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STRONG DUALITY CONDITIONS IN SEMIDEFINITE PROGRAMMING

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Semidefinite optimization problems unify and generalize 'standard' problems, such as linnear or quadratic programming. They have linear objective function and the variable is a symmetric matrix, which is required to be positive semidefinite, or (in dual case) the constraint is that an affine combination of symmetric matrices is positive semidefinite. An additional regularity property, that is, an existence of an interior point in the feasible set is needed to prove strong duality in SDP. It is known that this property implies that the dual optimal solution set is nonempty and bounded. We show that it is not only the sufficient but also the necessary condition. So, as a consequence we obtain two new conditions of strong duality.

K e y w o r d s: convex programming, semidefinite programming, strong duality, interior point methods

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1 INTRODUCTION

In 1984 Karamarkar published the polynomial "projective" algorithm for solving linear programming (LP) problems, which was very close to interior point methods (IPM). This was the beginning of an IPM developement and a suggestion for an uprise of new mathematical programming classes such as semidefinite programming (SDP) or cone programming.

Moreover, more duality results were developed. One of these works was [2], where it was shown that the duality theory in LP can be derived using IPM. In this paper it was also stated and proved that

(i) $\mathcal{P} \neq \emptyset, \mathcal{D}^0 \neq \emptyset \Leftrightarrow \mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ is bounded;

(ii) $\mathcal{D} \neq \emptyset, \mathcal{P}^0 \neq \emptyset$ and the constraint matrix rank is maximal $\Leftrightarrow \mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ is bounded;

where we denote $\mathcal{P}, \mathcal{P}^0, \mathcal{P}^*(\mathcal{D}, \mathcal{D}^0, \mathcal{D}^*)$ the primal (dual) feasible/strictly feasible/optimal solution set.

Many results in LP analogously hold in SDP, but some do not. (see [1],[4]) The reason is that the feasible set in SDP is an intersection of a finite number of hyperplanes and a convex, but nonpolyhedral (like in LP) cone. We were motivated by the fact, that in SDP only the implications from the left to the right in statements (i), (ii) above are known. Our goal is to show that the equivalences hold.

2 DEFINITIONS

We denote S^n the vector space of $n \times n$ real symmetric matrices, $dim(S^n) = n(n+1)/2$. We write " $\bullet$ " for the inner product defined on S^n as

$$\mathbf{X} \bullet \mathbf{Y} := trace(\mathbf{XY}).$$

$\mathbf{X} \succeq 0, (\mathbf{X} \succ 0)$ is notation for positive semidefinite (positive definite) matrix. The set of all positive semidefinite matrices forms a closed convex cone. The interior of this cone consists of all positive definite matrices (that is all nonsingular positive semidefinite matrices).

Let $\mathbf{A}_1, \dots, \mathbf{A}_m, \mathbf{C} \in S^n$ and a vector $b \in R^m$. Then

$$(P) \quad \min \left\{ \mathbf{C} \bullet \mathbf{X} \mid \mathbf{A}_i \bullet \mathbf{X} = b_i;\ i = 1, \dots, m;\ \mathbf{X} \succeq 0 \right\}$$

is primal SDP problem in standard form, and

$$(D) \qquad \max \left\{ b^T y \mid \mathbf{C} - \sum_{i=1}^{m} \mathbf{A}_i y_i \succeq 0 \right\}$$

is dual SDP problem in standard form. We denote

$$\mathcal{P} = \{\mathbf{X} \in S^n | \mathbf{A}_i \bullet \mathbf{X} = b_i,\ i = 1, \dots, m;\ \mathbf{X} \succeq 0\},$$

$$\mathcal{P}^0 = \{\mathbf{X} \in S^n | \mathbf{A}_i \bullet \mathbf{X} = b_i,\ i = 1, \dots, m;\ \mathbf{X} \succ 0\}$$

primal feasible/ primal strictly feasible set, and similarly

$$\mathcal{D} = \{y \in R^m | \sum_{i=1}^{m} \mathbf{A}_i y_i \preceq \mathbf{C}\},$$

$$\mathcal{D}^0 = \{y \in R^m | \sum_{i=1}^{m} \mathbf{A}_i y_i \prec \mathbf{C}\}$$

dual feasible/ dual strictly feasible set. We write p^*, d^* for the primal and the dual optimal value, that is

$$p^* = \inf\{\mathbf{C} \bullet \mathbf{X} \mid \mathbf{X} \in \mathcal{P}\},$$

$$d^* = \sup\{b^T y \mid y \in \mathcal{D}\},$$

and define $p^* = +\infty$ if $\mathcal{P} = \emptyset$ and $d^* = -\infty$ if $\mathcal{D} = \emptyset$. Finaly we denote

$$\mathcal{P}^* = \{\mathbf{X} \in \mathcal{P} | \mathbf{C} \bullet \mathbf{X} = p^*\}$$

$$\mathcal{D}^* = \{y \in \mathcal{D} | b^T y = d^*\}$$

the primal and dual optimal solution set.

In SDP the weak duality holds, that is $d^* \leq p^*$. If the equality $d^* = p^*$ holds, then we say that strong duality holds.

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3 THEOREMS OF ALTERNATIVES CONSEQUENCES

We show that the both equivalences, similar to the ones in LP we mentioned in the first section, hold. Particular results (stated in Theorem 1 and Theorem 2) can be found in [4].

Theorem 1. If $\mathcal{P} \neq \emptyset, \mathcal{D}^0 \neq \emptyset$, then $\mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ is bounded.

Theorem 2. If $\mathcal{D} \neq \emptyset, \mathcal{P}^0 \neq \emptyset$ and the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ are linearly independent, then $\mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ is bounded.

Now, we prove that the reverse holds. First, we introduce an auxiliary lemma, which is a kind of theorem of alternatives and can be found in generalized form in [6].

Lemma 1. Let $\mathbf{G}_i \in S^n,\ i = 0, 1, \ldots, m$. Then exactly one of the statements is true:

1. $\exists u \in R^m :\ \mathbf{G}_0 - \sum_{i=1}^m \mathbf{G}_i u_i \succ 0$
2. $\exists \mathbf{Z} \in S^n,\ \mathbf{Z} \succeq 0,\ \mathbf{Z} \neq \mathbf{0}\ :\ \mathbf{Z} \bullet \mathbf{G}_i = 0\ \forall i = 1, \ldots m, \quad \mathbf{Z} \bullet \mathbf{G}_0 \leq 0$.

Proof. Suppose that both alternatives hold at once. Compute

$$\mathbf{Z} \bullet (\mathbf{G}_0 - \sum_{i=1}^m \mathbf{G}_i u_i) = \mathbf{Z} \bullet \mathbf{G}_0 \leq 0.$$

But the properties of the matrices at the lefthand side guarantee that the inner product must be positive. Next we will show that at least one of the alternatives holds. Let

$$C = \{\mathbf{G}_0 - \sum_{i=1}^m \mathbf{G}_i u_i \mid u \in R^m\}, \quad D = \{\mathbf{X} \in S^n \mid \mathbf{X} \succ 0\}.$$

If $C \cap D \neq \emptyset$, then alternative 1 holds. Suppose $C \cap D = \emptyset$. Obviously, C is a convex subset of S^n and D is an open convex cone. From the separating hyperplane theorem we obtain that

$$\exists \mathbf{Z} \in S^n, \mathbf{Z} \neq 0 :\ \mathbf{Z} \bullet \mathbf{X} \leq 0,\ \forall \mathbf{X} \in C,\ \mathbf{Z} \bullet \mathbf{X} \geq 0,\ \forall \mathbf{X} \in D.$$

From the second inequality it follows that $\mathbf{Z} \succeq 0$. From the first inequality we obtain

$$0 \geq \mathbf{Z} \bullet (\mathbf{G}_0 - \sum_{i=1}^m \mathbf{G}_i u_i) = \mathbf{Z} \bullet \mathbf{G}_0 - \sum_{i=1}^m (\mathbf{Z} \bullet \mathbf{G}_i) u_i,\ \forall u \in R^m.$$

Assume that there exists $j \in \{1, \ldots, m\}$ so that $\mathbf{Z} \bullet \mathbf{G}_j \neq 0$. Then we can find $u = (0, \ldots, 0, u_j, 0, \ldots, 0) \in R^m$ which satisfies

$$\mathbf{Z} \bullet \mathbf{G}_0 - (\mathbf{Z} \bullet \mathbf{G}_j) u_j > 0.$$

Hence $\mathbf{Z} \bullet \mathbf{G}_i = 0,\ \forall i = 1, 2, \ldots, m$ and we immediately obtain $\mathbf{Z} \bullet \mathbf{G}_0 \leq 0$.
■

Theorem 3. If $\mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ is bounded, then $\mathcal{D}^0 \neq \emptyset$.

Proof. Suppose $\mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ is bounded. We show that $\mathcal{D}^0 \neq \emptyset$. Since $\mathcal{P}^* \neq \emptyset$, there exists $\mathbf{X}^* \in \mathcal{P}^*$, and so $p^* = \mathbf{C} \bullet \mathbf{X}^*$ is finite. A point in $\mathcal{P}^*$ is primal optimal if and only if it satisfies the following conditions:

$$\mathbf{A}_i \bullet \mathbf{X}^* = b_i,\ \forall i = 1, 2, \ldots m, \quad \mathbf{C} \bullet \mathbf{X}^* \leq p^*. \qquad (1)$$

Because the value p^* is minimal possible, we could replace the equality optimality condition $p^* = \mathbf{C} \bullet \mathbf{X}^*$ with the inequality condition in the second relation of (1). Now, suppose $\mathcal{D}^0 = \emptyset$. That means

$$\nexists y \in R^m :\ \mathbf{C} - \sum_{i=1}^m \mathbf{A}_i y_i \succ 0.$$

We can apply Lemma 1 and obtain:

$$\exists \mathbf{Z} \in S^n,\ \mathbf{Z} \succeq 0,\ \mathbf{Z} \neq \mathbf{0} :$$

$$\mathbf{Z} \bullet \mathbf{A}_i = 0\ \forall i = 1, \ldots, m,\ \mathbf{Z} \bullet C \leq 0$$

Consider the ray $\{\mathbf{W}_t = \mathbf{X}^* + t\mathbf{Z} \mid t \geq 0\}$. We have $\mathbf{A}_i \bullet \mathbf{W}_t = b_i$ for all $i = 1, 2, \ldots, m$ and $t \geq 0$. Further we have $\mathbf{C} \bullet \mathbf{W}_t = \mathbf{C} \bullet \mathbf{X}^* + t\mathbf{C} \bullet \mathbf{Z} \leq p^*$. Hence the ray $\{\mathbf{W}_t \mid t \geq 0\}$ lies in the set $\mathcal{P}^*$. But this contradicts the fact that $\mathcal{P}^*$ is bounded, so the set $\mathcal{D}^0$ must be nonempty.
■

The following lemma of alternatives will be useful for proving our next result stated in Theorem 4.

Lemma 2. Assume $\mathbf{G}_i \in S^n,\ i = 1, \ldots, m$ and $g \in R^m$. Then exactly one of the statements is true:

1. $\exists \mathbf{Z} \in S^n,\ \mathbf{Z} \succ 0 : \quad \mathbf{Z} \bullet \mathbf{G}_i = g_i,\ \forall i = 1, \ldots m \quad \wedge$ $\mathbf{G}_1, \ldots, \mathbf{G}_m$ are lineary independent
2. $\exists u \in R^m,\ u \neq 0 : \quad \sum_{i=1}^m \mathbf{G}_i u_i \succeq 0\ \wedge\ u^T g \leq 0$

Proof. First, we prove that both alternatives cannot hold at once. Compute

$$g^T u = \sum_{i=1}^m (\mathbf{G}_i \bullet \mathbf{X}) u_i = \mathbf{X} \bullet \sum_{i=1}^m \mathbf{G}_i u_i > 0,$$

where the last inequality follows from the fact that $\mathbf{X} \succ 0$ and that $\sum_{i=1}^m \mathbf{G}_i u_i \succeq 0$ is nonzero. Now, we show that at least one of the alternatves hold. Let

$$C = \{h \in R^m \mid \exists \mathbf{X} \succ 0 :\ \mathbf{G}_i \bullet \mathbf{X} = h_i,\ i = 1, 2, \ldots, m\},$$

$$D = \{g\}.$$

If $C \cap D \neq \emptyset$, then the first part of alternative 1 holds. If $C \cap D = \emptyset$, then from the separating hyperplane theorem and the fact that C is a convex cone we obtain

$$\exists u \in R^m\ u \neq 0 :\ u^T g \leq 0\ \wedge\ u^T h \geq 0,\ \forall h \in C.$$

From the second inequality we have

$$0 \leq u^T h = \sum_{i=1}^{m} u_i(\mathbf{G}_i \bullet \mathbf{X}) = \mathbf{X} \bullet \sum_{i=1}^{m} u_i \mathbf{G}_i, \qquad \forall \mathbf{X} \succ 0.$$

This implies $\sum_{i=1}^{m} u_i \mathbf{G}_i \succeq 0$ and so the alternative 2 must hold. Finally suppose that the alternative 2 does not hold. That means

$$\forall u \in R^m \ u \neq 0 : \ \sum_{i=1}^{m} u_i \mathbf{G}_i \succeq 0 \Rightarrow u^T g > 0. \qquad (2)$$

If the matrices $\mathbf{G}_1, \ldots, \mathbf{G}_m$ were not linearly independent, then $\exists y \in R^m, y \neq 0 : \sum_{i=1}^{m} y_i \mathbf{G}_i = 0$. But $\mathbf{0} \succeq 0$ and so (2) implies $y^T g > 0$.
■

Theorem 4. If $\mathcal{D}^* \neq \emptyset$ and $\mathcal{D}^*$ is bounded, then $\mathcal{P}^0 \neq \emptyset$ and the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ are linearly independent.

Proof. Suppose $\mathcal{D}^* \neq \emptyset$ and $\mathcal{D}^*$ is bounded. We show that $\mathcal{P}^0 \neq \emptyset$ and matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ are linearly independent. If $\mathcal{D}^* \neq \emptyset$, then there exists $y^* \in \mathcal{D}^*$ and thus the value $d^* = b^T y^*$ is finite. A point is dual feasible if and only if it satisfies the conditions:

$$\mathbf{C} - \sum_{i=1}^{m} \mathbf{A}_i y_i^* \succeq 0, \qquad b^T y^* \geq d^*. \qquad (3)$$

Because the value d^* is maximal possible, we could replace the equality optimality condition $d^* = b^T y^*$ with the second relation in (3). Suppose $\mathcal{P}^0 = \emptyset$ or $\mathbf{A}_1, \ldots, \mathbf{A}_m$ are not linearly independent. If $\mathcal{P}^0 = \emptyset$, then

$$\nexists \mathbf{X} \succ 0 : \ \mathbf{A}_i \bullet \mathbf{X} = b_i \ \forall i = 1, \ldots, m.$$

Applying Lemma 2 we obtain:

$$\exists u \in R^m, \ u \neq 0 : \ \sum_{i=1}^{m} \mathbf{A}_i u_i \succeq 0 \ \wedge \ u^T g \leq 0.$$

So we can construct a ray $\{p^t = y^* - tu \mid t \geq 0\}$ which lies in the set $\mathcal{D}^*$. This is a contradiction with the fact that $\mathcal{D}^*$ is bounded.
■

4 STRONG DUALITY IN SDP

In the next theorem we recall two well-known sufficient conditions of strong duality in SDP (see [4]). They can be considered as a special case of the Slater condition known in convex programming.

Theorem 5. a)If $\mathcal{P} \neq \emptyset$ and $\mathcal{D}^0 \neq \emptyset$, then $p^* = d^*$.

b)If $\mathcal{D} \neq \emptyset$, $\mathcal{P}^0 \neq \emptyset$ and $\mathbf{A}_1, ..., \mathbf{A}_m$ are linearly independent, then $p^* = d^*$.

As a consequence of Theorem 3 and Theorem 4 established in this paper we obtain two new sufficient conditions of strong duality.

Corollary 1. c) If $\mathcal{P}^* \neq \emptyset, \mathcal{P}^*$ is bounded, then $p^* = d^*$.

d)If $\mathcal{D}^* \neq \emptyset, \mathcal{D}^*$ is bounded, then $p^* = d^*$.

5 CONCLUSION

In semidefinite programming, the existence of a feasible point and the existence of an interior point in the feasible set of the dual counterpart is equivalent to the fact that the optimal solution set is nonempty and bounded. This result is an analogy to the known statement in linear programming (see [2]) and gives us a solid view on some aspects of the duality theory in SDP.

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