and also $\langle x \quad = \langle m \rangle h = 0$, which is not surprising, given the symmetry of the walk. Furthermore

$$\langle m^2 \quad = 4 \left\langle k^2 \right\rangle - 4N \langle k \rangle + N^2 = \quad {}^2 + N - 2N^2 + N^2 = N$$

from which

$$\sqrt{\langle x^2 \rangle} = \sqrt{N} h \tag{2.79}$$

which is the *standard deviation of* x, since $\langle x \rangle = 0$. Formula (2.79) contains a key information: at time $N\tau$, the distance from the origin is of order $\sqrt{N}h$, that is **the order of the time scale is the square of the space scale**. In other words, if we want to leave the standard deviation unchanged in the limit process, we must rescale the time as the square of the space, that is we must use a *space-time parabolic dilation*!

But let us proceed step by step. The next one is to deduce a *difference equation* for the transition probability $p = p(x,t)$. It is on this equation that we will carry out the limit procedure.

2.4.2 The limit transition probability

The particle motion has no memory since each move is independent from the previous one. If the particle location at time $t + \tau$ is x, this means that at time t its location was at $x - h$ or at $x + h$, with equal probability. The total probability formula then gives

$$p(x, t + \tau) = \frac{1}{2} p(x - h, t) + \frac{1}{2} p(x + h, t) \tag{2.80}$$

with the initial conditions

$$p(0,0) = 1 \qquad \text{and} \qquad p(x,0) = 0 \quad \text{if } x \neq 0.$$

Keeping fixed x and t, let us examine what happens when $h \to 0, \tau \to 0$. It is convenient to think of p as a smooth function, defined in the whole half plane $\mathbb{R} \times (0, +\infty)$ and not only at the discrete set of points $(mh, N\tau)$. In addition, by passing to the limit, we will find a continuous probability distribution so that $p(\quad)$, being the probability to find the particle at (x,t), should be zero. If we interpret p as a *probability density*, this inconvenience disappears. Using Taylor's formula we can write[26]

$$p(x, t + \tau) = p(x,t) + p_t(x,t)\tau + o(\tau),$$

$$p(x \pm h, t) = p(x,t) \pm p_x(x,t) h + \frac{1}{2} p_{xx}(x,t) h^2 + o\left(h^2\right).$$

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[26] The symbol $o(z)$, (“little o of ”) denotes a quantity of lower order with respect to z; precisely

$$\frac{(z)}{z} \to 0 \qquad \text{when } z \to 0$$

Substituting into (2.80), after some simplifications, we find

$$p_t\tau + o(\tau) = \frac{1}{2}p_{xx}h^2 + o\left(h^2\right).$$

Dividing by τ,

$$p_t + o(1) = \frac{1}{2}\frac{h^2}{\tau}p_{xx} + o\left(\frac{h^2}{\tau}\right). \tag{2.81}$$

This is the crucial point; in the last equation we meet again the combination $\frac{h^2}{\tau}$!!

If we want to obtain something non trivial when $h, \tau \to 0$, **we must require that h^2/τ has a finite and positive limit**; the simplest choice is to keep

$$\frac{h^2}{\tau} = 2D \tag{2.82}$$

for some number $D > 0$ (the number 2 is there for aesthetic reasons only).

Passing to the limit in (2.81), we get for p the equation

$$p_t = Dp_{xx} \tag{2.83}$$

while the initial condition becomes

$$\lim_{t\to 0^+} p(x,t) = \delta. \tag{2.84}$$

We have already seen that the unique solution of (2.83), (2.84) is

$$p(x,t) = \Gamma_D(x,t)$$

since

$$\int_{\mathbb{R}} p(x,t)\,dx = 1.$$

Thus, the constant D in (2.82) is precisely the *diffusion coefficient.* Recalling that

$$h^2 = \frac{\langle x^2 \rangle}{N}, \qquad \tau = \frac{t}{N}$$

we have

$$\frac{h^2}{\tau} = \frac{\langle x^2 \rangle}{t} = 2D$$

that means: *in unit time, the particle diffuses an average distance of* $\sqrt{2}$. It is worthwhile to recall that the dimensions of D are

$$[D] = [length]^2 \times [time]^{-1}$$

and that the combination x^2/Dt is *dimensionless*, not only invariant by parabolic dilations. Also, from (2.82) we deduce

$$\frac{h}{\tau} = \frac{2D}{h} \to +\infty. \tag{2.85}$$

This shows that the average speed h/τ of the particle at each step becomes un-

average distance is purely due to the rapid fluctuations of its motion.