Existence of Weighted Interior-Point Paths in Semidefinite Programming

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Outline

- Semidefinite programming
- The central path
- The weighted central path
- The existence of weighted central paths

Notation

S^n	vector space of real symmetric $n \times n$ matrices, $\dim S^n = \frac{n(n+1)}{2}$
"$\bullet$"	inner product, defined on S^n as $\mathbf{A} \bullet \mathbf{B} = tr(\mathbf{AB}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{ij}$
$\mathbf{X} \succeq 0, \mathbf{X} \in S_{+}^{n}$	$\mathbf{X}$ is positive semidefinite
$\mathbf{X} \succ 0, \mathbf{X} \in S_{++}^{n}$	$\mathbf{X}$ is positive definite

Semidefinite Programming Problem

Data:

$$\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m, \mathbf{C} \in S^n$$
$$b \in R^m$$

$$\begin{array}{ll} \text{minimize} & \mathbf{X} \bullet \mathbf{C} \\ \text{subject to} & \mathbf{A}_i \bullet \mathbf{X} = b_i, \\ & i = 1, \ldots, m, \\ & \mathbf{X} \succeq 0. \end{array}$$

Primal and Dual SDP

PRIMAL	DUAL
minimize $\mathbf{X} \bullet \mathbf{C}$ subject to $\mathbf{A}_i \bullet \mathbf{X} = b_i,$ $i = 1, \ldots, m,$ $\mathbf{X} \succeq 0.$	maximize $b^T y$ subject to $\sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C},$ $\mathbf{S} \succeq 0.$

$\mathbf{X} \in S^n$ primal variable

$(y, \mathbf{S}) \in R^m \times S^n$ dual variables

Importance of SDP

- SDP contains important classes of problems as special cases, e.g.
  - linear programming
  - convex quadratic programming
  - second-order cone programming
- SDP problems can be solved in polynomial time using interior point methods .

SDP Applications

- **Direct SDP applications:**
  - Quasiconvex nonlinear programming
  - Eigenvalue problems
  - System and control theory
  - Statistics (experimental design)
- **SDP Relaxations:**
  - Combinatorial optimization
  - Quadratic programming (nonconvex)

Assumptions

Assumption 1: The matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ are linearly independent, i.e.

$$\sum_{i=1}^{m} \mathbf{A}_i u_i = 0 \quad \Rightarrow \quad u_i = 0,\ i = 1, \ldots, m.$$

Assumption 2: There exists an **Interior point** $(\mathbf{X}, y, \mathbf{S}) \in S^n \times R^m \times S^n$, i.e.

$$\begin{array}{lll} \mathbf{A}_i \bullet \mathbf{X} = b_i, & i = 1, \ldots, m & \mathbf{X} \succ 0 \\ \sum_{i=1}^{m} \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C} & & \mathbf{S} \succ 0 \end{array}$$

Optimality Conditions

$(\mathbf{X}, y, \mathbf{S})$ is **optimal** if and only if

(primal feasibility) $\mathbf{A}_i \bullet \mathbf{X} = b_i, \; i = 1, \ldots, m, \quad \mathbf{X} \succeq 0$

(dual feasibility) $\sum_{i=1}^{m} \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, \quad \mathbf{S} \succeq 0,$

(complementarity) $\mathbf{X}\mathbf{S} = 0.$

Central Path

Perturbed optimality conditions

$$\mathbf{A}_i \bullet \mathbf{X} = b_i,\ i = 1, \ldots, m, \quad \mathbf{X} \succeq 0 \quad \text{(primal feasibility)}$$

$$\sum_{i=1}^{m} \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, \quad \mathbf{S} \succeq 0 \quad \text{(dual feasibility)}$$

$$\mathbf{XS} = \mu \mathbf{I}. \quad \text{(perturbed complementarity)}$$

- For any $\mu > 0$ there exists unique solution $(\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu))$ of the system above
- The **central path** in SDP is defined as the map

$$R_{++} \rightarrow S^n \times R^m \times S^n, \qquad \mu \rightarrow (\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu)).$$

Implementation Problems

1. The product $\mathbf{XS} \notin S^n$ in general, even if $\mathbf{X}, \mathbf{S} \in S^n$
2. The interior point does not exist or is unknown

Solution:

1. So called "complementarity condition symmetrization", e.g.

$$\mathbf{XS} \longrightarrow \frac{\mathbf{XS}+\mathbf{SX}}{2}$$

2. Feasibility condition perturbation

Complementarity Condition Symmetrization

$\mathbf{XS}$ is replaced by a symmetrization matrix $\Phi(\mathbf{X},\mathbf{S}) \in S^n$:

If $\mathbf{X} \succeq 0,\ \mathbf{S} \succeq 0$, then $\mathbf{XS} = 0 \ \Leftrightarrow\ \Phi(\mathbf{X},\mathbf{S}) = 0$.

Symmetrization maps:

$$\begin{aligned}
\Phi_1(\mathbf{X},\mathbf{S}) &= (\mathbf{XS}+\mathbf{SX})/2 \\
\Phi_2(\mathbf{X},\mathbf{S}) &= \mathbf{X}^{\frac{1}{2}}\mathbf{S}\mathbf{X}^{\frac{1}{2}} \\
\Phi_3(\mathbf{X},\mathbf{S}) &= \mathbf{L}_{\mathbf{X}}^T\mathbf{S}\mathbf{L}_{\mathbf{X}} \\
\Phi_4(\mathbf{X},\mathbf{S}) &= (\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}}+\mathbf{S}^{\frac{1}{2}}\mathbf{X}^{\frac{1}{2}})/2 \\
\Phi_5(\mathbf{X},\mathbf{S}) &= (\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}+\mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}})/2
\end{aligned}$$

Infeasible Central Path

- Feasibility condition perturbation

$$\begin{array}{rcl} \mathbf{A}_i \bullet \mathbf{X} = b_i & \longrightarrow & \mathbf{A}_i \bullet \mathbf{X} = b_i + \mu \triangle b_i \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C} & \longrightarrow & \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C} + \mu \triangle \mathbf{C} \end{array}$$

- The Assumption 2 (existence of an interior point) is replaced with:

Assumption 3: There exists an optimal point $(\mathbf{X}, y, \mathbf{S}) \in S^n \times R^m \times S^n$ such that

$$\begin{array}{rcr} \mathbf{A}_i \bullet \mathbf{X} = b_i, & i = 1, \ldots, m, & \mathbf{X} \succeq 0, \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, & & \mathbf{S} \succeq 0, \\ \mathbf{X}\mathbf{S} = 0. & & \end{array}$$

Weighted central path

Motivation:

- Several numerical approaches in interior point methods prefer not to work with the identity matrix $\mathbf{I}$, but with a positive definite matrix $\mathbf{W}$ in the perturbed complementarity condition:

$$\Phi(\mathbf{X}, \mathbf{S}) = \mu \mathbf{I} \longrightarrow \Phi(\mathbf{X}, \mathbf{S}) = \mu \mathbf{W}$$

- Implementation problems 1,2

Weighted central path

The weighted central path in SDP is defined as the map

$$R_{++} \to S^n \times R^m \times S^n, \qquad \mu \to (\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu)).$$

of the solutions of the system

$$\left.\begin{array}{r} \mathbf{A}_i \bullet \mathbf{X} = b_i + \mu \triangle b_i, \quad i = 1, \ldots, m, \quad \mathbf{X} \succ 0, \\ \sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C} + \mu \triangle \mathbf{C}, \quad \mathbf{S} \succ 0, \qquad\qquad\qquad\quad \\ \Phi_j(\mathbf{X}, \mathbf{S}) = \phi_j(\mu) \mathbf{W}, \qquad\qquad\qquad\qquad\qquad\qquad\quad \end{array}\right\} \qquad (\star)$$

where $\triangle b \in R^m, \triangle \mathbf{C} \in S^n$ are fixed, $\mathbf{W} \succ 0$ is the weight, $\Phi_j(\mathbf{X}, \mathbf{S})$ is one of the symmetrization maps and

$$\phi_j(\mu) = \mu, \quad j = 1, 2, 3; \qquad \phi_j(\mu) = \sqrt{\mu}, \quad j = 4, 5.$$

Goal

- For any $j \in \{1, \ldots, 5\}$ find the set $\mathcal{W}_j$ of suitable weights
- Having $j \in \{1, \ldots, 5\}$, prove that for any $\mathbf{W} \in \mathcal{W}_j$ and any $\mu > 0$ there exists unique solution of the system $(\star)$.

Related Papers

- **R.D.C. Monteiro, P. Zanjacomo**, 2000.

  - Nonlinear semidefinite complementarity problems
  - All five symmetrizations
  - Nonlinear analysis - theory of local homeomorphic maps

- **M. Preiss, J. Stoer**, 2003.

  - Linear complementarity problems
  - The symmetrization $(\mathbf{XS}+\mathbf{SX})/2$
  - Analytic continuation, Implicit function theorem

Existence of the Weighted Central Path - Proof

Define $\mathcal{A} : S^n \to R^m, \qquad \mathcal{A}(\mathbf{X}) = [\mathbf{A}_1 \bullet \mathbf{X}, \ldots, \mathbf{A}_m \bullet \mathbf{X}]$

$\mathcal{A}^* : R^m \to S^n, \qquad \mathcal{A}^*(y) = \sum_{i=1}^m \mathbf{A}_i y_i.$

and

$$F^j_{\mu,\mathbf{W}} : S^n \times R^m \times S^n \to R^m \times S^n \times S^n$$

$$F^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S}) = \begin{bmatrix} \mathcal{A}(\mathbf{X}) - b - \mu\triangle b \\ \mathcal{A}^*(y) + \mathbf{S} - \mathbf{C} - \mu\triangle\mathbf{C} \\ \Phi_j(\mathbf{X}, \mathbf{S}) - \phi_j(\mu)\mathbf{W} \end{bmatrix}.$$

$$\left[\mathbf{X} \succ 0, \quad \mathbf{S} \succ 0, \quad F^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S}) = 0\right] \iff (\star)$$

Existence of the Weighted Central Path - Proof (continued)

Idea of the proof:

- Identify the set of suitable weights = the set with the property:

  $DF^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S})$ is a nonsingular linear map for all $(\mathbf{X}, y, \mathbf{S})$ satisfying $(\star)$.

- Choose the parameters $\triangle b, \triangle \mathbf{C}$
- Boundedness of the set of solutions of $(\star)$
- Apply The implicit function theorem and analytic continuation technique on the system $F^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S}) = 0$ (which is analytic in all variables and parameters $\mu, \mathbf{W}$
- Uniqueness of the solutions

Existence of the Weighted Central Path - Proof (continued)

Nonsingularity of the Fréchet Derivatives

If $\mathbf{X}, \mathbf{S} \in S^n_{++}$, then

$$DF^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S})[\triangle\mathbf{X}, \triangle y, \triangle\mathbf{S}] = \begin{bmatrix} \mathcal{A}(\triangle\mathbf{X}) \\ \mathcal{A}^*(\triangle y) + \triangle\mathbf{S} \\ D\Phi_j(\mathbf{X}, \mathbf{S})[\triangle\mathbf{X}, \triangle\mathbf{S}] \end{bmatrix}$$

Lemma 1: Let $\mathbf{X} \succ 0$, $\mathbf{S} \succ 0$. The following implication holds

$\Phi_j(\mathbf{X}, \mathbf{S}) \in \mathcal{W}_j \;\Rightarrow\; DF^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S})$ is a nonsingular linear map.

Existence of the Weighted Central Path - Proof (continued)

For $\varepsilon \in (0,1)$ denote

$$\mathcal{M}_\varepsilon = \{\, \mathbf{Z} \succ 0; \exists \nu : \ \|\mathbf{Z}-\nu\mathbf{I}\| < \varepsilon\nu\} = \left\{ \mathbf{Z} \succ 0; \frac{\lambda_{max}(\mathbf{Z})}{\lambda_{min}(\mathbf{Z})} < \frac{1+\varepsilon}{1-\varepsilon} \right\}.$$

j	$\Phi_j(\mathbf{X},\mathbf{S})$	$\mathcal{W}_j$
1	$(\mathbf{XS}+\mathbf{SX})/2$	S^n_{++}
2	$\mathbf{X}^{\frac{1}{2}}\mathbf{S}\mathbf{X}^{\frac{1}{2}}$	$\mathcal{M}_{\frac{1}{\sqrt{2}}}$
3	$\mathbf{L}_{\mathbf{X}}^T\mathbf{S}\mathbf{L}_{\mathbf{X}}$	$\mathcal{M}_{\frac{1}{\sqrt{2}}}$ or D^n_{++}
4	$(\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}}+\mathbf{S}^{\frac{1}{2}}\mathbf{X}^{\frac{1}{2}})/2$	$\mathcal{M}_\tau$
5	$(\mathbf{U}_{\mathbf{S}}^T\mathbf{L}_{\mathbf{X}}+\mathbf{L}_{\mathbf{X}}^T\mathbf{U}_{\mathbf{S}}^T)/2$	$\mathcal{M}_\tau$

($\tau \doteq 0,124848$)

Existence of the Weighted Central Path - Proof (continued)

Assumption 4: For any $j \in \{1, \ldots, 5\}$ let $\triangle b, \triangle \mathbf{C}$ be such that there exists $\mathbf{W}^0 \in \mathcal{W}_j$ and $\mu_0 > 0$ so that the system $(\star)$ is solvable for $\mathbf{W} = \mathbf{W}^0$ and $\nu = \mu_0$.

Remark: There always exist $\triangle b, \triangle \mathbf{C}$ such that they satisfy Assumption 4:

Choose arbitrary $\mathbf{W}^0 \in \mathcal{W}_j$, $\mu_0 > 0$ and $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ such that

$$\mathbf{X}^0 \succ 0, \ \mathbf{S}^0 \succ 0 \qquad \Phi_j(\mathbf{X}^0, \mathbf{S}^0) = \phi_j(\mu^0)\mathbf{W}^0.$$

Let

$$\triangle b = \frac{\mathcal{A}(\mathbf{X}^0) - b}{\phi_j(\mu^0)}, \qquad \triangle \mathbf{C} = \frac{\mathcal{A}^*(y^0) + \mathbf{S}^0 - \mathbf{C}}{\phi_j(\mu^0)},$$

Existence of the Weighted Central Path - Proof (continued)

Recall:

- **Assumption 1:** $\mathbf{A}_i$ are linearly independent, $i = 1, \ldots, m$
- **Assumption 3:** Existence of an optimal solution
- **Assumption 4:** Appropriately chosen $\triangle b, \triangle \mathbf{C}$

Boundedness:

Lemma 2: Let $\mathcal{O}(\mathbf{W}^0) \subset S^n_{++}$ be a bounded neighborhood of $\mathbf{W}^0$. Then the set

$$\mathcal{M} = \{(\ \mathbf{X}_\mathbf{W}(\mu), y_\mathbf{W}(\mu), \mathbf{S}_\mathbf{W}(\mu)\ ) \mid 0 < \mu \leq \mu_0, \mathbf{W} \in \mathcal{O}(\mathbf{W}^0)\}$$

is bounded.

Existence of the Weighted Central Path - Proof (continued)

- Let $j \in \{1, \ldots, 5\}$
- Let $(\mu_0, \mathbf{W}_0)$ be given in Assumption 4
- Let $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ be the solution of the system $(\star)$ for $\mathbf{W} = \mathbf{W}^0$ and $\mu = \mu_0$.
- Let $\psi : \langle 0, 1 \rangle \to (0, \mu_0\rangle \times \mathcal{W}_j$ be a continous path (e.g. a line segment):

$$\psi(0) = (\mu_0, \mathbf{W}^0) \longrightarrow \psi(1) = (\mu_1, \mathbf{W}^1), \qquad \psi(t) = (\mu_t, \mathbf{W}^t)$$

Existence of the Weighted Central Path - Proof (continued)

Nonsingularity of $DF^j_{\mu,\mathbf{W}}(\mathbf{X}, y, \mathbf{S})$ Implicit function theorem Boundedness	$\Longrightarrow$	We can make the analytic continuation along ψ from $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ to the solution $(\mathbf{X}^1, y^1, \mathbf{S}^1)$ of the system $(\star)$ for $\mathbf{W} = \mathbf{W}^1$ and $\mu = \mu_1$.

- There exists (an analytic) function g: $g(\psi(t)) = (\mathbf{X}^t, y^t, \mathbf{S}^t)$, where $(\mathbf{X}^t, y^t, \mathbf{S}^t)$ is the solution of $(\star)$ for for $\mathbf{W} = \mathbf{W}^t$ and $\mu = \mu_t$.
- For all $t \in \langle 0, 1 \rangle$ the function $g(\psi(t))$ is uniquely determined by the path ψ and the starting value $g(\psi(0))$.

Existence of the Weighted Central Path - Proof (continued)

EXISTENCE

Corollary: For any $\mu \in (0, \mu_0\rangle$ and $\mathbf{W} \in \mathcal{W}_j$ there exists a solution of $(\star)$

UNIQUENESS

Lemma 3: Let $\mathbf{W} = \mathbf{I}$. If the system $(\star)$ has a solution for some $\mu > 0$ then this solution is unique.

- Lemma 3 + uniqueness of $g(\psi(t))$ imply

Corollary: If the system $(\star)$ has a solution for some $\mu > 0$ and $\mathbf{W} \in \mathcal{W}_j$ then this solution is unique.

Existence of the Weighted Central Path - Proof (the end)

Let $j \in \{1,\dots,5\}$ and $\mathbf{W} \in \mathcal{W}_j$.

Assumption 1 **Assumption 3** **Assumption 4**	For any $\mu \in (0, \mu_0\rangle$ there exists unique solution of the system $\mathbf{A}_i \bullet \mathbf{X} = b_i + \mu \triangle b_i, \quad i = 1,\dots,m,$ $\sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C} + \mu \triangle \mathbf{C}, \quad \mathbf{S} \succ 0,$ $\Phi_j(\mathbf{X},\mathbf{S}) = \phi_j(\mu)\mathbf{W}, \quad \mathbf{X} \succ 0,$
Assumption 1 **Assumption 2**	For any $\mu > 0$ there exists unique solution of the system $\mathbf{A}_i \bullet \mathbf{X} = b_i \quad i = 1,\dots,m,$ $\sum_{i=1}^m \mathbf{A}_i y_i + \mathbf{S} = \mathbf{C}, \quad \mathbf{S} \succ 0,$ $\Phi_j(\mathbf{X},\mathbf{S}) = \phi_j(\mu)\mathbf{W}, \quad \mathbf{X} \succ 0,$

Conclusion

- this result **is not new**- the (weighted) central path in semidefinite programming can be considered as a special case of the (weighted) central path in the semidefinite complementarity problem
- we used more elementary technique to show the existence of the more complicated types of weighted paths in SDP
- the result **is new** for the path associated with $\mathbf{L}_{\mathbf{X}}^{T}\mathbf{S}\mathbf{L}_{\mathbf{X}}$ and the set of all positive definite diagonal weights
- to show the existence of the weighted paths is the first step for studying its properties (e.g. the limiting behavior)