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Existence of Weighted Interior-Point Paths in Semidefinite Programming

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Abstract

Semidefinite programming (SDP) is a special class of convex programming, which has been recently intensively studied because of its applicability to various areas, such as combinatorial optimization, system and control theory or mechanical and electrical engineering. Moreover, SDP problems can be efficiently solved by interior point methods (IPM). The most important concept in the IPM theory is the central path. It is an analytic curve in the interior of the feasible set which tends to an optimal point at the boundary. The properties of the central path are important for designing and analyzing of the IPM algorithms. In this paper we will study the existence of different types of so-called weighted interior point paths.

Keywords: semidefinite programming, interior point methods, weighted central path

1 *INTRODUCTION*

Semidefinite programming is a special and relatively new field of mathematical programming. It contains important classes of problems as special cases, such as linear programming, general convex quadratic programming or so called second-order-cone programming. SDP has many interesting applications in combinatorial optimization (MAX-CUT problem), quasiconvex programming, spectral analysis (min-max eigenvalue problem), engineering (system and control theory, optimal truss design). SDP problems can be solved in polynomial time by interior point algorithms. More about semidefinite programming theory and applications can be found in [4, 13].

The central path is crucial in the study of IMP. Most interior point methods follow the central path to reach an optimal solution. Since the behavior of the central path is important for the interior point algorithms, its properties are intensively studied (see e.g. [1, 2, 3]).

1.1 *Semidefinite programming problems*

Consider S^n - the vector space of $n \times n$ symmetric matrices, with the inner product "$\bullet$" defined as $\mathbf{X} \bullet \mathbf{Y} = tr(\mathbf{XY})$. Denote S^n_+ (S^n_{++}) the closed (open) convex cone of all positive semidefinite (positive definite) matrices. For $\mathbf{X} \in S^n$ we will write $\mathbf{X} \succeq 0$ ($\mathbf{X} \succ 0$) if $\mathbf{X} \in S^n_+$ ($\mathbf{X} \in S^n_{++}$).

Let $\mathbf{A}_1, \ldots, \mathbf{A}_m, \mathbf{C} \in S^n$ and $b \in R^m$ are given. Then the primal semidefinite programming problem can be expressed in the form

$$\begin{array}{lrcl} minimize & \mathbf{X} \bullet \mathbf{C} & & \\ subject\ to & \mathbf{A}_i \bullet \mathbf{X} & = & b_i, \quad for\ all\ i = 1, \ldots, m, \\ & \mathbf{X} & \succeq & 0, \end{array} \tag{1}$$

where $\mathbf{X} \in S^n$ is the variable. The dual semidefinite programming problem is

$$\begin{array}{rrcl} \textit{maximize} & b^T y & & \\ \textit{subject to} & \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} & = & \mathbf{C}, \\ & \mathbf{S} & \succeq & 0, \end{array} \tag{2}$$

where $(\mathbf{S}, y) \in S^n \times R^m$ are the variables. We will denote

$$\mathcal{P}^0 = \{\mathbf{X} \in S^n \mid \mathbf{A}_i \bullet \mathbf{X} = b_i, i = 1, \dots, m; \mathbf{X} \succ 0\}$$

and

$$\mathcal{D}^0 = \{(\mathbf{S}, y) \in S^n \times R^m \mid \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}; \mathbf{S} \succ 0\}$$

the primal and the dual strictly feasible set, respectively.

1.2 Central path in semidefinite programming

The following two assumptions are usually made in semidefinite programming:
(A1) The matrices $\mathbf{A}_1, \dots, \mathbf{A}_m$ are linearly independent.
(A2) $\mathcal{P}^0 \neq \emptyset, \mathcal{D}^0 \neq \emptyset$.
The assumption A1 ensures the one-to one correspondence between the dual variables y and $\mathbf{S}$. The assumption A2 (also referred to as the interior point assumption) follows from the duality theorem and the both assumptions together are equivalent to the fact, that the optimal solution sets are nonempty and bounded (see [12]).
Under these assumptions the well known necessary and sufficient conditions of optimality hold:

$(\mathbf{X}, y, \mathbf{S})$ is optimal if and only if

$$\begin{array}{rl} \mathbf{A}_i \bullet \mathbf{X} = b_i, & i = 1, \dots, m, \quad \mathbf{X} \succeq 0, \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, & \mathbf{S} \succeq 0, \\ \mathbf{X}\mathbf{S} = 0. & \end{array} \tag{3}$$

The first condition is the primal feasibility, the second condition is the dual feasibility and the third is the complementarity condition. Interior point methods usually work with the system

$$\begin{array}{rl} \mathbf{A}_i \bullet \mathbf{X} = b_i, & i = 1, \dots, m, \quad \mathbf{X} \succ 0, \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, & \mathbf{S} \succ 0, \\ \mathbf{X}\mathbf{S} = \mu \mathbf{I}. & \end{array} \tag{4}$$

where, in comparison with (3), the complementarity condition is perturbed. It is well known that (under the assumptions (A1), (A2)) for any $\mu > 0$ there exists unique solution $(\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu))$ of (4). [1] Then the central path for semidefinite programming can be well defined as the set

$$\{\ (\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu)) \mid \mu > 0\}$$

or alternatively as the map

$$R_{++} \to S^n \times R^m \times S^n, \qquad \mu \to (\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu)).$$

[1] This result can be proved by defining the primal and dual logarithmic barrier problems associated with (1) and (2) and follows from the strict convexity (concavity) of the primal (dual) barrier function. For the proof see e.g. [4].

1.3 Symmetrization of the complementarity condition

It is well known that the product of two symmetric matrices is not necessary symmetric. This may cause problems in the interior point algorithms, which are based on solving the system (4). Therefore the matrix $\mathbf{XS}$ is replaced with a symmetrization matrix $\Phi(\mathbf{X}, \mathbf{S}) \in S^n$ and they are equivalent in the following way:

$$\text{If } \mathbf{X} \succeq 0, \mathbf{S} \succeq 0 \text{ then } \mathbf{XS} = 0 \text{ if and only if } \Phi(\mathbf{X}, \mathbf{S}) = 0.$$

In semidefinite programming and semidefinite complementarity problems the following symmetrization maps are discussed (see [1], [5]-[10]):

$$\begin{array}{rcl} \Phi_1(\mathbf{X}, \mathbf{S}) & = & (\mathbf{XS} + \mathbf{SX})/2 \\ \Phi_2(\mathbf{X}, \mathbf{S}) & = & \mathbf{X}^{\frac{1}{2}}\mathbf{S}\mathbf{X}^{\frac{1}{2}} \\ \Phi_3(\mathbf{X}, \mathbf{S}) & = & \mathbf{L}_{\mathbf{X}}^T\mathbf{S}\mathbf{L}_{\mathbf{X}}^T \\ \Phi_4(\mathbf{X}, \mathbf{S}) & = & (\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}} + \mathbf{S}^{\frac{1}{2}}\mathbf{X}^{\frac{1}{2}})/2 \\ \Phi_5(\mathbf{X}, \mathbf{S}) & = & (\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}} + \mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}}^T)/2 \end{array} \tag{5}$$

where $\mathbf{X}^{\frac{1}{2}}, \mathbf{S}^{\frac{1}{2}}$ are the square roots of the matrices $\mathbf{X}, \mathbf{S}$; $\mathbf{L}_{\mathbf{X}}$ is the lower Cholesky factor of the matrix $\mathbf{X}$ and $\mathbf{U}_{\mathbf{S}}$ is the upper Cholesky factor of the matrix $\mathbf{S}$. Let us note, that if $\mathbf{X} \succeq 0, \mathbf{S} \succeq 0$, then the matrices $\Phi_2(\mathbf{X}, \mathbf{S}), \Phi_3(\mathbf{X}, \mathbf{S})$ are positive semidefinite, however the other are not in general.

1.4 Motivation and goal

In linear programming, the concept of the central path can be easily extended to the concept of the weighted central path—by defining weighted logarithmic barrier functions. However this technique can not be applied to semidefinite programming. There were more approaches of how to define the weighted central path in SDP. One of them was developed by Monteiro et al. (see [5], [8]) originally for nonlinear semidefinite complementarity problems. Following this approach one can define the weighted central path for SDP as the set $\{\ (\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu)) \mid \mu > 0\}$ of the solutions of the parameterized systems

$$\begin{array}{c} \mathbf{A}_i \bullet \mathbf{X} = b_i + \mu\triangle b_i, \quad i = 1, \ldots, m, \quad \mathbf{X} \succ 0, \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C} + \mu\triangle\mathbf{C}, \quad \mathbf{S} \succ 0, \\ \Phi_j(\mathbf{X}, \mathbf{S}) = \phi_j(\mu)\mathbf{W}, \end{array} \tag{6}$$

where $\triangle b \in R^m, \triangle\mathbf{C} \in S^n$ are fixed, $\mathbf{W} \succ 0$ is the weight, $\Phi_j(\mathbf{X}, \mathbf{S})$ is one of the symmetrization maps in (5) and

$$\phi_j(\mu) = \mu, \quad j = 1, 2, 3; \qquad \phi_j(\mu) = \sqrt{\mu}, \quad j = 4, 5.$$

Therefore, according to the symmetrization map we will distinguish five types of weighted paths in semidefinite programming.

The authors Monteiro and Zanjacomo ([8]) have proved the existence of the weighted paths in nonlinear semidefinite complementarity problems using deep results from nonlinear analysis, based on the theory of the local homeomorphic maps. The another approach, used by Preiss and Stoer ([9]) was more elementary, it was essentially based on the implicit function theorem. However, latter authors proved the existence of the weighted path in linear complementarity problem associated only with the symmetrization $\Phi_1(\mathbf{X}, \mathbf{S})$. The same symmetrization and technique was used in [11] for the existence of the weighted central path in SDP. In this paper we extend the result [11] to all five symmetrizations defined in (5).

2 PRELIMINARIES

All the symmetrization maps in (5) have very useful properties, which we state in the following lemmas.

Lemma 1 *If* $\mathbf{X} \succeq 0, \mathbf{S} \succeq 0$*, then*
a) $\mathbf{X} \bullet \mathbf{S} = tr(\Phi_j(\mathbf{X}, \mathbf{S})),\ j = 1, 2, 3$*;*
b) $\mathbf{X} \bullet \mathbf{S} \leq 2\ tr(\Phi_j(\mathbf{X}, \mathbf{S})^2),\ j = 4, 5.$

Proof. The statement a) follows directly from the properties of the trace. We will prove the statement b). From the assumptions of the lemma it follows that the matrices $\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}}$ and $\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}$ have nonnegative eigenvalues. But for every square matrix $\mathbf{A}$ with nonnegative eigenvalues it holds that

$$tr\Big(\frac{\mathbf{A}+\mathbf{A}^T}{2}\Big)^2 = tr\Big(\frac{\mathbf{A}^2+\mathbf{A}\mathbf{A}^T+\mathbf{A}^T\mathbf{A}+(\mathbf{A}^T)^2}{4}\Big) = tr\Big(\frac{\mathbf{A}^2+\mathbf{A}\mathbf{A}^T}{2}\Big) \geq tr\Big(\frac{\mathbf{A}\mathbf{A}^T}{2}\Big).$$

The rest of the proof follows from the fact that

$$\mathbf{X} \bullet \mathbf{S} = tr(\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}}\mathbf{X}^{\frac{1}{2}}) = tr(\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}\mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}}^T).$$

■

Lemma 2 *Let* $j \in \{1, \ldots, 5\}$ *be arbitrary,* $\mathbf{X} \succeq 0$*,* $\mathbf{S} \succeq 0$ *and* $\Phi_j(\mathbf{X}, \mathbf{S}) \succ 0$*. Then* $\mathbf{X} \succ 0$*,* $\mathbf{S} \succ 0$*.*

Proof. The statement for j=2,3 is obvious. If j=1, and $\mathbf{X} \succeq 0$ is singular, then $\mathbf{Q}^T\mathbf{X}\mathbf{Q} = \mathbf{D} = diag(d_1, \ldots, d_k, 0, \ldots, 0)$ for some orthogonal matrix $\mathbf{Q}$ and hence the matrix

$$\mathbf{Q}^T(\mathbf{X}\mathbf{S} + \mathbf{S}\mathbf{X})\mathbf{Q} = \mathbf{D}\mathbf{Q}^T\mathbf{S}\mathbf{Q} + \mathbf{Q}^T\mathbf{S}\mathbf{Q}\mathbf{D}$$

is singular. However this contradicts the assumption. If j=4, the proof is the same. The statement for j=5 follows from the fact, that if $\mathbf{X}$ is singular, then there exists an index i such that $(\mathbf{L}_{\mathbf{X}})_{ii} = 0$. Then also $(\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}} + \mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}})_{ii} = 0$, but this contradicts the assumption.

■

Lemma 3 *Let* $\nu > 0$ *and* $\mathbf{X} \succ 0$*,* $\mathbf{S} \succ 0$*. Then* $\mathbf{X}\mathbf{S} = \nu\mathbf{I}$ *if and only if*
a) $\Phi_j(\mathbf{X}, \mathbf{S}) = \nu\mathbf{I},\ j = 1, 2, 3$
b) $\Phi_j(\mathbf{X}, \mathbf{S}) = \sqrt{\nu}\mathbf{I},\ j = 4, 5.$

Proof. $\Rightarrow$ The statement for j=1,2,3 is obvious. Consider j=4. The matrices $\mathbf{X}, \mathbf{S}$ commute and therefore are simultaneously diagonalizable, that is, there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{X} = \mathbf{Q}\mathbf{D}_{\mathbf{X}}\mathbf{Q}^T$ and $\mathbf{S} = \mathbf{Q}\mathbf{D}_{\mathbf{S}}\mathbf{Q}^T$. Therefore

$$(\mathbf{X}\mathbf{S})^{\frac{1}{2}} = \mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}} = \sqrt{\nu}\mathbf{I} = \mathbf{S}^{\frac{1}{2}}\mathbf{X}^{\frac{1}{2}} = (\mathbf{S}\mathbf{X})^{\frac{1}{2}}$$

Let j=5. The matrix $\mathbf{L} = \mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}$ is lower triangular with positive diagonal entries. We have that $\mathbf{X}\mathbf{S} = \nu\mathbf{I}$ if and only if $\mathbf{U}_{\mathbf{S}}^T\mathbf{X}\mathbf{U}_{\mathbf{S}} = \nu\mathbf{I}$. On the other hand $\mathbf{L}\mathbf{L}^T = \mathbf{U}_{\mathbf{S}}^T\mathbf{X}\mathbf{U}_{\mathbf{S}} = (\sqrt{\nu}\mathbf{I})(\sqrt{\nu}\mathbf{I})$ and therefore from the uniqueness of the Cholesky factor we obtain that $\mathbf{L} = \mathbf{L}^T = \sqrt{\nu}\mathbf{I}$.

$\Leftarrow$ The statement is obvious for j=2,3. For j=1 assume that $\mathbf{XS}+\mathbf{SX}=2\nu\mathbf{I}$. For any symmetric matrix $\mathbf{A}$ and positive diagonal matrix $\mathbf{D}$ we have that

$$(\mathbf{AD}+\mathbf{DA})_{ij}=(\mathbf{D}_{ii}+\mathbf{D}_{jj})\mathbf{A}_{ij}=0 \;\Leftrightarrow\; \mathbf{A}_{ij}=0.$$

We have that $\mathbf{X}=\mathbf{QDQ}^T$ for some orthogonal matrix $\mathbf{Q}$ and positive diagonal matrix $\mathbf{D}$ and hence

$$\mathbf{XS}+\mathbf{SX}=\mathbf{QDQ}^T\mathbf{S}+\mathbf{SQDQ}^T=2\nu\mathbf{I} \;\Leftrightarrow\; \mathbf{DQ}^T\mathbf{SQ}+\mathbf{Q}^T\mathbf{SQDQ}^T=2\nu\mathbf{I}.$$

Therefore $\mathbf{Q}^T\mathbf{SQ}$ must be diagonal. We obtain that $\mathbf{X}$, $\mathbf{S}$ are simultaneously diagonalizable and so they commute. The proof for j=4 is similar. Finally assume that j=5 and $\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}+\mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}}^T=2\nu\mathbf{I}$. Because the matrix $\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}$ is lower triangular, $\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}=\mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}}^T=\sqrt{\nu}\mathbf{I}$. So we obtain $\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}\mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}}^T=\mathbf{U}_{\mathbf{S}}^T\mathbf{X}\mathbf{U}_{\mathbf{S}}^T=\nu\mathbf{I}$ that is equivalent to $\mathbf{XS}=\nu\mathbf{I}$.

■

3 WEIGHTED CENTRAL PATH IN SEMIDEFINITE PROGRAMMING

In this section we will prove that the weighted paths can be well defined (for appropriately chosen weights). To this aim we need to show that for fixed $\triangle b, \triangle\mathbf{C}$, properly chosen weight $\mathbf{W}$ and any $\mu>0$ there exists a unique solution of the system (6). [2] Obviously, such weighted central paths does not lie in the in the interior of the feasible set in general, and hence they can be useful if the interior point does not exist or is unknown. In the next we will consider the assumption (A1) and instead of (A2) we will consider a weaker assumption:

(A3) The system (3) is solvable.

The main tool we will use in the proof of the existence of the weighted paths is the analytic version of the implicit function theorem (see e.g. [4]).

Define the linear map

$$\mathcal{A}: S^n \to R^m, \qquad \mathcal{A}(\mathbf{X})=[\mathbf{A}_1\bullet\mathbf{X},\ldots,\mathbf{A}_m\bullet\mathbf{X}],$$

and its adjoint

$$\mathcal{A}^*: R^m \to S^n, \qquad \mathcal{A}^*(y)=\sum_{i=1}^m \mathbf{A}_i y_i.$$

For fixed $\triangle b\in R^m, \triangle\mathbf{C}\in S^n$ consider the maps $F^j_{\mu,\mathbf{W}}: S^n\times R^m\times S^n \to R^m\times S^n\times S^n$ with parameters $\mu>0$ and $\mathbf{W}\succ 0$ (j=1,..,5):

$$F^j_{\mu,\mathbf{W}}(\mathbf{X},y,\mathbf{S})=\begin{bmatrix}\mathcal{A}(\mathbf{X})-b-\mu\triangle b\\ \mathcal{A}^*(y)+\mathbf{S}-\mathbf{C}-\mu\triangle\mathbf{C}\\ \Phi_j(\mathbf{X},\mathbf{S})-\phi_j(\mu)\end{bmatrix}. \tag{7}$$

Clearly the system (6) is equivalent to

$$F^j_{\mu,\mathbf{W}}(\mathbf{X},y,\mathbf{S})=0.$$

[2]To prove the existence of the solution of (6) can not be performed the same way as in the case of the system (4)—it seems not to be possible to characterize the weighted central path in SDP using weighted logarithmic barrier problems.

3.1 Regularity of the Fréchet derivatives

In the context of applying the implicit function theorem we will be interested in the Fréchet derivative of the maps $F^j_{\mu,\mathbf{W}}$. It can be derived that if $\mathbf{X} \succ 0, \mathbf{S} \succ 0$, it is the linear map

$$DF^j_{\mu,\mathbf{W}}(\mathbf{X},y,\mathbf{S})[\triangle\mathbf{X},\triangle y,\triangle\mathbf{S}] = \begin{bmatrix} \mathcal{A}(\triangle\mathbf{X}) \\ \mathcal{A}^*(\triangle y)+\triangle\mathbf{S} \\ D\Phi_j(\mathbf{X},\mathbf{S})[\triangle\mathbf{X},\triangle\mathbf{S}] \end{bmatrix}$$

with the variables $[\triangle\mathbf{X},\triangle y,\triangle\mathbf{S}] \in S^n \times R^m \times S^n$ where

- $D\Phi_1(\mathbf{X},\mathbf{S})[\triangle\mathbf{X},\triangle\mathbf{S}] = \frac{1}{2}(\triangle\mathbf{X}\mathbf{S}+\mathbf{S}\triangle\mathbf{X}+\triangle\mathbf{S}\mathbf{X}+\mathbf{X}\triangle\mathbf{S})$
- $D\Phi_2(\mathbf{X},\mathbf{S})[\triangle\mathbf{X},\triangle\mathbf{S}] = \langle\langle\triangle\mathbf{X}\rangle\rangle_{\mathbf{X}^{\frac{1}{2}}}\mathbf{S}\mathbf{X}^{\frac{1}{2}}+\mathbf{X}^{\frac{1}{2}}\mathbf{S}\langle\langle\triangle\mathbf{X}\rangle\rangle_{\mathbf{X}^{\frac{1}{2}}}+\mathbf{X}^{\frac{1}{2}}\triangle\mathbf{S}\mathbf{X}^{\frac{1}{2}}$
- $D\Phi_3(\mathbf{X},\mathbf{S})[\triangle\mathbf{X},\triangle\mathbf{S}] = [[\triangle\mathbf{X}]]^T_{\mathbf{L_X}}\mathbf{S}\mathbf{L_X}+\mathbf{L_X}^T\mathbf{S}[[\triangle\mathbf{X}]]_{\mathbf{L_X}}+\mathbf{L_X}^T\triangle\mathbf{S}\mathbf{L_X}$
- $D\Phi_4(\mathbf{X},\mathbf{S})[\triangle\mathbf{X},\triangle\mathbf{S}] = \frac{1}{2}(\langle\langle\triangle\mathbf{X}\rangle\rangle_{\mathbf{X}^{\frac{1}{2}}}\mathbf{S}^{\frac{1}{2}}+\mathbf{S}^{\frac{1}{2}}\langle\langle\triangle\mathbf{X}\rangle\rangle_{\mathbf{X}^{\frac{1}{2}}}+\langle\langle\triangle\mathbf{S}\rangle\rangle_{\mathbf{S}^{\frac{1}{2}}}\mathbf{X}^{\frac{1}{2}}+\mathbf{X}^{\frac{1}{2}}\langle\langle\triangle\mathbf{S}\rangle\rangle_{\mathbf{S}^{\frac{1}{2}}})$
- $D\Phi_5(\mathbf{X},\mathbf{S})[\triangle\mathbf{X},\triangle\mathbf{S}] = \frac{1}{2}([[\triangle\mathbf{X}]]^T_{\mathbf{L_X}}\mathbf{U_S}+\mathbf{U_S}^T[[\triangle\mathbf{X}]]_{\mathbf{L_X}}+[[\triangle\mathbf{S}]]^T_{\mathbf{U_S}}\mathbf{L_X}+\mathbf{L_X}^T[[\triangle\mathbf{S}]]_{\mathbf{U_S}})$

and $\langle\langle\mathbf{B}\rangle\rangle_{\mathbf{A}} \in S^n$ means the solution $\mathbf{H}$ of the equation $\mathbf{AH}+\mathbf{HA}=\mathbf{B}$ (which exists and is unique for every $\mathbf{A}\succ 0$ and $\mathbf{B}\in S^n$), and $[[\mathbf{B}]]_{\mathbf{L}} \in S^n$ is the solution $\mathbf{H}$ of the equation $\mathbf{LH}^T+\mathbf{HL}^T=\mathbf{B}$ (which exists and is unique for any $\mathbf{L}\in L^n$ with positive diagonal entries and any $\mathbf{B}\in S^n$, where L^n is the real vector space of all $n\times n$ lower triangular matrices).

For $\varepsilon > 0$ denote

$$\mathcal{M}_\varepsilon = \{\ \mathbf{Z}\succ 0; \exists\nu:\ \ \|\mathbf{Z}-\nu\mathbf{I}\| < \varepsilon\nu\}.$$

and for all $j=1,\dots,5$ define the sets $\mathcal{W}_j$ in the following way:

$$\begin{aligned} &\mathcal{W}_1 = S^n_{++} \\ &\mathcal{W}_2 = \mathcal{M}_{\frac{1}{\sqrt{2}}} \\ &\mathcal{W}_3 = \mathcal{M}_{\frac{1}{\sqrt{2}}}\ \textit{or}\ \mathcal{W}_3 = D^n_{++} \\ &\mathcal{W}_4 = \mathcal{M}_\tau \\ &\mathcal{W}_5 = \mathcal{M}_\tau \end{aligned}$$

where

$$\tau = \frac{1+\sqrt{2}-\sqrt{4\sqrt{2}+2}}{5-4\sqrt{2\sqrt{2}+1}}$$

and D^n_{++} is the notation for the open convex set of all diagonal matrices with positive diagonal entries.

Theorem 1 *Let $\mathbf{X}\succ 0$, $\mathbf{S}\succ 0$. Then*

$$\Phi_j(\mathbf{X},\mathbf{S}) \in \mathcal{W}_j\ \Rightarrow\ DF^j_{\mu,\mathbf{W}}(\mathbf{X},y,\mathbf{S})\ \textit{is regular linear map.}$$

Proof. The proof of the result for j=1 can be found in [9] or can be easily shown using the properties of the symmetric Kronecker product. The results for j=2, j=4 and j=5 are consequences of Lemma 2.3 of [6] and Propositions 4 and 5 of [8], respectively. The result for j=3, the case $\mathcal{W}_3 = \mathcal{M}_{\frac{1}{\sqrt{2}}}$ is a consequence of Lemma 2.4 of [7] and we will prove the case $\mathcal{W}_3 = D^n_{++}$.

Assume that $\mathbf{X} \succ 0, \mathbf{S} \succ 0$ and $\mathbf{L}_{\mathbf{X}}^T \mathbf{S} \mathbf{L}_{\mathbf{X}}^T \in D^n_{++}$. We will show, that the system

$$\begin{array}{c} \mathcal{A}(\triangle \mathbf{X}) = 0 \\ \mathcal{A}^*(\triangle y) + \triangle \mathbf{S} = 0 \\ [[\triangle \mathbf{X}]]_{\mathbf{L_X}}^T \mathbf{S} \mathbf{L_X} + \mathbf{L_X}^T \mathbf{S} [[\triangle \mathbf{X}]]_{\mathbf{L_X}} + \mathbf{L_X}^T \triangle \mathbf{S} \mathbf{L_X} = 0 \end{array} \tag{8}$$

has only the solution $\triangle \mathbf{X} = \triangle \mathbf{S} = 0$. It can be easily seen that the first two equations in (8) imply $\triangle \mathbf{X} \bullet \triangle \mathbf{S} = 0$. Denote $\mathbf{U} = [[\triangle \mathbf{X}]]_{\mathbf{L_X}} \in L^n$. Then from the definition of $[[\triangle \mathbf{X}]]_{\mathbf{L_X}}$ and (8) we have that

$$\begin{array}{c} \mathbf{U}^T \mathbf{S} \mathbf{L_X} + \mathbf{L_X}^T \mathbf{S} \mathbf{U} + \mathbf{L_X}^T \triangle \mathbf{S} \mathbf{L_X} = 0, \\ \mathbf{L_X} \mathbf{U}^T + \mathbf{U} \mathbf{L}_{\mathbf{X}}^T = \triangle \mathbf{X} \end{array} .$$

We can express

$$\triangle \mathbf{S} = -(\mathbf{L_X}^{-T} \mathbf{U} \mathbf{S} + \mathbf{S} \mathbf{U} \mathbf{L}_{\mathbf{X}}^{-1})$$

and obtain

$$0 = -\triangle \mathbf{X} \bullet \triangle \mathbf{S} = (\mathbf{L_X} \mathbf{U}^T + \mathbf{U} \mathbf{L}_{\mathbf{X}}^T) \bullet (\mathbf{L_X}^{-T} \mathbf{U} \mathbf{S} + \mathbf{S} \mathbf{U} \mathbf{L}_{\mathbf{X}}^{-1}) =$$

$$2tr(\mathbf{U}^T \mathbf{S} \mathbf{U}) + 2tr[(\mathbf{L}_{\mathbf{X}}^T \mathbf{S} \mathbf{L}_{\mathbf{X}}^T)(\mathbf{U} \mathbf{L}_{\mathbf{X}}^{-1})^2].$$

¿From the assumptions we have that $tr(\mathbf{U}^T \mathbf{S} \mathbf{U}) = 0$ and therefore also $\mathbf{U} = 0, \triangle \mathbf{X} = 0, \triangle \mathbf{S} = 0$.

■

The following lemma provides a nice description of $\mathcal{M}_\varepsilon$.

Lemma 4 *Let $\varepsilon \in (0,1)$. The set $\mathcal{M}_\varepsilon$ is a convex cone, moreover,*

$$\mathbf{Z} \in \mathcal{M}_\varepsilon \;\Leftrightarrow\; \kappa(\mathbf{Z}) = \frac{\lambda_{max}(\mathbf{Z})}{\lambda_{min}(\mathbf{Z})} < \frac{1+\varepsilon}{1-\varepsilon}.$$

(where $\kappa(\mathbf{Z})$ means the condition number of $\mathbf{Z}$.)

Proof. The proof that $\mathcal{M}_\varepsilon$ is a convex cone is straightforward. We will prove the second part of the lemma. Denote

$$\lambda_{max}(\mathbf{Z}) = \lambda_1(\mathbf{Z}) \geq \lambda_2(\mathbf{Z}) \geq \cdots \geq \lambda_n(\mathbf{Z}) = \lambda_{min}(\mathbf{Z})$$

the eigenvalues of $\mathbf{Z}$. It holds that $\mathbf{Z} \in \mathcal{M}_\varepsilon$ if and only if $\mathbf{Z} \succ 0$ and there exists $\nu > 0$ such that

$$\|\mathbf{Z} - \nu \mathbf{I}\| = \max_i |\lambda_i(\mathbf{Z} - \nu| < \nu\varepsilon. \tag{9}$$

The inequality (9) is equivalent to

$$(1-\varepsilon)\nu < \lambda_{min}(\mathbf{Z}) \leq \lambda_{max}(\mathbf{Z}) < (1+\varepsilon)\nu.$$

$$\mathbf{Z} \in \mathcal{M}_\varepsilon \;\Leftrightarrow\; \frac{\lambda_{max}(\mathbf{Z})}{\lambda_{min}(\mathbf{Z})} < \frac{1+\varepsilon}{1-\varepsilon}.$$

■

3.2 Boundedness of the weighted path

In the next, besides the assumptions A1 and A3 we will assume the following:

(A4) For any $j \in \{1,\dots,5\}$ let $\triangle b, \triangle \mathbf{C}$ be such that there exists $\mathbf{W}^0 \in \mathcal{W}_j$ and $\mu_0 > 0$ such that the system (6) is solvable for $\mathbf{W} = \mathbf{W}^0$ and $\nu = \mu_0$.

In what follows, by $\mathbf{W}^0$ and μ_0 we will denote an arbitrarily chosen weight $\mathbf{W}^0 \in \mathcal{W}_j$ and $\mu_0 > 0$ for which the system (6) is solvable. Let us remark that for $j \in \{1,\dots,5\}$ there always exist $\triangle b, \triangle \mathbf{C}$ such that they satisfy (A4). In fact, we can choose a weight $\mathbf{W}^0 \in \mathcal{W}_j$ and $\mu_0 > 0$ and pick up $(\mathbf{X}^0, y^0, \mathbf{S}^0) \in S^n_{++} \times R^m \times S^n_{++}$ such that

$$\Phi_j(\mathbf{X}^0, \mathbf{S}^0) = \phi_j(\mu^0)\mathbf{W}^0.$$

Then if we let

$$\triangle b = \frac{\mathcal{A}(\mathbf{X}^0) - b}{\phi_j(\mu^0)}, \qquad \triangle \mathbf{C} = \frac{\mathcal{A}^*(y^0) + \mathbf{S}^0 - \mathbf{C}}{\phi_j(\mu^0)},$$

then $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ is a solution of the system (6) for $\mu = \mu_0$ and $\mathbf{W} = \mathbf{W}^0$. On the other hand, if the assumption (A2) holds, $\triangle b = 0, \triangle \mathbf{C} = 0$ satisfy (A4) with $\mathbf{W}^0 = \mathbf{I}$ and any $\mu_0 > 0$, since the central path exists.

For $\mu > 0$ and $\mathbf{W} \succ 0$ we will denote

$$(\mathbf{X}_{(\mu,\mathbf{W})}, y_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})})$$

a solution of the system (6) (for some $j \in \{1,\dots,5\}$). Obviously, the solution needs not exist nor be unique. Nevertheless, we can prove the following lemma which states that the set of all solutions for some μ and $\mathbf{W}$ is bounded.

Lemma 5 *Let $\mathcal{O}(\mathbf{W}^0) \subset S^n_{++}$ be a bounded neighborhood of $\mathbf{W}^0$. Then the set*

$$\mathcal{M} = \{(\mathbf{X}_{(\mu,\mathbf{W})}, y_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})}) \mid 0 < \mu \le \mu_0, \mathbf{W} \in \mathcal{O}(\mathbf{W}^0)\}$$

is bounded.

Proof. Let $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ be the solution of (6) for $\mu = \mu_0$ and $\mathbf{W} = \mathbf{W}^0$. Let $0 < \mu \le \mu_0$ and $\mathbf{W} \in \mathcal{O}(\mathbf{W}^0)$ be arbitrary, such that there exist a solution $(\mathbf{X}_{(\mu,\mathbf{W})}, y_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})})$ of the system (6). From (A3) we have that there exists $(\mathbf{X}^*, y^*, \mathbf{S}^*)$ such that:

$$\mathbf{A}_i \bullet \mathbf{X}^* = b_i, \quad \sum_{i=1}^m \mathbf{A}_i y_i^* + \mathbf{S}^* = \mathbf{C}, \quad \mathbf{X}^* \succeq 0, \quad \mathbf{S}^* \succeq 0, \quad \mathbf{X}^*\mathbf{S}^* = 0.$$

Define

$$\begin{pmatrix} \hat{\mathbf{X}} \\ \hat{y} \\ \hat{\mathbf{S}} \end{pmatrix} = \frac{\mu}{\mu_0} \begin{pmatrix} \mathbf{X}^0 \\ y^0 \\ \mathbf{S}^0 \end{pmatrix} + \left(1 - \frac{\mu}{\mu_0}\right) \begin{pmatrix} \mathbf{X}^* \\ y^* \\ \mathbf{S}^* \end{pmatrix}.$$

Clearly

$$\mathbf{A}_i \bullet \hat{\mathbf{X}} = \frac{\mu}{\mu_0} \mathbf{A}_i \bullet \mathbf{X}^0 + \left(1 - \frac{\mu}{\mu_0}\right) \mathbf{A}_i \bullet \mathbf{X}^* = b_i + \mu \triangle b_i, \quad \forall i = 1,\dots,m,$$

$$\sum_{i=1}^m \mathbf{A}_i \hat{y}_i + \hat{\mathbf{S}} = \frac{\mu}{\mu_0} \left(\sum_{i=1}^m \mathbf{A}_i y_i^0 + \mathbf{S}^0\right) + \left(1 - \frac{\mu}{\mu_0}\right) \left(\sum_{i=1}^m \mathbf{A}_i y_i^0 + \mathbf{S}^0\right) = \mathbf{C} + \mu \triangle \mathbf{C}$$

and hence

$$\mathbf{A}_i \bullet (\hat{\mathbf{X}} - \mathbf{X}_{(\mu,\mathbf{W})}) = 0, \quad \sum_{i=1}^{m} \mathbf{A}_i(\hat{y}_i - (y_{(\mu,\mathbf{W})})_i) + (\hat{\mathbf{S}} - \mathbf{S}_{(\mu,\mathbf{W})}) = 0.$$

Therefore

$$(\hat{\mathbf{X}} - \mathbf{X}_{\mu,\mathbf{W}}) \bullet (\hat{\mathbf{S}} - \mathbf{S}_{\mu,\mathbf{W}}) = 0.$$

This gives

$$\hat{\mathbf{X}} \bullet \mathbf{S}_{(\mu,\mathbf{W})} + \mathbf{X}_{(\mu,\mathbf{W})} \bullet \hat{\mathbf{S}} = \hat{\mathbf{X}} \bullet \hat{\mathbf{S}} + \mathbf{X}_{(\mu,\mathbf{W})} \bullet \mathbf{S}_{(\mu,\mathbf{W})}. \tag{10}$$

We first observe, that

$$\begin{aligned}
\hat{\mathbf{X}} \bullet \hat{\mathbf{S}} &= \left(\frac{\mu}{\mu_0}\right)^2 \mathbf{X}^0 \bullet \mathbf{S}^0 + \left(1 - \frac{\mu}{\mu_0}\right)^2 \mathbf{X}^* \bullet \mathbf{S}^* + \frac{\mu}{\mu_0}\left(1 - \frac{\mu}{\mu_0}\right)(\mathbf{X}^0 \bullet \mathbf{S}^* + \mathbf{S}^0 \bullet \mathbf{X}^*) = \\
&= \left(\frac{\mu}{\mu_0}\right)^2 \mathbf{X}^0 \bullet \mathbf{S}^0 + \frac{\mu}{\mu_0}\left(1 - \frac{\mu}{\mu_0}\right)(\mathbf{X}^0 \bullet \mathbf{S}^* + \mathbf{S}^0 \bullet \mathbf{X}^*) \\
&\begin{cases} \le \mu_0 \, tr(\mathbf{W}^0) + (\mathbf{X}^0 \bullet \mathbf{S}^* + \mathbf{S}^0 \bullet \mathbf{X}^*) & j = 1,2,3, \\ \le 2\mu_0 tr((\mathbf{W}^0)^2) + (\mathbf{X}^0 \bullet \mathbf{S}^* + \mathbf{S}^0 \bullet \mathbf{X}^*) & j = 4,5, \end{cases}
\end{aligned} \tag{11}$$

where the inequalities follow from Lemma 1. According to the same lemma we have

$$\begin{aligned}
\mathbf{X}_{(\mu,\mathbf{W})} \bullet \mathbf{S}_{(\mu,\mathbf{W})} &= \mu \, tr(\mathbf{W}) \le \beta, \quad j = 1,2,3, \\
\mathbf{X}_{(\mu,\mathbf{W})} \bullet \mathbf{S}_{(\mu,\mathbf{W})} &\le 2tr(\mathbf{W}^2) \le \beta, \quad j = 4,5
\end{aligned} \tag{12}$$

for some $\beta > 0$, since $0 < \mu < \mu_0$ a $\mathbf{W} \in \mathcal{O}(\mathbf{W}_0)$, which is bounded. Finally, from (10), (11), (12) we obtain, that

$$\hat{\mathbf{X}} \bullet \mathbf{S}_{(\mu,\mathbf{W})} + \mathbf{X}_{(\mu,\mathbf{W})} \bullet \hat{\mathbf{S}} \le \gamma$$

for some $\gamma > 0$ and hence the set

$$\mathcal{M}_1 = \left\{ (\mathbf{X}_{(\mu,\mathbf{W})}, \mathbf{S}_{(\mu,\mathbf{W})}) \mid \mu \in (0, \mu_0\rangle, \ \mathbf{W} \in \mathcal{O}(\mathbf{W}_0) \right\}$$

is included in the simplex

$$\left\{ (\mathbf{X}, \mathbf{S}) \mid \mathbf{X} \succeq 0, \ \mathbf{S} \succeq 0, \ \hat{\mathbf{X}} \bullet \mathbf{S} + \mathbf{X} \bullet \hat{\mathbf{S}} \le \gamma \right\}$$

which is bounded, since $\hat{\mathbf{X}} \succ 0, \hat{\mathbf{S}} \succ 0$. The boundedness of the set $\mathcal{M}$ now follows from the boundedness of $\mathcal{M}_1$ and the assumption (A1).

■

3.3 The existence of the weighted path

Let $j \in \{1, \ldots, 5\}$. Consider the map

$$G_j : S^n \times R^m \times S^n \times R \times S^n \to R^m \times S^n \times S^n$$

such that

$$G_j(\mathbf{X}, y, \mathbf{S}, \mu, \mathbf{W}) = F^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S})$$

Obviously $G_j(\mathbf{X}^0, y^0, \mathbf{S}^0, \mu_0, \mathbf{W}^0) = 0$. The following technique is called the analytic continuation and was used by Preiss and Stoer to prove the existence of the weighted path in linear complementarity problem associated with the symmetrization $(\mathbf{XS} + \mathbf{SX})/2$ (see Lemma 3.5, Lemma 3.6, Lemma 3.7 in [9]).

Lemma 6 *Let $j \in \{1, 2, \dots, 5\}$. Assume $\mu_1 \in (0, \mu_0\rangle, \mathbf{W}^1 \in \mathcal{W}_j$ and let*

$$\psi : \langle 0, 1\rangle \to (0, \mu_0\rangle \times \mathcal{W}_j, \qquad \psi(t) = (\mu_t, \mathbf{W}^t)$$

be a continuous path from $\psi(0) = (\mu_0, \mathbf{W}^0)$ to $\psi(1) = (\mu_1, \mathbf{W}^1)$. Then for all $t \in \langle 0, 1\rangle$ the system

$$G_j(\mathbf{X}, y, \mathbf{S}, \mu_t, \mathbf{W}^t) = 0$$

has a locally unique solution $(\mathbf{X}^t, y^t, \mathbf{S}^t)$, where $\mathbf{X}^t \succ 0, \mathbf{S}^t \succ 0$. Moreover, there exists a function

$$g_j : R_{++} \times S^n_{++} \to S^n_{++} \times R^m \times S^n_{++},$$

which is defined and analytic on some neighborhood of $\psi(t)$, satisfies $g(\psi(t)) = (\mathbf{X}^t, y^t, \mathbf{S}^t)$ and

$$G_j(g_j(\psi(t)), \psi(t)) = 0.$$

Proof. For $t \in \langle 0, 1\rangle$ consider the system

$$G_j(\mathbf{X}, y, \mathbf{S}, \psi(t)) = 0, \quad \mathbf{X} \succ 0, \ \mathbf{S} \succ 0. \tag{13}$$

The point $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ is the solution of this system for $t = 0$. From Theorem 3 it follows that the partial Fréchet derivative $DG_j(\mathbf{X}, y, \mathbf{S}, \phi(t))$ concerning the variables $(\mathbf{X}, y, \mathbf{S})$ is nonsingular in $(\mathbf{X}^0, y^0, \mathbf{S}^0, \phi(0))$. From the implicit function theorem we obtain that there exists an analytic function g_j defined on some neighborhood of $\psi(0) = (\mu_0, \mathbf{W}^0)$ such that

$$g_j(\psi(0)) = g_j(\mu_0, \mathbf{W}^0) = (\mathbf{X}^0, y^0, \mathbf{S}^0)$$

and

$$G_j(g_j(\psi(t)), \psi(t)) = G_j(g_j(\mu_t, \mathbf{W}^t), \mu_t, \mathbf{W}^t) = 0$$

on some neighborhood of $t = 0$. Actually, there is a maximal $\bar{t} \in (0, 1\rangle$ such that

$$g_j(\psi(t)) = g_j(\mu_t, \mathbf{W}^t) = (\mathbf{X}^t, y^t, \mathbf{S}^t), \quad \forall t \in \langle 0, \bar{t}).$$

That means $(\mathbf{X}^t, y^t, \mathbf{S}^t)$ is a locally unique solution of

$$F^j_{\mu_t, \mathbf{W}^t}(\mathbf{X}, y, \mathbf{S}) = 0, \qquad \forall\ t \in \langle 0, \bar{t}).$$

Moreover, from Lemma 2 we have that $\mathbf{X}^t \succ 0, \mathbf{S}^t \succ 0$. From the continuity of ψ it follows that $\psi(\langle 0, 1\rangle)$ is a compact subset of $(0, \mu_0\rangle \times \mathcal{W}_j$, therefore, according to Lemma 5 the set

$$\{g(\psi(t)) = (\mathbf{X}^t, y^t, \mathbf{S}^t), \ | \ t \in \langle 0, \bar{t})\}$$

is bounded. Let $t_k \in \langle 0, \bar{t})$ for $k = 1, 2, \dots$ and $\lim_{k\to\infty} t_k = \bar{t}$. Then there exists a sequence $\{t_{k_j}\}_{j=1}^{\infty}$ chosen from $\{t_k\}_{k=1}^{\infty}$ such that

$$\lim_{j\to\infty} g_j(\psi(t_{k_j})) = (\bar{\mathbf{X}}, \bar{y}, \bar{\mathbf{S}}).$$

Because

$$\begin{aligned} \mathbf{A}_i \bullet \mathbf{X}^{t_{k_j}} &= b_i + \mu_{t_{k_j}} \triangle b_i, \quad i = 1, \dots, m, \quad \mathbf{X}^{t_{k_j}} \succ 0, \\ \textstyle\sum_{i=1}^m \mathbf{A}_i y_i^{t_{k_j}} + \mathbf{S}^{t_{k_j}} &= \mathbf{C} + \mu_{t_{k_j}} \triangle \mathbf{C}, \quad \mathbf{S}^{t_{k_j}} \succ 0, \\ \Psi_j(\mathbf{X}^{t_{k_j}}, \mathbf{S}^{t_{k_j}}) &= \phi_j(\mu_{t_{k_j}})\mathbf{W}^{t_{k_j}}, \end{aligned}$$

by taking limit $j \to \infty$ we obtain

$$\begin{aligned} \mathbf{A}_i \bullet \bar{\mathbf{X}} &= b_i + \mu_{\bar{t}} \triangle b_i, \quad i = 1, \ldots, m, \quad \bar{\mathbf{X}} \succeq 0, \\ \sum_{i=1}^m \mathbf{A}_i \bar{y}_i + \bar{\mathbf{S}} &= \mathbf{C} + \mu_{\bar{t}} \triangle \mathbf{C}, \quad \bar{\mathbf{S}} \succeq 0 \\ \Psi_j(\bar{\mathbf{X}}, \bar{\mathbf{S}}) &= \phi_j(\mu_{\bar{t}}) \mathbf{W}^{\bar{t}}. \end{aligned}$$

Applying Lemma 2 again we have that $\bar{\mathbf{X}}, \bar{\mathbf{S}}$ are positive definite. Therefore $(\bar{\mathbf{X}}, \bar{y}, \bar{\mathbf{S}})$ is the solution of the system (13) for $t = \bar{t}$. The partial Fréchet derivative $DG_j(\mathbf{X}, y, \mathbf{S}, \psi(t))$ concerning the variables $(\mathbf{X}, y, \mathbf{S})$ is nonsingular in $(\bar{\mathbf{X}}, \bar{y}, \bar{\mathbf{S}})$ and $(\bar{\mathbf{X}}, \bar{y}, \bar{\mathbf{S}})$ is locally unique solution of the system

$$F^j_{\mu_{\bar{t}}, \mathbf{W}^{\bar{t}}}(\mathbf{X}, y, \mathbf{S}) = 0.$$

By applying the implicit function theorem again and from the maximality of $\bar{t}$ we obtain that $\bar{t} = 1$.

■

Corollary 1 *For any $\mu \in (0, \mu_0\rangle$ and $\mathbf{W} \in \mathcal{W}_j$ there exists a solution of* (6).

Proof. It suffices to prove that having $\mathbf{W} \in \mathcal{W}_j$ and $\mu \in (0, \mu_0\rangle$ one can find a continuous path from $(\mu_0, \mathbf{W}^0)$ to $(\mu, \mathbf{W})$. However, we can define $\psi(t) = (t\mu + (1-t)\mu_0, t\mathbf{W} + (1-t)\mathbf{W}^0)$. Obviously $t\mu + (1-t)\mu_0 \in (0, \mu_0\rangle$ for all $t \in \langle 0, 1\rangle$ and since $\mathcal{W}_j$ is convex, $t\mathbf{W} + (1-t)\mathbf{W}^0 \in \mathcal{W}_j$.

■

Having the existence result stated in Corollary 1 we turn our attention to the uniqueness of the solutions. As a consequence of Lemma 6 we obtain the following result that will be useful later.

Corollary 2 *For all $t \in \langle 0, 1\rangle$ the function $g(\psi(t))$ from Lemma 6 is uniquely determined by the path ψ and the starting value $g(\psi(0))$.*

First, we prove the uniqueness of (6) for a special choice of the weight matrix $\mathbf{W} = \mathbf{I}$. This result will be used then in the proof of Lemma 8.

Lemma 7 *Let $j \in \{1, 2, \ldots, 5\}$ be arbitrary. If the system*

$$\left. \begin{aligned} \mathcal{A}(\mathbf{X}) &= b + \mu \triangle b, \quad \mathbf{X} \succ 0, \\ \mathcal{A}^*(y) + \mathbf{S} &= \mathbf{C} + \mu \triangle C, \quad \mathbf{S} \succ 0, \\ \Phi_j(\mathbf{X}, \mathbf{S}) &= \phi_j(\mu) \mathbf{I} \end{aligned} \right\} \tag{14}$$

has a solution for some $\mu > 0$ then this solution is unique.

Proof. Suppose there are two solutions $(\mathbf{X}_1, y_1, \mathbf{S}_1)$, $(\mathbf{X}_2, y_2, \mathbf{S}_2)$ of the system (14). Let $(\triangle\mathbf{X}, \triangle y, \triangle\mathbf{S}) = (\mathbf{X}_1, y_1, \mathbf{S}_1) - (\mathbf{X}_2, y_2, \mathbf{S}_2)$. Then $\mathcal{A}(\triangle\mathbf{X}) = 0, \tilde{\mathcal{A}}(\triangle y) + \triangle\mathbf{S} = 0$ and hence $\triangle\mathbf{X} \bullet \triangle\mathbf{S} = 0$.
Lemma 3 states that

$$\Phi_j(\mathbf{X}_i, \mathbf{S}_i) = \phi_j(\mu)\mathbf{I} \iff \mathbf{X}_i \mathbf{S}_i = \mu \mathbf{I} \qquad i = 1, 2.$$

Therefore

$$\mu\mathbf{I} = \mathbf{X}_1\mathbf{S}_1 = (\mathbf{X}_2 + \triangle\mathbf{X})(\mathbf{S}_2 + \triangle\mathbf{S}) = \mathbf{X}_2\mathbf{S}_2 + \mathbf{X}_2\triangle\mathbf{S} + \triangle\mathbf{X}\mathbf{S}_2 + \triangle\mathbf{X}\triangle\mathbf{S},$$

$$\mu \mathbf{I} = \mathbf{X}_2\mathbf{S}_2 = (\mathbf{X}_1 - \triangle\mathbf{X})(\mathbf{S}_1 - \triangle\mathbf{S}) = \mathbf{X}_1\mathbf{S}_1 - \mathbf{X}_1\triangle\mathbf{S} - \triangle\mathbf{X}\mathbf{S}_1 + \triangle\mathbf{X}\triangle\mathbf{S}$$

and by subtracting the equations above we obtain that

$$(\mathbf{X}_1 + \mathbf{X}_2)\triangle\mathbf{S} + \triangle\mathbf{X}(\mathbf{S}_1 + \mathbf{S}_2) = 0. \;\Leftrightarrow\; \triangle\mathbf{S} = -(\mathbf{X}_1 + \mathbf{X}_2)^{-1}\triangle\mathbf{X}(\mathbf{S}_1 + \mathbf{S}_2).$$

We can express $\triangle\mathbf{S}$ as

$$\triangle\mathbf{S} = -(\mathbf{X}_1 + \mathbf{X}_2)^{-1}\triangle\mathbf{X}(\mathbf{S}_1 + \mathbf{S}_2) \tag{15}$$

and hence

$$0 = \triangle\mathbf{X} \bullet \triangle\mathbf{S} = tr(\triangle\mathbf{X}\triangle\mathbf{S}) = -tr(\triangle\mathbf{X}(\mathbf{X}_1 + \mathbf{X}_2)^{-1}\triangle\mathbf{X}(\mathbf{S}_1 + \mathbf{S}_2)) =$$

$$= -tr((\mathbf{S}_1 + \mathbf{S}_2)^{\frac{1}{2}}\triangle\mathbf{X}(\mathbf{X}_1 + \mathbf{X}_2)^{-1}\triangle\mathbf{X}(\mathbf{S}_1 + \mathbf{S}_2)^{\frac{1}{2}}).$$

The trace of the positive semidefinite matrix is zero if and only if it is the zero matrix. That's why $\triangle\mathbf{X} = 0$ and from (15) also $\triangle\mathbf{S} = 0$. Finally, the assumption A1 gives $\triangle y = 0$.

■

We now prove the uniqueness for the general weight matrix $\mathbf{W}$.

Lemma 8 *If the system*

$$\left.\begin{aligned} \mathcal{A}(\mathbf{X}) = b + \mu\triangle b, \quad \mathbf{X} \succ 0,\\ \mathcal{A}^*(y) + \mathbf{S} = \mathbf{C} + \mu\triangle\mathbf{C}, \quad \mathbf{S} \succ 0,\\ \Phi_j(\mathbf{X}, \mathbf{S}) = \phi_j(\mu)\mathbf{W} \qquad\quad \end{aligned}\right\} \tag{16}$$

has a solution for some $\mu > 0$, then this solution is unique.

Proof. Let $\mu > 0$ and suppose there are two solutions $(\mathbf{X}_1, y_1, \mathbf{S}_1), (\mathbf{X}_2, y_2, \mathbf{S}_2)$. Consider the path

$$\psi : \langle 0, 1\rangle \to R_{++} \times \mathcal{W}_j, \quad \psi(t) = (\mu, t\mathbf{I} + (1-t)\mathbf{W}).$$

Lemma 6 states that there exist analytic continuations from $(\mathbf{X}_1, y_1, \mathbf{S}_1)$ and $(\mathbf{X}_2, y_2, \mathbf{S}_2)$ along ψ to the solution of the system (14), which is unique (Lemma 7). Denote this solution $(\mathbf{X}_I, y_I, \mathbf{S}_I)$. The analytic continuation from $(\mathbf{X}_I, y_I, \mathbf{S}_I)$ along the inverse path $\phi^{-1}(t) = \phi(1-t)$ leads to both $(\mathbf{X}_1, y_1, \mathbf{S}_1)$ and $(\mathbf{X}_2, y_2, \mathbf{S}_2)$. The uniqueness of the analytic continuation (Corollary
2) implies $(\mathbf{X}_1, y_1, \mathbf{S}_1) = (\mathbf{X}_2, y_2, \mathbf{S}_2)$.

■

Now we can formulate the main result of the paper, which is a simple consequence of Corollary 1 and Lemma 8. Let us recall that it was proved under the assumptions (A1), (A3), (A4).

Theorem 2 *Let $j \in \{1, 2, \ldots, 5\}$. Then for any $\mu \in (0, \mu_0\rangle$ and $\mathbf{W} \in \mathcal{W}_j$ there exists unique solution of the system* (6).

As it was mentioned above, under the assumption (A2) the choice $\triangle b = 0, \triangle\mathbf{C} = 0$ satisfies (A4). Hence we obtain the following corollary of Theorem 2.

Corollary 3 *Assume (A1) and (A2) and let $j \in \{1, 2, \ldots, 5\}$. Then for any $\mu > 0$ and $\mathbf{W} \in \mathcal{W}_j$ there exists unique solution of the system*

$$\left.\begin{aligned} \mathcal{A}(\mathbf{X}) = b, \quad \mathbf{X} \succ 0,\\ \mathcal{A}^*(y) + \mathbf{S} = \mathbf{C}, \quad \mathbf{S} \succ 0,\\ \Phi_j(\mathbf{X}, \mathbf{S}) = \phi_j(\mu)\mathbf{W}. \qquad\quad \end{aligned}\right\}$$

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