)$ is expressed in the following lognormal form

$$\frac{dB}{B} = \mu_B(r,t)\,dt + \quad {}_B(r,t)\,dZ,$$

the drift rate $\mu_B(r,t)$ and volatility $\sigma_B(r,t)$ of the bond price process are found to be

$$\mu_B(r,t) = \frac{1}{B}\left(\frac{\partial B}{\partial t} + \mu\frac{\partial B}{\partial r} + \frac{\rho^2}{2}\frac{\partial^2 B}{\partial r^2}\right) \tag{7.2.5a}$$

$$\sigma_B(r,t) = \frac{\rho}{B}\frac{\partial B}{\partial r}. \tag{7.2.5b}$$

Since the short rate is not a traded security, it cannot be used to hedge with the bond, like the role of the underlying asset in an equity option. Instead, we try to hedge bonds of different maturities. This is possible because the instantaneous returns on bonds of varying maturities are correlated as there exists the common underlying stochastic short rate that drives the bond prices. The following portfolio is constructed: we buy a bond of dollar value V_1 with maturity T_1 and sell another bond of dollar value V_2 with maturity T_2. The portfolio value Π is given by

$$\Pi = V_1 - V_2.$$

According to the bond price dynamics defined by (7.2.4), the change in portfolio value in time dt is

$$\begin{aligned} d\Pi = &\left[V_1\mu_B(r,t;T_1) - V_2\mu_B(r,t;T_2)\right]dt \\ &+ \left[V_1\sigma_B(r,t;T_1) - V_2\sigma_B(r,t;T_2)\right]dZ. \end{aligned}$$

Suppose V_1 and V_2 are chosen such that

$$V_1 = \frac{\sigma_B(r,t;T_2)}{\sigma_B(r,t;T_2) - \sigma_B(r,t;T_1)}\Pi \quad \text{and} \quad V_2 = \frac{\sigma_B(r,t;T_1)}{\sigma_B(r,t;T_2) - \sigma_B(r,t;T_1)}\Pi,$$

then the stochastic term in $d\Pi$ vanishes and the equation becomes

$$\frac{d\Pi}{\Pi} = \frac{\mu_B(r,t;T_1)\sigma_B(r,t;T_2) - \mu_B(r,t;T_2)\sigma_B(r,t;T_1)}{\sigma_B(r,t;T_2) - \sigma_B(r,t;T_1)}dt. \tag{7.2.6a}$$

Since the portfolio is instantaneously riskless, in order to avoid arbitrage opportunities, it must earn the riskless short rate so that

$$d\Pi = r\Pi\, dt. \tag{7.2.6b}$$

Combining (7.2.6a,b), we obtain

$$\frac{\mu_B(r,t;T_1) - r}{\sigma_B(r,t;T_1)} = \frac{\mu_B(r,t;T_2) - r}{\sigma_B(r,t;T_2)}.$$

The above relation is valid for arbitrary maturity dates T_1 and T_2, so the ratio $\frac{\mu_B(r,t)-r}{\sigma_B(r,t)}$ should be independent of maturity T. Let the common ratio be defined by $\lambda(r,t)$, that is,

$$\frac{\mu_B(r,t) - r}{\sigma_B(r,t)} = \lambda(r,t). \tag{7.2.7}$$

The quantity $\lambda(r,t)$ is called the *market price of risk* of the short rate (see Problem 7.4), since it gives the extra increase in expected instantaneous rate of return on a bond per an additional unit of risk. In a market that admits no arbitrage opportunity, bonds that are hedgeable among themselves should have the same market price of risk, regardless of maturity. If we substitute $\mu_B(r,t)$ and $\sigma_B(r,t)$ into (7.2.7), we obtain the following governing differential equation for the price of a zero-coupon bond

$$\frac{\partial B}{\partial t} + \frac{\rho^2}{2}\frac{\partial^2 B}{\partial r^2} + (\mu - \lambda\rho)\frac{\partial B}{\partial r} - rB = 0, \quad t < T, \tag{7.2.8}$$