Convex optimization

Convex analysis - sets

Topic 2: Convex analysis - sets

- Convex sets
- Operations preserving convexity
- Separation of convex sets

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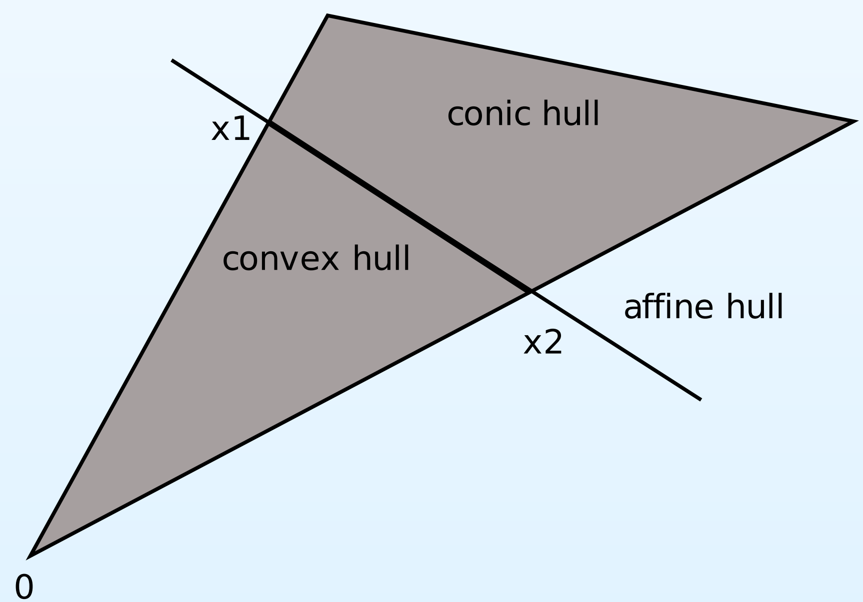
$$y = \theta_1 x_1 + \cdots + \theta_k x_k$$

	$\sum_{i=1}^{k} \theta_k = 1$	$\theta_i \geq 0, \forall i$	
linear combination			vector subspace
afinne combination	$\surd$		afinne sets
conic combination		$\surd$	convex cone
convex combination	$\surd$	$\surd$	convex set



Examples of convex sets

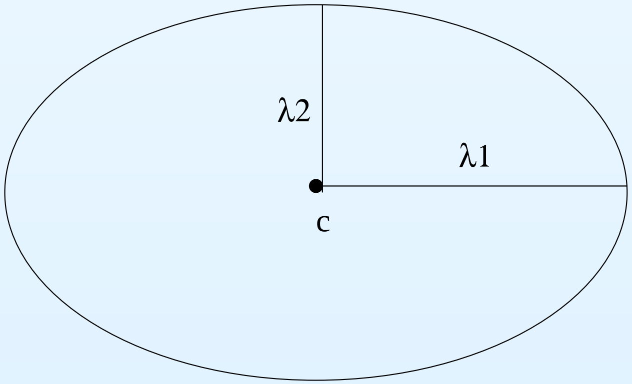
empty set	$\emptyset$
point	$\{x_0\}$
vector space	$\mathbb{R}^n$
line	$\{x_0 + \alpha v \mid \alpha \in \mathbb{R}\},\ v \neq 0$
halfline (ray)	$\{x_0 + \alpha v \mid \alpha \geq 0\},\ v \neq 0$
line segent	$\{x_0 + \alpha v \mid \alpha \in [0, 1]\},\ v \neq 0$
halfspace	$\{x \mid a^T x \leq b\}$

hyperplane	$\{x \mid a^T x = b\}$
polyhedral set	$\{x \mid a_j^T x \leq b_j, j = 1, \dots, m\}$
unit simplex	$\{x \geq 0 \mid \sum_{i=1}^n x_i \leq 1\}$
probability simplex	$\{x \geq 0 \mid \sum_{i=1}^n x_i = 1\}$
ball	$\mathcal{B}(c, r) = \{x \mid \|x - c\| \leq r\}$
ellipsoid	$\mathcal{E}(c, P) = \{x \mid (x - c)^T P^{-1} (x - c) \leq 1\}$

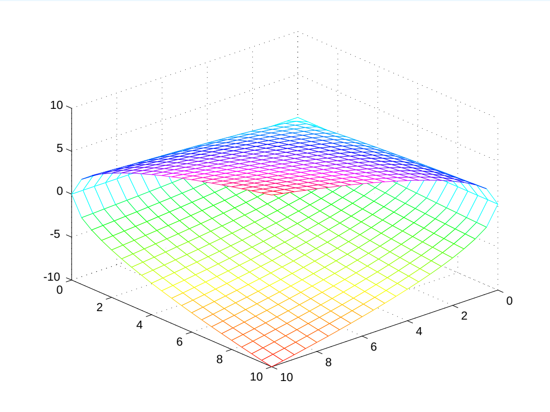
Elipsoid

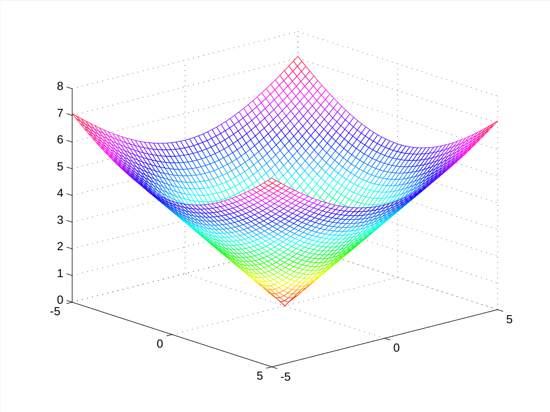
$$\mathcal{E}(c,P) = \{x \mid (x-c)^T P^{-2}(x-c) \leq 1\}$$

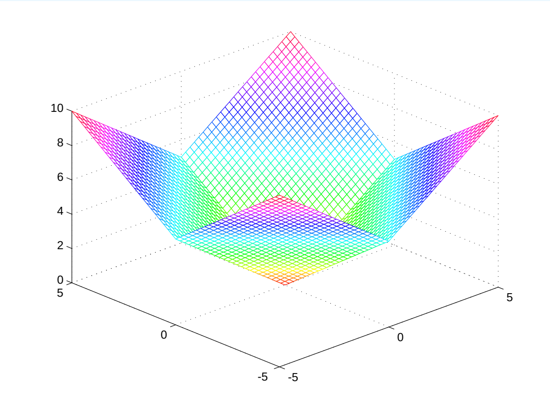
- $P \succ 0, \; P = Q\Lambda Q^T, \; \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$
- λ_i - lenghts of the semiaxes of the ellipsoid
- $\mathcal{E}(c, rI) = \mathcal{B}(c,r)$
- Different representation: $\mathcal{E}(c,P) = \{c + Au \mid \|u\|_2 \leq 1\}$, $P^2 = AA^T$
- $\mathcal{B}(c,r) = \{c + ru \mid \|u\|_2 \leq 1\}$



norm cone	$\mathcal{K}_{norm} = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$
second order cone	$\mathcal{K}_2 = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq t\}$
positive semidefinite matrices	$\mathcal{S}^n_+ = \{X \in \mathcal{S}^n \mid z^T X z \geq 0 \; \forall z\}$ $(\mathcal{S}^n = \{X \in \mathbb{R}^{n\times n}, X = X^T\})$







Operations preserving convexity

- intersection $S_1 \cap S_2$, $\bigcap_{\alpha \in A} S_\alpha$
- sum $S_1 + S_2 = \{x_1 + x_2 \mid x_1 \in S_1, x_2 \in S_2\}$
- cartsian product $S_1 \times S_2 = \{(x_1, x_2) \mid x_1 \in S_1, x_2 \in S_2\}$

- affine transformation $f : \mathbb{R}^n \to \mathbb{R}^m,\ f(x) = Ax + b$

$$f(S) = \{f(x) \mid x \in S)\}, \quad f^{-1}(S) = \{x \mid f(x) \in S\}$$

- perspective $P : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$

$$P(x, t) = \frac{x}{t}$$

- linear fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \mathcal{D}(f) = \{x \mid c^T x + d > 0\}$$

Separation of convex sets

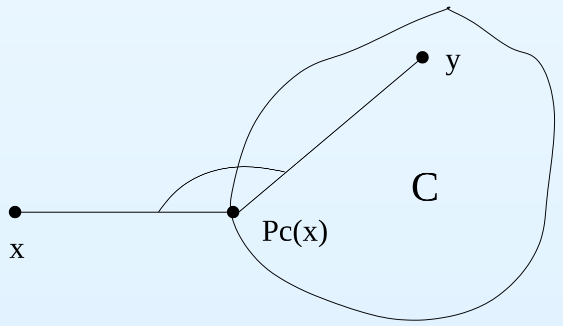
Lemma 1. Assume $C \subseteq \mathbb{R}^n$ is nonempty, convex and closed set. Then $\forall x \in C$ there exists an unique vector

$$P_C(x) = \arg\min_{z \in C} \|z - x\|_2$$

and

$$(y - P_C(x))^T (x - P_C(x)) \leq 0, \ \forall y \in C.$$

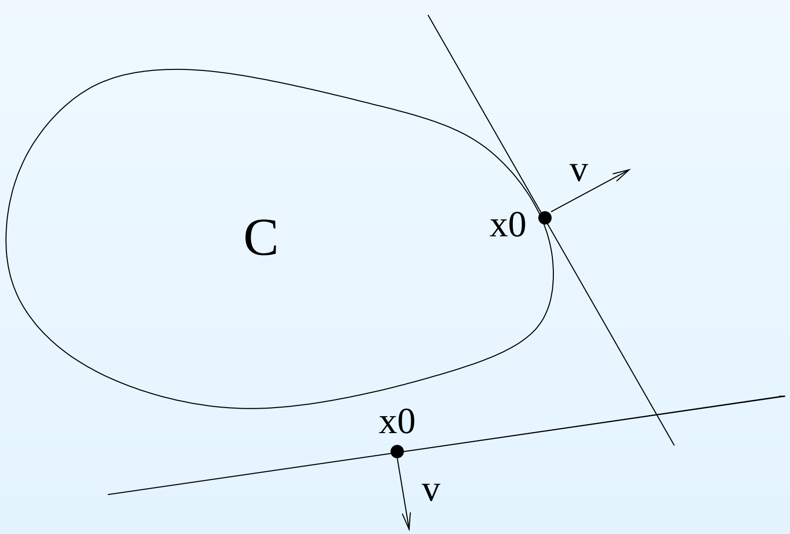
The vector $P_C(x)$ is called **projection** of the vector x onto the set C.



Supporting hyprplane theorem

Theorem 1. Assume $C \neq \emptyset$ is a convex set and $x_0 \notin \mathbf{int}(C)$. Then there exists a vector $v \neq 0$ such that

$$v^T x_0 \geq v^T x, \quad \forall x \in C.$$



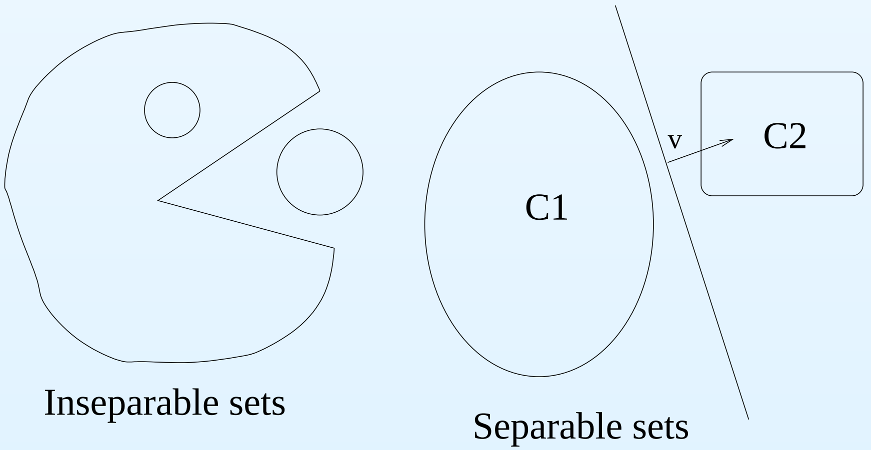
Separating hyperplane theorem

Theorem 2. Assume $\emptyset \neq A$, $\emptyset \neq B$ are convex sets, $A \cap B = \emptyset$. Then there exists a vector $v \neq 0$ such that the followin implication holds:

$$x \in A,\ y \in B \ \Rightarrow\ v^T x \leq v^T y.$$

Analogously: there exists a vector $v \neq 0$ and a constant $\gamma \in \mathbb{R}$ such that

$$v^T x \leq \gamma,\ \forall x \in A, \qquad v^T y \geq \gamma,\ \forall y \in B.$$



Separating hyperplane theorem - strict separation

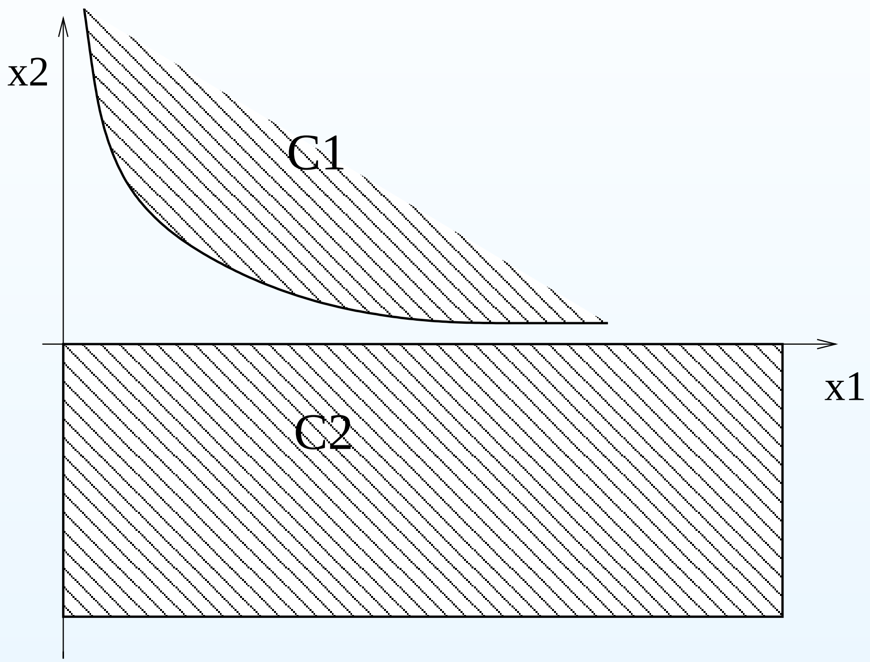
Theorem 3. Assume $C \neq \emptyset$ is a convex and closed set and $x_0 \notin C$. Then there exists a vector $v \neq 0$ and a constant $\gamma \in \mathbb{R}$ such that

$$v^T x_0 < \gamma, \qquad v^T z > \gamma, \ \forall z \in C.$$

Separating hyperplane theorem - strict separation

Theorem 4. Assume $\emptyset \neq A$, $\emptyset \neq B$ are closed, convex sets, B is bounded and $A \cap B = \emptyset$. Then there exists a vector $v \neq 0$ and a constant $\gamma \in \mathbb{R}$ such that

$$v^T x > \gamma, \ \forall x \in A, \qquad v^T y < \gamma, \ \forall y \in B.$$



2 convex and closed sets are not necessarily separable

- $C_1 = \{(x_1, x_2) \mid x_1 x_2 \geq 1\}$
- $C_2 = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \leq 0\}$

Theorem 5. Assume C is a convex cone and $x_0 \notin C$. Then there exists a vector $v \neq 0$ such that

$$v^T x_0 < 0, \qquad v^T z \geq 0, \ \forall z \in C.$$