The *Markov property* can be stated as follows: given any condition H, related to the behavior of the particle before time $s \geq 0$, the process $Y(t) = B^x(t+s)$ is a Brownian motion with initial distribution[32]

$$\mu(I) = P^x \{B^x(s) \in I \,|H\}.$$

This property establishes the independence of the *future* process $B^x(t+s)$ from the *past* (absence of memory) when the *present* $B^x(\ )$ is known and reflects the *absence of memory* of the random walk.

In the strong *Markov property*, s is substituted by a random time τ, depending only on the behavior of the particle in the interval $[0, \tau]$. In other words, to decide whether or not the event $\{\tau \leq t\}$ is true, it is enough to know the behavior of the particle up to time t. These kinds of random times are called *stopping times*. An important example is the *first exit time* from a domain, that we will consider in the next chapter. Instead, the random time defined by

$$\tau = \inf\{t : B(t) > 10 \text{ and } B(t+1) < 10\}$$

is *not* a stopping time. Indeed (measuring time in *seconds*), τ is "the smallest" among the times t such that the Brownian path is above level 10 at time t, and after one second is below 10. Clearly, to decide whether $\tau \leq 3$, say, it is not enough to know the path up to time $t = 3$, since τ involves the behavior of the path up to the *future* time $t = 4$.

• *Expectation*. Given a sufficiently smooth function $g = g(\ )$, $y \in \mathbb{R}$, we can define the random variable

$$Z(t) = (g \circ B^x)(t) = g(B^x(t)).$$

Its expected value is given by the formula

$$E^x[Z(t)] = \int_{\mathbb{R}} g(y) P(x, t, dy) = \int_{\mathbb{R}} g(y) \Gamma(y - x, t)\, dy.$$

We will meet this formula in a completely different situation later on.

2.5 Diffusion, Drift and Reaction

2.5.1 Random walk with drift

The hypothesis of symmetry of our random walk can be removed. Suppose our unit mass particle moves along the x axis with space step $h > 0$, every time interval of duration $\tau > 0$, according to the following rules (Fig. 2.9).

1. The particle starts from $x = 0$.
2. It moves to the right with probability $p_0 \neq \frac{1}{2}$ and to the left with probability $q_0 = 1 - p_0$ independently of the previous step.

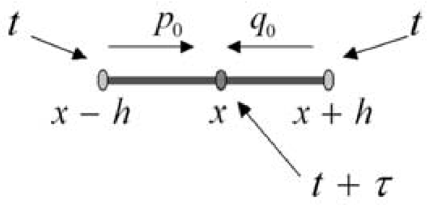

Fig. 2.9. Random walk with drift

Rule 2 breaks the symmetry of the walk and models a particle tendency to move to the right or to the left, according to the sign of $p_0 - q_0$ being positive or negative, respectively. Again we denote by $p = p(x,t)$ the probability that the particle location is $x = mh$ at time $t = N\tau$. From the total probability formula we have:

$$p(x,t+\tau) = \ _0p(x-h,t) + q_0 p(x+h,t) \tag{2.87}$$

with the usual initial conditions

$$p(0,0) = 1 \quad \text{and} \quad p(x,0) = 0 \quad \text{if } x \neq 0.$$

As in the symmetric case, keeping x and t fixed, we want to examine what happens when we pass to the limit for $h \to 0, \tau \to 0$. From Taylor formula, we have

$$p(x,t+\tau) = p(x,t) + p_t(x,t)\tau + o(\tau),$$

$$p(x \pm h,t) = p(x,t) \pm p_x(x,t)h + \frac{1}{2}p_{xx}(x,t)h^2 + o\left(h^2\right).$$

Substituting into (2.87), we get

$$p_t\tau + o(\tau) = \frac{1}{2}p_{xx}h^2 + (q_0 - p_0)hp_x + o\left(h^2\right). \tag{2.88}$$

A new term appears: $(q_0 - p_0)hp_x$. Dividing by τ, we obtain

$$p_t + o(1) = \frac{1}{2}\frac{h^2}{\tau}p_{xx} + \boxed{\frac{(q_0 - p_0)h}{\tau}p_x} + o\left(\frac{h^2}{\tau}\right). \tag{2.89}$$

Again, here is the crucial point. If we let $h, \tau \to 0$, we realize that the assumption

$$\frac{h^2}{\tau} = 2D \tag{2.90}$$

alone is not sufficient anymore to get something non trivial from (2.89): indeed, if we keep p_0 and $\ _0$ constant, we have

$$\frac{(q_0 - p_0)h}{\tau} \to \infty$$

[32] $P(A|H)$ denotes the conditional probability of A, given .

and from (2.89) we get a contradiction. What else we have to require? Writing

$$\frac{(q_0 - p_0)\,h}{\tau} = \frac{(q_0 - p_0)}{h}\frac{h^2}{\tau}$$

we see we must require, in addition to (2.90), that

$$\frac{q_0 - p_0}{h} \to \beta \tag{2.91}$$

with β finite. Notice that, since $q_0 + p_0 = 1$, (2.91) is equivalent to

$$p_0 = \frac{1}{2} - \frac{\beta}{2}h + o(h) \quad \text{and} \quad q_0 = \frac{1}{2} + \frac{\beta}{2}h + o(h)\,, \tag{2.92}$$

that could be interpreted as a *symmetry of the motion at a microscopic scale.*

With (2.91) at hand, we have

$$\frac{(q_0 - p_0)}{h}\frac{h^2}{\tau} \to 2D\beta \equiv b$$

and (2.89) becomes in the limit,

$$p_t = Dp_{xx} + bp_x. \tag{2.93}$$

We already know that Dp_{xx} models a diffusion phenomenon. Let us *unmask* the term bp_x, by first examining the dimensions of b. Since $q_0 - p_0$ is dimensionless, being a difference of probabilities, the dimensions of b are those of h/τ, namely of a **velocity**.

Thus the coefficient b codifies the tendency of the limiting continuous motion, to move towards a privileged direction with speed $|b|$: to the right if $b < 0$, to the left if $b > 0$. In other words, there exists a *current of intensity* $|b|$ driving the particle. *The random walk has become a* **diffusion process with drift**.

The last point of view calls for an analogy with the diffusion of a substance transported along a channel.

2.5.2 Pollution in a channel

In this section we examine a simple convection-diffusion model of a pollutant on the surface of a narrow channel. A water stream of constant speed v transports the pollutant along the positive direction of the x axis. We can neglect the depth of the water (thinking to a floating pollutant) and the transverse dimension (thinking of a very narrow channel).

Our purpose is to derive a mathematical model capable of describing the evolution of the concentration[33] $= c\,(x,t)$ of the pollutant. Accordingly, the integral

$$\int_x^{x+\Delta x} c\,(y,t)\,dy \tag{2.94}$$

[33] $[c] = [mass] \times [\quad]^{-1}$.