$$\frac{\partial b}{\partial t}(t,T) + \mu_1(t)b(t,T) - \frac{\alpha_1(t)}{2}b^2(t,T) + 1 = 0, \quad b(T,T) = 0, \tag{7.2.19a}$$

$$\frac{\partial a}{\partial t}(t,T) - \mu_0(t)b(t,T) + \frac{\alpha_0^2(t)}{2}b^2(t,T) = 0, \quad a(T,T) = 0. \tag{7.2.19b}$$

The nonlinear differential equation for $b(t,T)$ is called the *Ricatti equation*. For some special cases of $\mu_1(t)$ and $\alpha_1(t)$, it is possible to derive a closed form solution to $b(t,T)$. Once the analytic solution to $b(t,T)$ is available, we can obtain $a(t,T)$ by direct integration of (7.2.19b). In the next two sections, we consider two renowned short rate models that admit the bond price solution in an affine form.

7.2.2 Vasicek Mean Reversion Model

Vasicek (1977) proposed the stochastic process for the short rate r_t under the physical measure to be governed by the Ornstein–Uhlenbeck process:

$$dr_t = \alpha(\gamma - r_t)\,dt + \rho\,dZ_t, \quad \alpha > 0. \tag{7.2.20}$$

The above process is sometimes called the elastic random walk or *mean reversion process*. The instantaneous drift $\alpha(\gamma - r_t)$ represents the effect of pulling the process toward its long-term mean   with magnitude proportional to the deviation of the process from the mean. The mean reversion assumption agrees with the economic phenomenon that interest rates appear over time to be pulled back to some long-run average value. To explain the mean reversion phenomenon, we argue that when interest rates increase, the economy slows down and there is less demand for loans; this leads to the tendency for rates to fall. The stochastic differential equation (7.2.20) can be integrated to give

$$r(T) = \gamma + [r(t) - \gamma]e^{-\alpha(T-t)} + \rho\int_t^T e^{-\alpha(T-t)}\,dZ(t). \tag{7.2.21}$$

Due to the Brownian term in the stochastic integral, it is possible that the short rate may become negative under the Vasicek model. Conditional on the current level of short rate $r(t)$, the mean of the short rate at T is found to be

$$E[r(T)|r(t)] = \quad + [r(t) - \gamma]e^{-\alpha(T-t)}. \tag{7.2.22}$$

The variance of the mean reversion process is governed by

$$\frac{d}{dt}\,\mathrm{var}(r(t)) = -2\alpha\,\mathrm{var}(r(t)) + \;^2.$$

By observing the initial condition that the variance at the current time is zero (see Problem 7.11), we obtain

$$\mathrm{var}(r(T)|r(t)) = \frac{\rho^2}{2\alpha}\left[1 - e^{-2\alpha(T-t)}\right., \quad t < T. \tag{7.2.23}$$

Analytic Bond Price Formula

Suppose we assume the market price of risk λ to be constant, independent of r and t, then it is possible to derive an analytic formula for the bond price under the Vasicek model. The Vasicek mean reversion model corresponds to $\mu_0 = \alpha\gamma - \lambda\rho, \mu_1 = -\alpha, \alpha_0 = \rho$ and $\alpha_1 = 0$ in (7.2.18). We obtain the following pair of differential equations for $a(t,T)$ and $b(t,T)$:

$$\frac{da}{dt} + (\lambda\rho - \alpha\gamma)b + \frac{\rho^2}{2}b^2 = 0, \quad t < T$$
$$\frac{db}{dt} - \alpha b + 1 = 0, \quad t < T,$$

with final conditions: $a(T,T) = 0$ and $b(T,T) = 0$. Solving the coupled system of differential equations, we obtain

$$B(r,t;T) = \exp\left(\frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right](R_\infty - r) - R_\infty(T-t) - \frac{\rho^2}{4\alpha^3}\left[1 - e^{-\alpha(T-t)}\right]^2\right), \quad t < T, \tag{7.2.24}$$

where $R_\infty = \gamma - \frac{\rho\lambda}{\alpha} - \frac{\rho^2}{2\alpha^2}$ [R_∞ is actually equal to $\lim_{T\to\infty} R(t,T)$, see (7.2.26)]. Using (7.2.5a,b), the mean and standard deviation of the instantaneous rate of return of a bond maturing at time T are found to be

$$\mu_B(r,t;T) = r(t) + \frac{\rho\lambda}{\alpha}\left[1 - e^{-\alpha(T-t)}\right] \tag{7.2.25a}$$
$$\sigma_B(r,t;T) = \frac{\rho}{\alpha}\left[1 - e^{-\alpha(T-t)}\right]. \tag{7.2.25b}$$

The yield to maturity is found to be

$$R(t,T) = R_\infty + \frac{[r(t) - R_\infty][1 - e^{-\alpha(T-t)}]}{\alpha(T-t)} + \frac{\rho^2}{4\alpha^3(T-t)}[1 - e^{-\alpha(T-t)}]^2. \tag{7.2.26}$$

By taking T , the last two terms in (7.2.26) vanish so that the long-term internal rate of return is seen to be constant. Note that $R(t,T)$ and $\ln B(r,t;T)$ are linear functions of $r(t)$. Since $r(t)$ is normally distributed, it then follows that $R(t,T)$ is also normally distributed and $B(r,t;T)$ is lognormally distributed. Suppose we set $T = T_1$ and $T = T_2$ in (7.2.26), and subsequently eliminate $r(t)$, we obtain a relation between $R(t,T_1)$ and $R(t,T_2)$ that is dependent only on the parameter values.

Readers are invited to explore additional properties of the term structures of the yield curve associated with the Vasicek model in Problem 7.12. Also, a discrete version of the Vasicek model is presented in Problem 7.13.

7.2.3 Cox–Ingersoll–Ross Square Root Diffusion Model

Recall that the short rate may become negative under the Vasicek model due to its Gaussian nature. To rectify the problem, Cox, Ingersoll and Ross (1985) proposed the following square root diffusion process for the short rate:

$$dr_t = \alpha(\gamma - r_t)\,dt + \rho\sqrt{r_t}\,dZ_t, \quad \alpha, \gamma > 0. \tag{7.2.27}$$

With an initially nonnegative interest rate, r_t will never be negative. This is attributed to the mean-reverting drift rate that tends to pull r_t towards the long-run average γ and the diminishing volatility as r_t declines to zero (recall that volatility is constant in the Vasicek model). It can be shown that r_t can reach zero only if $\rho^2 > 2$; while the upward drift is sufficiently strong to make $r_t = 0$ impossible when $2\alpha\gamma \geq \rho^2$ [for a rigorous proof, see Cairns, 2004]. A heuristic argument is presented below. Define $L_t = \ln r_t$, then by Ito's lemma, the differential of L_t is found to be

$$dL = \left[\left(\alpha\gamma - \frac{\rho^2}{2}\right)e^{-L} - \alpha\right]dt + \rho e^{-L/2}\,dZ. \tag{7.2.28}$$

The drift and volatility coefficients are well behaved for positive L but they may blow up for large negative L. If $2\alpha\gamma < \rho^2$, the drift becomes negative for large negative L, pulling L further toward $-\infty$. This indicates that $2\alpha\gamma \geq \rho^2$ is a necessary condition for the short rate process to remain strictly positive.

The probability density of the short rate at time T, conditional on its value at the current time t, is given by

$$g(r(T); r(t)) = ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q\big(2(uv)^{1/2}\big), \tag{7.2.29}$$

where

$$c = \frac{2\alpha}{\rho^2\left[1 - e^{-\alpha(T-t)}\right]}, \quad u = cr(t)e^{-\alpha(T-t)}, \quad v = cr(T), \quad q = \frac{2\alpha\gamma}{\rho^2} - 1,$$

and I_q is the modified Bessel function of the first kind of order q [see Feller, 1951 for details]. The mean and variance of $r(T)$ conditional on $r(t)$ are given by (see Problem 7.11)

$$E[r(T)|r(t)] = r(t)e^{-\alpha(T-t)} + \gamma\big[1 - e^{-\alpha(T-t)}\big] \tag{7.2.30a}$$

$$\operatorname{var}(r(T)|r(t)) = r(t)\left(\frac{\rho^2}{\alpha}\right)\big[e^{- \quad -t)} - e^{-2\alpha(T-t)}\big] + \frac{\gamma\rho^2}{2\alpha}\big[1 - e^{-\alpha(T-t)}\big]^2. \tag{7.2.30b}$$

The distribution of the future short rates has the following properties:

(i) as $\alpha \to \infty$, the mean tends to γ and the variance to zero;
(ii) as $\alpha \to 0^+$, the mean tends to $r(t)$ and the variance to $^2(T - t)r(t)$.

The Cox–Ingersoll–Ross model falls within the class of affine term structure models, so the price of the discount bond assumes the same form as in (7.2.14).

The corresponding pair of differential equations for $a(t, T)$ and $b(t, T)$ are given by

$$\frac{da}{dt} - \alpha\gamma b = 0, \qquad t < T, \tag{7.2.31a}$$

$$\frac{db}{dt} - (\alpha + \lambda\rho)b - \frac{\rho^2}{2}b^2 + 1 = 0, \quad t < T, \tag{7.2.31b}$$

where the market price of risk is taken to be $\lambda\sqrt{r}$, and λ is assumed to be constant. The final conditions are

$$a(T, T) = 0 \quad \text{and} \quad b(T, T) = 0.$$

The solutions to the above equations are found to be (Cox, Ingersoll and Ross, 1985)

$$a(t, T) = \frac{2\alpha\gamma}{\rho^2} \ln\left(\frac{2\theta e^{(\theta+\psi)(T-t)/2}}{(\theta + \psi)\left[e^{\theta(T-t)} - 1\right] + 2\theta}\right) \tag{7.2.32a}$$

$$b(t, T) = \frac{2\left[e^{\theta(T-t)} - 1\right]}{(\theta + \psi)\left[e^{\theta(T-t)} - 1\right] + 2\theta}, \tag{7.2.32b}$$

where

$$\psi = \alpha + \lambda\rho, \quad \theta = \sqrt{\psi^2 + 2\ \ ^2}.$$

Note that the market price of risk λ appears only through the sum ψ in the above solution. The properties of the comparative statics for the bond price and the yield to maturity of the Cox–Ingersoll–Ross model are addressed in Problems 7.15–7.17.

7.2.4 Generalized One-Factor Short Rate Models

Besides the Vasicek and Cox–Ingersoll–Ross models, several other one-factor short rate models have also been proposed in the literature. Many of these models can be nested within the stochastic process represented by

$$dr_t = (\alpha + \beta r_t)dt + \rho r_t^{\ } \, dZ_t, \tag{7.2.33}$$

where the parameters α, β, γ and   are constants. For example, the Vasicek and Cox–Ingersoll–Ross models correspond to $\gamma = 0$ and $\gamma = 1/2$, respectively, and the Geometric Brownian model corresponds to $\alpha = 0$ and $\gamma = 1$. The stochastic interest rate model used by Merton (1973, Chap. 1) can be nested within the Vasicek model with $\beta = 0$ and $\gamma = 0$. Other examples of one-factor interest rate models nested within the stochastic process of (7.2.33) are:

Dothan model (1978)	$dr = \rho r\, dZr$
Brennan–Schwartz model (1980)	$dr = (\alpha + \beta r)dt + \rho r\, dZ$
Cox–Ingersoll–Ross variable rate model (1980)	$dr = \rho r^{3/2} dZ$
Constant elasticity of variance model	$dr = \beta r\, dt + \rho r^{\ } \, dZ$