Convex optimization

Convex optimization problems

Topic 4: Convex optimization problems

- Basic terminology
- Formulation of equivalent problems
- Local and global optimum
- Optimality conditions for differentiable functions
- Bisection method for solving quasi-convex problems
- Convex optimization classes

Convex optimization problem in standard form

$$\left.\begin{array}{ll} Min & f_0(x) \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p. \end{array}\right\} \quad (CO)$$

$f_i(x) : \mathbb{R}^n \to \mathbb{R},\ i = 0, 1, \ldots, m$ - convex functions
$h_i(x) : \mathbb{R}^n \to \mathbb{R},\ i = 1, \ldots, p$ - affine functions

If f_0 is quasi-convex - **quasi-convex optimization problem**

Feasible solution set:

$$\mathcal{P} = \{x \mid f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p.\}$$

Optimal value:

$$p^* = \inf\{f_0(x) \mid x \in \mathcal{P}\} \in \mathbb{R} \cup \pm\infty, \quad p^* = +\infty \ \Leftrightarrow \ \mathcal{P} = \emptyset.$$

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- The point $x^* \in \mathcal{P}$ is called **optimal** if

$$f_0(x^*) = p^*, \text{ resp. } f_0(x^*) \leq f_0(x) \; \forall x \in \mathcal{P}.$$

- $\mathcal{P}^*$ - optimal solutions set - convex
- The point $x_\varepsilon \in \mathcal{P}$ is called ε-suboptimal, if

$$f_0(x_\varepsilon) \leq p * + \varepsilon, \text{ resp. } f_0(x_\varepsilon) \leq f_0(x) + \varepsilon, \; \forall x \in \mathcal{P} \quad (\varepsilon > 0).$$

- Locally optimal solution $\hat{x} \in \mathcal{P}$:

$$\exists r > 0: \; f_0(\hat{x}) \leq f_0(x), \; \forall x \in \mathcal{P}, \|x - \hat{x}\|_2 \leq r$$

Feasibility problem

Find x: $\quad f_i(x) \leq 0, \; i = 1, \dots, m, \quad h_i(x) = 0, \; i = 1, \dots, p.$

$$\begin{aligned} Min \quad & 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p. \end{aligned}$$

Formulation of equivalent problems:

- **Transformation of variables**

  Assume $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective map. Define

  $$F_i(y) = f_i(\phi(y)),\ i = 0, 1, \ldots, m,\ H_i(y) = h_i(\phi(y)),\ i = 1, \ldots, p$$

  Problem (CO) is equivalent to

  $$\left.\begin{array}{lll} Min & F_0(y) & \\ & F_i(y) \leq 0, & i = 1, \ldots, m \\ & H_i(y) = 0, & i = 1, \ldots, p. \end{array}\right\} (E1)$$

  x^* is optimal for (CO) $\Rightarrow y^* = \phi^{-1}(x^*)$ is optimal for (E1)
  y^* is optimal for (E1) $\Rightarrow x^* = \phi(y^*)$ is optimal for (CO)

Formulation of equivalent problems

- **Transformation of functions**

  Assume $\psi_0, \psi_1, \ldots, \psi_m : \mathbb{R} \to \mathbb{R}$ have the following properties:
  - ψ_0 - is increasing and convex
  - $\psi_i(u)$ - *nondecreasing and convex OR non-increasing and concave*
  - $\psi_i(u) \leq 0 \;\Leftrightarrow\; u \leq 0, \quad \forall i = 1, \ldots, m$

  Define

  $$\tilde{f}_i(z) = \psi_i(f_i(x)),\; i = 0, 1, \ldots, m$$

  Problem (CO) is equivalent to

  $$\left.\begin{array}{ll} Min & \tilde{f}_0(x) \\ & \tilde{f}_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p. \end{array}\right\} (E2)$$

Formulation of equivalent problems

- **Elimination of linear constraints**

  Consider the constraints $h_i(x) = 0, \; i = 1, \ldots, p$ in the form $Ax = b$ where $A \in \mathbb{R}^{p \times x}, b \in \mathbb{R}^p$.
  - If $b \notin \mathcal{S}(A) \Rightarrow$ the problem is infeasible.
  - If $b \in \mathcal{S}(A) \Rightarrow$ then any solution of the system $Ax = b$ can be expressed as $Fz + x_0$, where $x_0 \in \mathbb{R}^n$ is a (fixed) solution of $Ax = b$, $F \in \mathbb{R}^{n \times k}$ ($k = n - h(A)$) is a matrix satisfying $\mathcal{S}(F) = \mathcal{N}(A)$ a $z \in \mathbb{R}^k$ is arbitrary.

  Problem (CO) is equivalent to

$$\left.\begin{array}{ll} Min & f_0(Fz + x_0), \\ & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m. \end{array}\right\} (E3)$$

Formulation of equivalent problems

- **Slack variables**
  - $f_k(x) \leq 0 \longrightarrow f_k(x) + s_k = 0,\ s_k \geq 0$
  - to preserve the convexity - f_k are assumed to be affine

- **Epigraph formulation**

  Problem (CO) is equivalent to

$$\left.\begin{array}{lll} Min & t & \\ & f_0(x) - t \leq 0, & \\ & f_i(x) \leq 0, & i = 1, \ldots, m, \\ & h_i(x) = 0, & i = 1, \ldots, p. \end{array}\right\} (E4)$$

Formulation of equivalent problems

- **Partial optimization**

  It holds
  $$\inf_{x_1, x_2} f(x_1, x_2) = \inf_{x_1} \inf_{x_2} f(x_1, x_2).$$

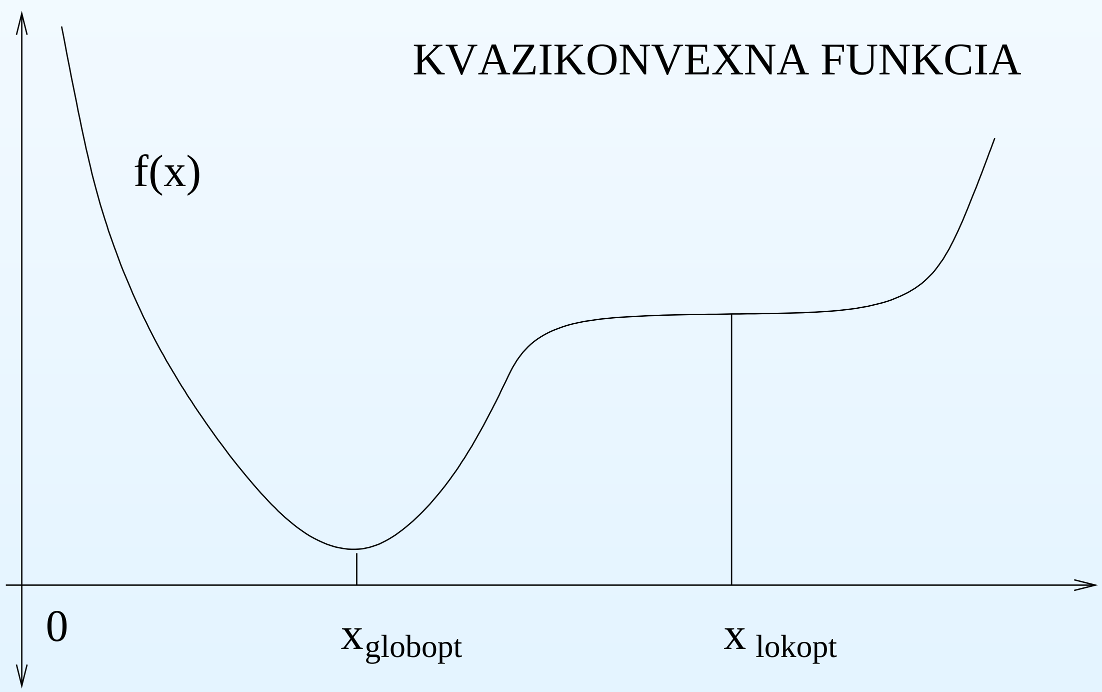
  Let $x = (x_1, x_2) \in \mathbb{R}^n$. Consider the problem
  $$\begin{array}{ll} Min_{x_1, x_2} & f_0(x_1, x_2) \\ & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

  The problem can be solved in 2 phases:

  1. $Min_{x_1} f_0(x_1, x_2)$ - find analytical solution x_2^*.
  2. $$\begin{array}{ll} Min_{x_1} & f_0(x_1, x_2^*) \\ & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

Local and global minimum

- Every local optimum of a convex problem is also a global optimum.
- Does not hold for quasi-convex optimization problems!!



Optimality conditions for differentiable functions

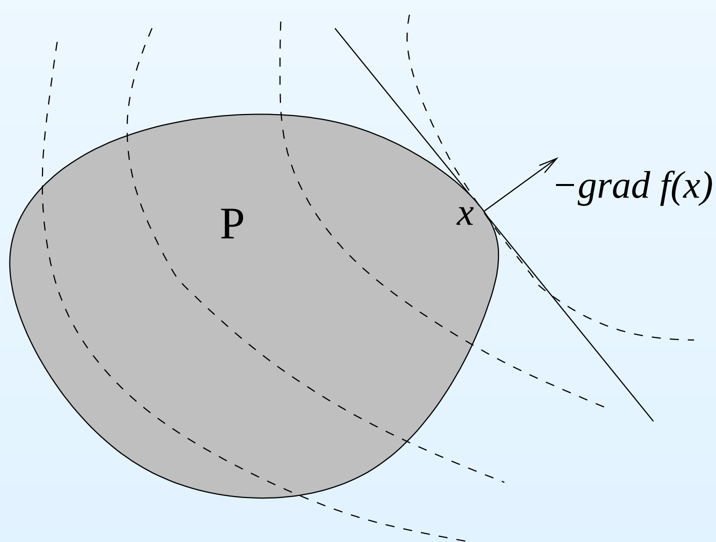
- If f_0 is convex and differentiable then

$$\forall x, y, \quad x \neq y : f_0(x) \geq f_0(y) + \nabla f_0(y)^T (x - y)$$

- $\hat{x}$ is optimal solution of the problem (CO) $\Leftrightarrow$ $\hat{x} \in \mathcal{P}$ and

$$\forall x \in \mathcal{P} \quad \nabla f_0(\hat{x})^T (x - \hat{x}) \geq 0 \qquad (1).$$

- **geometric interpretation**: vector $-\nabla f(\hat{x})$ defines the supporting hyperplane of the set $\mathcal{P}$ at the point $\hat{x}$



Unconstrained convex problems The condition (1) is reduced to

$$\nabla f_0(\hat{x}) = 0.$$

Example:

Minimizing quadratic function $f_0(x) = \frac{1}{2}x^T Px + q^T x + r$. Necessary and sufficient condition of optimality is

$$Px + q = 0.$$

- $q \notin \mathcal{S}(P)$ - f_0 is unbounded from below - the solution does not exist.
- $P \succ 0$ - unique solution $\hat{x} = -P^{-1}q$.
- $P \succeq 0$ - singular, $q \in \mathcal{S}(P)$ - the optimal solution set can be expressed as $\mathcal{P}^* = \{-P^{\dagger}q + \mathcal{N}(P)\}$.

Equality constraint convex problems

- The condition (1) is of the form

$$\nabla f_0(\hat{x})^T(x - \hat{x}) \geq 0, \quad \forall x : \; Ax = b$$

- Any solution of the system $Ax = b$ can be expressed as $x = \hat{x} + \mathcal{N}(A)$. The condition (1) has the form

$$\nabla f_0(\hat{x})^T v = 0, \quad \forall y : \; y \in \mathcal{N}(A) \quad (Ay = 0)$$

(since $\mathcal{N}(A)$ is a subspace)

- It holds $\nabla f_0(\hat{x}) \in \mathcal{N}(A)^{\perp} = \mathcal{S}(A^T)$. Optimality conditions for $\hat{x}$ are

$$\exists w \in \mathbb{R}^p : \quad A^T w = \nabla f_0(\hat{x}),$$
$$A\hat{x} = b.$$

- if f_0 is quasi-convex and differentiable, then

$$\forall x, y,\ x \neq y : f_0(x) \leq f_0(y) \ \Rightarrow\ \nabla f_0(y)^T(x-y) \leq 0$$

- **Quasi-convex optimization problem**: If $\hat{x} \in \mathcal{P}$ and

$$\forall x \in \mathcal{P} \quad \nabla f_0(\hat{x})^T(x-\hat{x}) > 0,$$

  then $\hat{x}$ is the optimal solution.

- For convex functions the condition $\nabla f_0(\hat{x}) = 0$ (together with feasibility) guarantees the optimality of $\hat{x}$. Does not hold in general for quasi-convex functions!

- For convex functions we have the **the necessary and sufficient condition** , for quasi-convex problems we have only the **sufficient** condition.

Solving quasi-convex problems via convex feasibility problems

Consider the quasi-convex optimization problem

$$\left.\begin{array}{ll} Min & f_0(x) \\ & f_i(x) \le 0, \quad i = 1, \dots, m \\ & Ax = b, \end{array}\right\} \quad (KKO)$$

i. e. the function f_0 is quasi-convex and the functions f_i are convex $(i = 1, \dots, m)$.

- Let $\phi_t(x) : \mathbb{R}^n \to \mathbb{R}$, $t \in \mathbb{R}$, be the class of convex functions such that

$$f_0(x) \le t \;\Leftrightarrow\; \phi_t(x) \le 0.$$

and for fixed x is $\phi_t(x)$ non-increasing in t.

- Example. $f_0(x) = \frac{c^T x + d}{e^T x + f}$, $\mathcal{D}(f_0) = \{x \mid e^T x + f > 0\}$.

$$\phi_t(x) = c^T x + d - t(e^T x + f)$$

- Denote p^* the optimal value of the problem (KKO). Consider the feasibility problem:

  Find x:
  $\phi_t(x) \leq 0,\ f_i(x) \leq 0,\ i = 1, \ldots, m, \quad Ax = b, \qquad (UP)$

  - If the problem is feasible, then $p^* \leq t$.
  - If the problem is infeasible, then $p^* \geq t$.

- Assume that the problem (KKO) is feasible and $p^* \in [a, b]$.

BISECTION METHOD FOR QUASI-CONVEX PROBLEMS

Input:	a, b, tolerance ε.
Repeat	1. $t := (a+b)/2$,
	2. Solve the feasibility problem,
	3. If (UP) is feasible , $b := t$, else $a := t$,
until	$b - a < \varepsilon$.

To find the ε-suboptimal solution we need to solve

$$N = \left\lceil \log_2 \left(\frac{b-a}{\varepsilon} \right) \right\rceil$$

convex feasibility problems.

Generalized convex optimization problem:

$$\begin{array}{ll} Min & f_0(x) \\ & f_i(x) \preceq_{\mathcal{K}_i} 0, \quad i = 1, \ldots, m \\ & Ax = b, \end{array}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$, $f_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$,

$\mathcal{K}_0 = \mathbb{R}_+, \mathcal{K}_i \subseteq \mathbb{R}^{n_i}, (i = 1, \ldots, m)$ are proper cones

the functions $f_i, (i = 0, 1, \ldots, m)$ are $\mathcal{K}_i$-convex.

If the functions $f_0, f_1, \ldots, f_m$ are linear $\Rightarrow$

$\mathcal{K}$	$\mathbb{R}^n_+$	$\mathcal{S}^n_+$	$\mathcal{K}_{\|\cdot\|_2}$
problem	LP	SDP	SOCP

Known convex optimization classes

- Linear programming
  $c \in \mathbb{R}^n,\ F \in \mathbb{R}^{m\times n},\ g \in \mathbb{R}^m,\ A \in \mathbb{R}^{p\times n},\ b \in \mathbb{R}^p$

$$\left.\begin{array}{ll} Min & c^T x \\ & Fx \le g \\ & Ax = b \end{array}\right\} \quad (LP)$$

- Quadratic programming
  $P_j \in \mathcal{S}^n_+,\ \ q_j \in \mathbb{R}^n,\ r_j \in \mathbb{R},\ j = 0, 1, \ldots, m,\ A \in \mathbb{R}^{p\times n},\ b \in \mathbb{R}^p$

$$\left.\begin{array}{ll} Min & x^T P_0 x + q_0^T x + r_0 \\ & x^T P_i x + q_i^T x + r_i \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}\right\} \quad (QP)$$

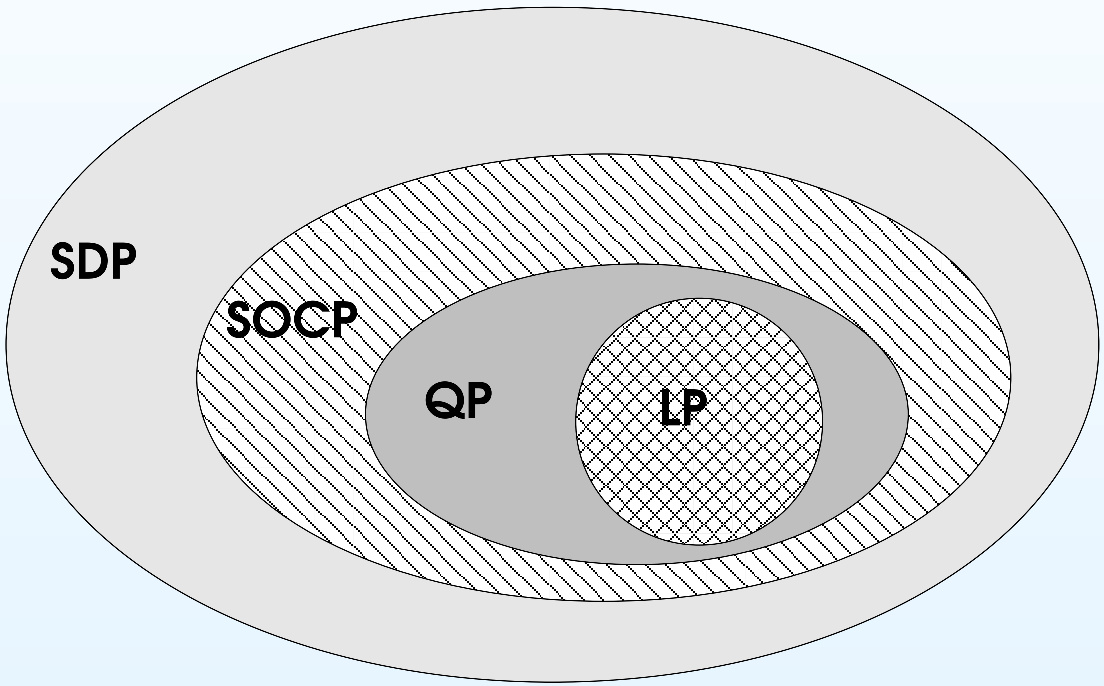
- Second order cone programming
  $c \in \mathbb{R}^n,\ F_i \in \mathbb{R}^{n_i \times n},\ g_i \in \mathbb{R}^{n_i},\ f_i \in \mathbb{R}^n,\ h_i \in \mathbb{R},\ i = i, \ldots, m, A \in \mathbb{R}^{p \times n},\ b \in \mathbb{R}^p$

$$\left.\begin{array}{ll} Min & c^T x \\ & \|F_i x + g_i\|_2 \leq f_i^T x + h_i, \quad i = 1, \ldots, m \\ & Ax = b \end{array}\right\} \quad (SOCP)$$

- Semidefinite programming
  $c \in \mathbb{R}^n,\ F_i \in \mathcal{S}^n,\ i = 1, \ldots, n,\ G \in \mathcal{S}^n,\ A \in \mathbb{R}^{p \times n},\ b \in \mathbb{R}^p$

$$\left.\begin{array}{ll} Min & c^T x \\ & \sum_{i=1}^n F_i x_i \preceq G \\ & Ax = b \end{array}\right\} \quad (SDP)$$

Relations between the classes



Geometric programming

The function $f : \mathbb{R}^n_{++} \to \mathbb{R}$,

$$f(x) = f(x_1, \ldots, x_n) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}},$$

where $c_k > 0$ and $a_{ik} \in \mathbb{R},\ i = 1, \ldots, n,\ k = 1, \ldots, K$, is called **posynomial function** of degree K.

Geometric programming problem

$$\left.\begin{array}{ll} Min & f_0(x) \\ & f_i(x) \leq 1, \quad i = 1, \ldots, m \\ & h_j(x) = 1, \quad j = 1, \ldots, p \end{array}\right\} \quad (GP)$$

where $f_0, f_1, \ldots, f_m$ are posynomials of degree K_i and $h_1, \ldots, h_p$ are monomials.

Convex formulations of the geometric programming problem

- transformation of variables - $\phi(x) = (e^{x_1}, \ldots, e^{x_n})$
- transformation of functions - $\psi(u) = \ln u$
- $f_i(x) = \sum_{k=1}^{K_i} c_{ki} x_1^{(a_{ik})_1} \ldots x_n^{(a_{ik})_n}$
- $h_j(x) = d_j x_1^{g_{j1}} \ldots x_n^{g_{jn}}$

$$\left.\begin{array}{lll} Min & \tilde{f}_0(x) = \ln\left(\sum_{k=1}^{K_0} e^{a_{0k}^T x + b_{0k}}\right) & \\ & \tilde{f}_i(x) = \ln\left(\sum_{k=1}^{K_i} e^{a_{ik}^T x + b_{ik}}\right) \leq 0, & i = 1, \ldots, m \\ & \tilde{h}_j(x) = g_j^T x + h_j = 0, & j = 1, \ldots, p, \end{array}\right\} \quad (KGP)$$

where
$a_{ik} = ((a_{ik})_1, \ldots, (a_{ik})_n)$, $g_j = (g_{j1}, \ldots, g_{jn})$, $b_{ki} = \ln c_{ki}$, $h_j = \ln d_j$,
$i = 1, \ldots, m$, $j = 1, \ldots, p$