Faculty of Mathematics and Physics, Comenius University

Mathematics of Economy and Finance

EQUILIBRIUM AND NO-ARBITRAGE MODELS OF THE TERM STRUCTURE

MASTER THESIS

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CONTENTS

I. INTRODUCTION

Theory about the modelling of the term structure of interest rates has evolved over the last twenty five years (perhaps the first and the simplest model of the term structure was introduced by Merton in 1973) and since then a number of different approaches have been developed. It represents one of the most dynamic areas of finance, where a lot of research is still going on, with interesting practical applications, and therefore is widely used by both academics and practitioners.

The aim of this thesis is to bring the reference for the different one-factor term structure models where the single source of uncertainty is assumed to be the instantaneous interest rate. The paper consists of two main parts. The first part deals with the theoretical models set in continuous time and concentrates on the most commonly used approaches, emphasising their differences, analysing their respective advantages and disadvantages and deriving the explicit representations for the prices of pure discount bonds. In the second part, after summarizing the theory underlying this approach to asset pricing, the practical econometric estimation of those models is presented and analysed. In order to simplify the analysis, models are reformulated in the discrete time. The focus of this part is less on theory and more on the properties of the bond yields implied by the model's structure and the choice of parameter values.

Although the major use of these fixed income models is their application to pricing of derivatives, it is not provided in this thesis, as it would make it a much longer paper. However, the results presented here are the necessary first steps in this direction.

II. THE NEED FOR THE TERM STRUCTURE MODELS

2.1 The motivation for term structure models

There is no single interest rate for any economy, and in addition, the structure of interest rates is interdependent. The interest rate is in general affected by a lot of factors, and one of those which impacts the interest rate for a particular security and which would be fundamental in the proceeding is the maturity. The relationship between yield on zero-coupon bonds and maturity (or term) will be referred to as the *term structure of interest rates*. Examination of the term structure is essential in analysis of the interest rate dependent securities and some of the uses include the following:

- analysis of the returns of fixed payment contracts of different maturities - the portfolios are varied along many dimensions but the most important is the maturity - it has the greatest influence on whether the portfolio will gain or loose in volatile interest rate environments,
- forecasting the future interest rates,

- pricing bonds and other fixed-payment contracts – in pricing financial obligations, it is crucial that consideration be given to the yields available on alternative investments with a similar length of commitment and the yield curve gives an idea of what that alternative yields are for coupon-bearing bonds,
- pricing contingent claims on fixed income securities – a large market has developed in pricing options on fixed income securities. The pricing of such contingent claim requires the evolution of the term structure over time to be modelled,
- arbitraging between bonds of different maturities – the term structure analysis can be used to make the yields more directly comparable and thereby simplify the analysis,
- forming expectations about the economy - the shape of the term structure appears to have an influence on future economic activity, including investment and consumption and it may also incorporate useful forecasts about future inflation.

There exist several theories or hypotheses explaining the dynamics of the term structure such as the market segmentation hypothesis, the liquidity premium hypothesis, etc. In recent years, a new way of modelling the term structure has evolved and it will be entitled as *the stochastic process modelling of the term structure*. This approach requires several assumptions: the term structure and bond prices are related to certain stochastic factors, these underlying factors are assumed to evolve over time according to a particular hypothesised stochastic process, and the interest rates and bond prices that result must satisfy no-arbitrage conditions. It should be noted that there is some parallel in a stochastic process generating the prices of fixed income securities and the process for generating a stock prices that underlies option pricing models such as well known Black-Scholes model. In addition, the valuation of options on fixed income securities all require some assumptions about the term structure generating process. Much of the research in the term structure field has been stimulated by the need to value such contingent claims.

2.2 Principal definitions

A default-free cash flow of a specified amount (*principal or face value*) at a known time in the future (*maturity date*) is called a *discount bond*. Discount bond which have a payoff of unity at maturity will be referred to as a *pure discount bond*. The price at time t of a discount bond maturing at time T will be denoted $P(t,T)$. Usually, many traded bonds are not pure discount bonds. The majority of government and corporate bonds promise to pay periodic payments in the interim, which are referred to as *coupons*, and a (usually larger) payment on the last payment date (the *maturity date*). These latter instruments are usually called *coupon-bearing bonds*. Since coupon-bearing bonds can be regarded as a suitable portfolio of discount bonds with payoffs and maturities that match the coupon payments, from the conceptual point of view, the collection of prices $P(t,T)$ for any $t \leq T$ fully describes the value to be associated to any collection of certain future cash flows. $P(t,T)$ will be referred to as *discount function.*

A security that entitles the holder to a given cash flow if a particular, prespecified, state of the world is attained at one or more future dates is called a *contingent claim*. In some situations, despite the fact that the cash flows are uncertain at time t, they can be replicated by entering suitable strategies which require only positive or negative holding of discount bonds. To avoid the possibility of arbitrage, the knowledge of the discount function completely determines the prices of this type of contingent claims (e.g. swaps, FRA). But for more complicated contingent claims, no-arbitrage and the knowledge of the discount function are not sufficient to determine their value. In this case also the specific assumptions about the probability distributions of the one or more random variables that determine the future cash flows need to be made.

The most common is to express the conditions, which indicate the future payments in terms of rates, and these are defined as follows:

The *yield to maturity*, YTM, is the rate of return that causes the market price of bond $P(t,T)$ to be equal to the present value of the future cash flows, i.e. coupon payments C_i and face value FV

$$P(t,T) = \sum_{i=t}^{T} \frac{C_i}{(1+YTM)^i} + \frac{FV}{(1+YTM)^T} \tag{2.2.1}$$

The time-*t* continuously compounded discrete *spot-rate* of maturity *T*, $R(t,T)$, is defined by

$$P(t,T) \equiv \exp[-R(t,T)(T-t)]$$

$$R(t,T) \equiv -\frac{\ln[P(t,T)]}{T-t} \tag{2.2.2}$$

The time-*t* continuously compounded discrete *forward-rate* spanning the period $[T, T+\Delta t]$ with $\Delta t \geq 0$, $f[t,T,T+\Delta t]$, is defined by

$$\frac{P(t,T+\Delta t)}{P(t,T)} \equiv \exp[-f(t,T,T+\Delta t)\Delta t]$$

$$f(t,T,T+\Delta t) \equiv -\frac{\ln[P(t,T+\Delta t)] - \ln[P(t,T)]}{\Delta t} \tag{2.2.3}$$

The limits as $T \to t$, $\Delta t \to 0$ in equations (2.2.2), (2.2.3) define the instantaneous *short-rate*, $r(t)$, and the instantaneous *forward-rate*, $f(t,T)$, as seen from the yield curve at time t, respectively

$$r(t) = \lim_{T \to t} R(t,T) \tag{2.2.4}$$

$$f(t,T) = \lim_{\Delta t \to 0}[-\frac{\ln[P(t,T+\Delta t)] - \ln[P(t,T)]}{\Delta t}] = -\frac{\partial \ln[P(t,T)]}{\partial T} \tag{2.2.5}$$

The short-rate, r, will be the critical instrument in the proceeding analysis - it will be used as the stochastic factor expressing the uncertainty and therefore driving bond prices and the term structure in all analysed models. From (2.2.5) it can be obtained

$$\int_t^T d\ln P(t,s) = -\int_t^T f(t,s)ds = \ln[P(t,T)] - \ln[P(t,t)] \tag{2.2.6}$$

and since $P(t,t) = 1$,

$$-\int_t^T f(t,s)ds = \ln P(t,T) \tag{2.2.7}$$

it leads to the formulae

$$P(t,T) = \exp[-\int_t^T f(t,s)ds]. \tag{2.2.8}$$

Formally, the *term structure* deals with the relationship between spot rates and term, while the *yield curve* deals with yield to maturity and term. In many instances the difference is not major, however, it can be important when the length of the commitment is long or the nature of cash flows is unusual - and therefore it is necessary to distinguish between the two. In the analysis of maturity - time relationship, it is preferable to work with spot-rates rather then yields to maturity because spot-rates do not depend on coupon effects. By coupon effect is denoted the fact that the yields of bonds of the same maturity depend on the pattern of their cash flows. Notice that the yield to maturity on a pure discount bond is equal to its spot-rate – there are no coupon effects with pure discount bonds and thus they will be essential in the following analysis.

The evolution of the yield curve can be described in terms of the dynamics of several equivalent financial quantities, such as, for instance spot-rates, forward-rates, or pure discount bonds. As long as the transformation laws from one quantity to another are known, the choice of the independent variables is just a matter of convenience. As shown above, for the three quantities just mentioned, the linking equations are:

$$R(t,T) \equiv -\frac{\ln[P(t,T)]}{T-t} \tag{2.2.2}$$

$$f(t,T) = -\frac{\partial \ln[P(t,T)]}{\partial T} \tag{2.2.5}$$

$$P(t,T) = \exp[-\int_t^T f(t,s)ds] \tag{2.2.8}$$

3.1 General form of one-factor models

In relation to the term structure of interest rates, arbitrage pricing theory has two purposes. The first, is to price pure discount bonds of varying maturities from a finite number of economic fundamentals (called state variables) and according to the amount (one or more) of state variable used, one- or multi- factor term structure models are recognised. The second target of the theory is to price interest rate sensitive contingent claims, taking as given all the prices of pure discount bonds.

In order to analyse the properties of individual one-factor models of the term structure it is assumed that the prices of all discount bonds can be expressed in terms of a single state variable. Usually, this random variable is assumed to be the short-rate *r*. A general representation for this class of models is given by

$$dr = \tilde{\mu}(t,r)dt + \tilde{v}(t,r)dz \tag{3.1.1}$$

where $\tilde{\mu}(t,r)$ and $\tilde{v}(t,r)$ represent the instantaneous drift (deterministic component) and variance (stochastic part) of the short-rate process, respectively. dz represents an increment in a Wiener process over a small time interval dt. Under the assumption of the stochastic process for *r* and using the arbitrage pricing theory (Rebonato (1996)), the value of a discount bond with maturity *T* ($t \leq T$), is the expectation of the payoff discounted at the future levels of the short-rate

$$P(t,T) = E_t\left[\exp(-\int_t^T r(s)ds)\right] \tag{3.1.2}$$

where E_t denotes the expected value in the risk-neutral world and the expectation is taken conditional on the information available at time t. The process for the short-rate is sufficient to determine the evolution of the term structure of interest rates, under the condition of absence of arbitrage opportunities.

The first step in the construction of a one-factor model is the specification of a reasonable form for the stochastic process of the driving factor. If this variable is the short-rate, the distributional properties implied by its stochastic process are of great interest, since they can give some indication as to how reasonable the chosen process is. The main features that one would like to observe in the short-rate behaviour predicted by any model are:

- the rates should not be allowed to become negative or to assume implausibly large values - i.e. the dispersion of the short-rate should be consistent with the expectations of likely values over the given time horizon,

- rates appear to be pulled back to some long-run average, i.e. very high levels of rates, on historical terms, tend to have a negative drift, and vice versa - this behaviour is known as a mean-reversion,
- rates of different maturity are imperfectly correlated,
- the volatility of rates of different maturity should be different, and the shorter rates usually displaying a higher volatility,
- the short-rate volatility has been observed to lack homoskedasticity, i.e. the level of volatility varies with the absolute level of the rates themselves.

No known one- or multi-factor model manages to capture all these features at the same time, and it depends for what purpose is the practical implementation designed for, what features are essential for it and which can be dispensed with. As there is a great number of different approaches that have been developed to model the term structure of interest rates, also the categorisation of the existing literature depends on which attributes of the models are taken to be the crucial.

The approach analysed here as the first is based on no-arbitrage considerations and is presented by the Vasicek's model (1977). It starts from assumptions about the stochastic evolution of interest rate and by imposing the constraint of the absence of arbitrage in economy derives the prices of all contingent claims. The approach pioneered by Cox, Ingersoll and Ross (1985b) usually starts with the description of the underlying economy and from assumptions about the stochastic evolution of one or more state variables in the economy and about the preferences of a representative investor. General equilibrium conditions are used to endogenize the interest rate and the price of all contingent claims. These two models represent traditional approach to model the term structure and are also known as *equilibrium models*. The problem is that they in general do not fit the initial yield curve. To avoid this, Ho and Lee (1986) originated the approach where the dynamics of the entire term structure is modelled in such a way that is automatically consistent with the initial (observed) yield curve. Those models are therefore labelled as *term structure consistent* or *no-arbitrage*.

In the discussed one-factor models, the short-rate process can be presented in the following general form

$$dr = \kappa(\theta - r)dt + \sigma r^{\beta} dz \tag{3.1.3}$$

where κ, θ, σ and β are positive constants and dz is an increment in a Wiener process over time interval dt. In these models, the interest rate, r, is pulled toward a level θ at rate κ. The situations where $\beta = 0$ or $\beta = \frac{1}{2}$ are of particular interest, because they lead to models which are analytically tractable. The first case results to the Vasicek model, the second one to the Cox, Ingersoll and Ross (CIR), which will be demonstrated later. Ho and Lee incorporated to their model the time-dependent parameter, the drift $\theta(t)$, so that the model fits the initial term structure of interest rates and parameter β was set equal to zero in this case.

The second aim of the theory, which is not provided in this paper, i.e. the evaluation of bond option prices is particularly interesting in Vasicek/CIR framework as well as in the case of Ho and Lee model, because in that it is possible to obtain a closed-form solutions for the case of European options on discount bonds.

In the next table are presented specifications for the short-rate process imposed by the respective models. Namely,

in Vasicek model	$\tilde{\mu}(t,r) = \kappa(\theta - r)$	and	$\tilde{v}(t,r) \equiv \sigma$
in Cox, Ingersoll and Ross model	$\tilde{\mu}(t,r) = \kappa(\theta - r)$	and	$\tilde{v}(t,r) = \sigma\sqrt{r}$
in Ho and Lee model	$\tilde{\mu}(t,r) = \theta(t)$	and	$\tilde{v}(t,r) \equiv \sigma$.

3.2 The derivation of the Vasicek and CIR models

A unified derivation is given of the Vasicek and CIR models. However, the original derivation of the CIR is different - their model was derived by applying the highly abstract general equilibrium asset pricing model of Cox, Ingersoll and Ross (1985a) in which the current prices, stochastic properties and also the market price of risk of all contingent claims (including bonds) are obtained endogenously.

The mean-reverting process of the short-rate r for both models will be described by using the relation presented in the previous section as

$$dr = \kappa(\theta - r)dt + \sigma r^{\beta} dz\,. \tag{3.1.3}$$

In the context of one-factor models, the price of a discount bond is the function of present time t, maturity T ($t \le T$), and the chosen factor, i.e. short-rate r

$$P = P(t,T,r)\,. \tag{3.2.1}$$

The bond price satisfies a stochastic differential equation

$$dP = P\mu(t,T,r)dt + Pv(t,T,r)dz \tag{3.2.2}$$

where the functions $\mu(t,T,r)$ and $v(t,T,r)$ are the mean and variance, respectively, of the instantaneous rate of return at time *t* on a bond with maturity at time *T*, given that the current short-rate is $r(t) = r$.

It follows from (3.1.3) and (3.2.2) by the Ito´s differential rule, that

$$dP = \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r}\kappa(\theta - r) + \frac{1}{2}(\sigma r^{\beta})^{2}\frac{\partial^{2} P}{\partial r^{2}}\right]dt + \sigma r^{\beta}\frac{\partial P}{\partial r}dz \tag{3.2.3}$$

and from equations (3.2.2), (3.2.3) that

$$v(t,T,r)=\frac{\partial P}{\partial r}\frac{1}{P}\sigma r^{\beta}. \quad (3.2.4)$$

A portfolio Π then can be created, composed of one unit of bond maturing at time T_1, and Θ units of bond maturing at time T_2

$$\Pi(t)=P(t,T_1,r)+\Theta P(t,T_2,r). \quad (3.2.5)$$

The process obeyed by the portfolio is then given by

$$d\Pi=\left[\mu(t,T_1,r)+\Theta\mu(t,T_2,r)\right]dt+\left[\frac{\partial P(t,T_1,r)}{\partial r}\sigma r^{\beta}+\Theta\frac{\partial P(t,T_2,r)}{\partial r}\sigma r^{\beta}\right]dz. \quad (3.2.6)$$

If Θ is chosen equal to

$$\Theta=-\frac{\dfrac{\partial P(t,T_1,r)}{\partial r}}{\dfrac{\partial P(t,T_2,r)}{\partial r}} \quad (3.2.7)$$

the portfolio is risk-free (the stochastic component vanishes).
The equation (3.2.5) can be rewritten

$$\Pi(t)=P(t,T_1,r)-\frac{\dfrac{\partial P(t,T_1,r)}{\partial r}}{\dfrac{\partial P(t,T_2,r)}{\partial r}}P(t,T_2,r) \quad (3.2.8)$$

where $v(t,T_1,r)$ and $v(t,T_2,r)$ is the variance of the bond maturing at time T_1 or T_2, respectively.
Over a small time interval dt the evolution of the portfolio is given by

$$d\Pi(t)=dP(t,T_1,r)-\frac{\dfrac{\partial P(t,T_1,r)}{\partial r}}{\dfrac{\partial P(t,T_2,r)}{\partial r}}dP(t,T_2,r). \quad (3.2.9)$$

The portfolio is now by construction risk-free; it can only earn over dt the instantaneous short-rate

$$d\Pi(t)=r(t)\Pi(t)dt. \quad (3.2.10)$$

Comparison of equations (3.2.9), (3.2.10) yields

$$\frac{\mu(t,T_1,r)-r(t)}{v(t,T_1,r)}=\frac{\mu(t,T_2,r)-r(t)}{v(t,T_2,r)}. \quad (3.2.11)$$

Since equation (3.2.11) is valid for arbitrary maturity dates T_1, T_2, therefore the ratio must be equal to a quantity, λ, possibly dependent on r and t, but which is independent of maturity T, i.e.

$$\lambda(t,r)=\frac{\mu(t,T,r)-r(t)}{v(t,T,r)} \tag{3.2.12}$$

The quantity $\lambda(t,r)$ is called the market price of risk, as it specifies the increase in expected instantaneous rate of return on a bond per additional unit of risk. Equation (3.2.12) will be used to derive an equation for the price of a discount bond. Writing (3.2.12) as

$$\mu(t,T,r)=r(t)+\lambda(t,r)v(t,T,r)\,. \tag{3.2.13}$$

Using equation (3.2.4), equation (3.2.12) can be rewritten

$$\mu(t,T,r)=r(t)+\lambda(t,r)\frac{\partial P}{\partial r}\frac{1}{P}\sigma r^{\beta}\,. \tag{3.2.14}$$

Equating terms in dt in equations (3.2.2) and (3.2.3) and using equation (3.2.14) yields

$$\frac{\partial P}{\partial t}+\kappa(\theta-r)\frac{\partial P}{\partial r}+\frac{1}{2}(\sigma r^{\beta})^{2}\frac{\partial^{2}P}{\partial r^{2}}=rP+\lambda(t,r)\sigma r^{\beta}\frac{\partial P}{\partial r}\,. \tag{3.2.15}$$

Partial differential equation of the second order (3.2.15) is called the term structure equation. Once the character of the short-rate process is described and the market price of risk specified, the prices of pure discount bonds are obtained by solving (3.2.15) subject to the boundary condition

$$P(T,T,r)=1, \tag{3.2.16}$$

which implies that the pure discount bond, according to the definition in the previous chapter, pays unity at the maturity time T.

At this point the two models impose different functional form for the market price of risk $\lambda(t,r)$ and for the exponent β. Namely,

in the Vasicek model $\quad \beta=0 \quad \lambda(t,r)=\lambda_0$

in the CIR model $\quad \beta=\frac{1}{2} \quad \lambda(t,r)=\lambda_0\sqrt{r}/\sigma$

where λ_0 is a different constant in both models. In Vasicek model λ_0 is constant across different interest rate securities (as a consequence of assuming no arbitrage). In contrary, CIR derived a model that allows the market price of risk to be determined endogenously. Comparing CIR to Vasicek model, it has the same mean-reverting drift, but the stochastic term has a standard deviation proportional to $\sqrt{r}$. This means that when the short-rate increases, its standard deviation increases, too. Therefore, in the case of CIR model, the negative interest rates are precluded.

Term structure equation (3.2.15) can be rearranged to give for the Vasicek model

$$\frac{\partial P}{\partial t}+\left[\kappa(\theta-r)-\lambda_0\sigma\right]\frac{\partial P}{\partial r}+\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}-rP=0 \tag{3.2.17}$$

and for the CIR model it yields

$$\frac{\partial P}{\partial t}+\kappa(\theta-r)\frac{\partial P}{\partial r}-\lambda_0 r\frac{\partial P}{\partial r}+\frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial r^2}-rP=0\,. \tag{3.2.18}$$

The solution to (3.2.17) and (3.2.18) subject to the boundary condition $P(T,T,r)=1$ has the same general form

$$P(t,T,r)=A(t,T)e^{-B(t,T)r(t)} \tag{3.2.19}$$

but the functions $A(t,T)$ and $B(t,T)$ are different for Vasicek and CIR models.

Concentrating first on the case of **<u>the Vasicek model</u>** suppose that the functions $A(t,T)$ and $B(t,T)$ are given by, respectively

$$B(t,T)=\frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{3.2.20}$$

$$A(t,T)=\exp\left[(B(t,T)-T+t)(\theta-\frac{\sigma^2}{2\kappa^2}-\frac{\sigma\lambda_0}{\kappa})-\frac{\sigma^2 B(t,T)^2}{4\kappa}\right]. \tag{3.2.21}$$

This can be straightforward verified by taking the appropriate derivatives:

$$\begin{aligned}\frac{\partial P}{\partial t}&=\frac{\partial A(t,T)}{\partial t}e^{-B(t,T)r(t)}-A(t,T)e^{-B(t,T)r(t)}r(t)\frac{\partial B(t,T)}{\partial t}\\&=P\left(\frac{1}{A(t,T)}\frac{\partial A(t,T)}{\partial t}-r(t)\frac{\partial B(t,T)}{\partial t}\right)\end{aligned} \tag{3.2.22}$$

$$\frac{\partial P}{\partial r}=-A(t,T)B(t,T)e^{-B(t,T)r(t)}=-B(t,T)P \tag{3.2.23}$$

$$\frac{\partial^2 P}{\partial r^2}=B(t,T)^2 P \tag{3.2.24}$$

and substituting equations (3.2.22) – (3.2.24) into term structure equation (3.2.17) gives

$$P\left(\frac{1}{A(t,T)}\frac{\partial A(t,T)}{\partial t}-r(t)\frac{\partial B(t,T)}{\partial t}\right)-\left[\kappa(\theta-r)-\lambda_0\sigma\right]B(t,T)P+\frac{1}{2}\sigma^2 B(t,T)^2 P-rP=0 \tag{3.2.25}$$

$$\begin{aligned}&\frac{1}{A(t,T)}\left[(-e^{-\kappa(T-t)}+1)(\theta-\tfrac{1}{2}\tfrac{\sigma^2}{\kappa^2}-\tfrac{\sigma\lambda_0}{\kappa})+\tfrac{1}{2}\tfrac{\sigma^2}{\kappa^2}(1-e^{-\kappa(T-t)})e^{-\kappa(T-t)}\right]A(t,T)\\&+re^{-\kappa(T-t)}-\left[\kappa(\theta-r)-\lambda_0\sigma\right]\frac{1-e^{-\kappa(T-t)}}{\kappa}+\frac{1}{2}\sigma^2\frac{(1-e^{-\kappa(T-t)})^2}{\kappa^2}-r=0\end{aligned} \tag{3.2.26}$$

which is an identity.

The term structure $R(t,T,r)$ of interest rates is then directly evaluated from the equation

$$R(t,T,r) = -\frac{1}{T-t}\ln P(t,T,r(t)) \tag{3.2.27}$$

and by using equations (3.2.20) and (3.2.21) yields

$$R(t,T,r) = -\frac{1}{T-t}\ln A(t,T) + \frac{1}{T-t}B(t,T)r(t)\,. \tag{3.2.28}$$

Equation (3.2.28) shows that $R(t,T,r)$ is linearly dependent on $r(t)$, i.e. the value of $r(t)$ determines the level of the term structure at time t, but the yield $R(t,\infty)$ on the long (infinite maturity) bond is independent from the current interest rate r

$$R(t,\infty) = \lim_{T\to\infty} R(t,T,r) \tag{3.2.29}$$

and since

$$\ln A(t,T) = -\tfrac{1}{\kappa}e^{-\kappa(T-t)} + \tfrac{1}{2}\tfrac{\sigma^2}{\kappa^3}e^{-\kappa(T-t)} - \theta(T-t) - \tfrac{\sigma\lambda_0}{\kappa}(T-t) + \tfrac{1}{2}\tfrac{\sigma^2}{\kappa^2}(T-t)$$

$$B(t,T) = \tfrac{1}{\kappa}[1-e^{-\kappa(T-t)}]$$

after substitution it gives

$$R(t,\infty) = \theta - \tfrac{1}{2}\tfrac{\sigma^2}{\kappa^2} - \tfrac{\sigma\lambda_0}{\kappa}\,. \tag{3.2.30}$$

The yield curve given by (3.2.28) starts at the current level of the short-rate $r(t)$ at $T=0$ and it approaches the asymptote $R(t,\infty)$ as $T\to\infty$. Analysing the intervals of monotonicity - if $r(t)$ is smaller or equal to $\theta - \tfrac{3}{4}\tfrac{\sigma^2}{\kappa^2}$, the yield curve is monotonically increasing. For the values of $r(t)$ equal to or larger θ, the curve is monotonically decreasing and for the intermediate values, it is humped.

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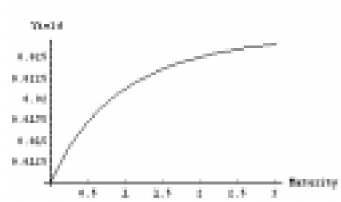
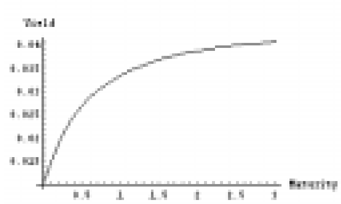
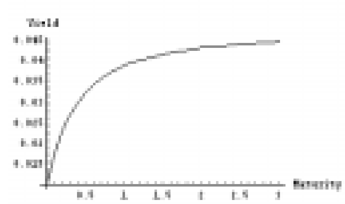
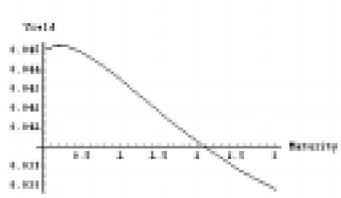
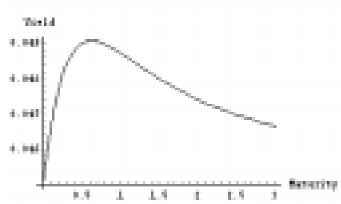
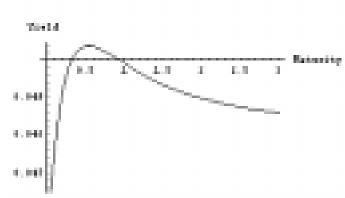
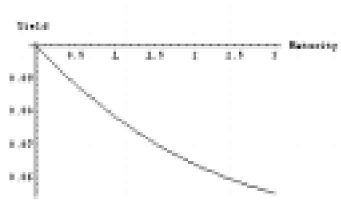
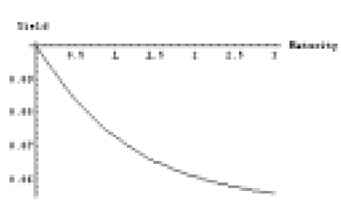
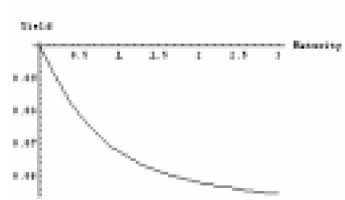
r=0.01			
r=0.045			
r=0.1			

Figure 3.1 Possible shapes of the yield curve when Vasicek model is used (σ=0.2, θ=0.05, λ=0 and τ=(T-t)=3)

Equation (3.2.28) can be rewritten (using equations (3.2.20) and (3.2.21)) and this yields

$$R(t,T,r) = R(t,\infty) + (r(t) - R(t,\infty))\frac{1}{\kappa(T-t)}(1-\exp(-\kappa(T-t))) + \frac{\sigma^2}{4\kappa^3(T-t)}(1-\exp(-\kappa(T-t)))^2 \tag{3.2.31}$$

By applying Ito's differential rule to the above equation and using equation (3.1.3), the volatility of $R(t,T,r)$ is given by the expression

$$\sigma_R = \frac{\sigma}{\kappa(T-t)}[1-\exp(-\kappa(T-t))]. \tag{3.2.32}$$

This curve is negative exponential declining to zero as $T \to \infty$, which implies that long-rates are less volatile than short-rates. The rate at which the standard deviation of the spot-rates decline is driven by the mean-reversion, i.e. by the parameter κ.

The strongest criticism of the latter model is that it allows negative interest rates, as the process for the short-rate is normal, with constant volatility. Also the possible shapes of the yield curve that can be obtained are rather limited which makes practical implementation difficult because the market yield curves often display very complex shapes.

Focusing now on **the model of Cox, Ingersoll and Ross**, the solution of partial differential equation (3.2.18) yields

$$P(t,T,r) = A(t,T)e^{-B(t,T)r(t)} \tag{3.2.19}$$

$$A(t,T) = \left[\frac{2\gamma\exp[(\kappa+\lambda+\gamma)(T-t)/2]}{(\kappa+\lambda+\gamma)(\exp[\gamma(T-t)]-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2} \tag{3.2.33}$$

$$B(t,T) = \frac{2(\exp[\gamma(T-t)]-1)}{(\kappa+\lambda+\gamma)(\exp[\gamma(T-t)]-1)+2\gamma} \tag{3.2.34}$$

$$\gamma = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}\,. \tag{3.2.35}$$

It can be noted from equation (3.2.19) that bond prices depend only on one random variable - the short-rate r. It is also interesting to notice that κ, θ and λ do not occur in these equations separately, but only in the combinations $\kappa\theta$ and $\kappa+\lambda$. Therefore, it is not possible to determine these quantities individually simply using prices of traded securities; in particular the market price of risk λ remains unrecoverable.

Now consider the yield to maturity $R(t,T,r)$ of a pure discount bond defined by equation (3.2.27). Similar analysis as in the case of Vasicek model concerning the yield $R(t,\infty)$ on the long (infinite maturity) bond shows that it is also independent from the current interest rate r

$$R(t,\infty) = \lim_{T\to\infty}\left[-\frac{1}{T-t}\ln A(t,T) + \frac{1}{T-t}B(t,T)r(t)\right] \qquad (3.2.28)$$

and since

$$\ln A(t,T) = \tfrac{2\kappa\theta}{\sigma^2}\left[\ln 2\gamma + \tfrac{1}{2}(\kappa+\lambda+\gamma)(T-t) - \ln\left((\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma\right)\right]$$

it follows by using L'Hospital rule that

$$\lim_{T\to\infty}[\tfrac{1}{T-t}rB(t,T)] = \lim_{T\to\infty}[\tfrac{1}{T-t}r\tfrac{2(e^{\gamma(T-t)}-1)}{(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}] = \tfrac{2}{\kappa+\lambda+\gamma}\lim_{T\to\infty}[\tfrac{1}{T-t}r] = 0$$

$$\lim_{T\to\infty}[-\tfrac{1}{T-t}\ln A(t,T)] = \tfrac{2\kappa\theta}{\sigma^2}\left[\gamma - \tfrac{1}{2}(\kappa+\lambda+\gamma)\right].$$

Using the results above it yields that

$$R(t,\infty) = \frac{2\kappa\theta}{\kappa+\lambda+\gamma}. \qquad (3.2.36)$$

When the value of the short-rate is below this long-term yield, the term structure is monotonically increasing, when it is above $\dfrac{\kappa\theta}{\kappa+\lambda}$, it is decreasing. For intermediate values of the short-rate, the yield curve is humped.

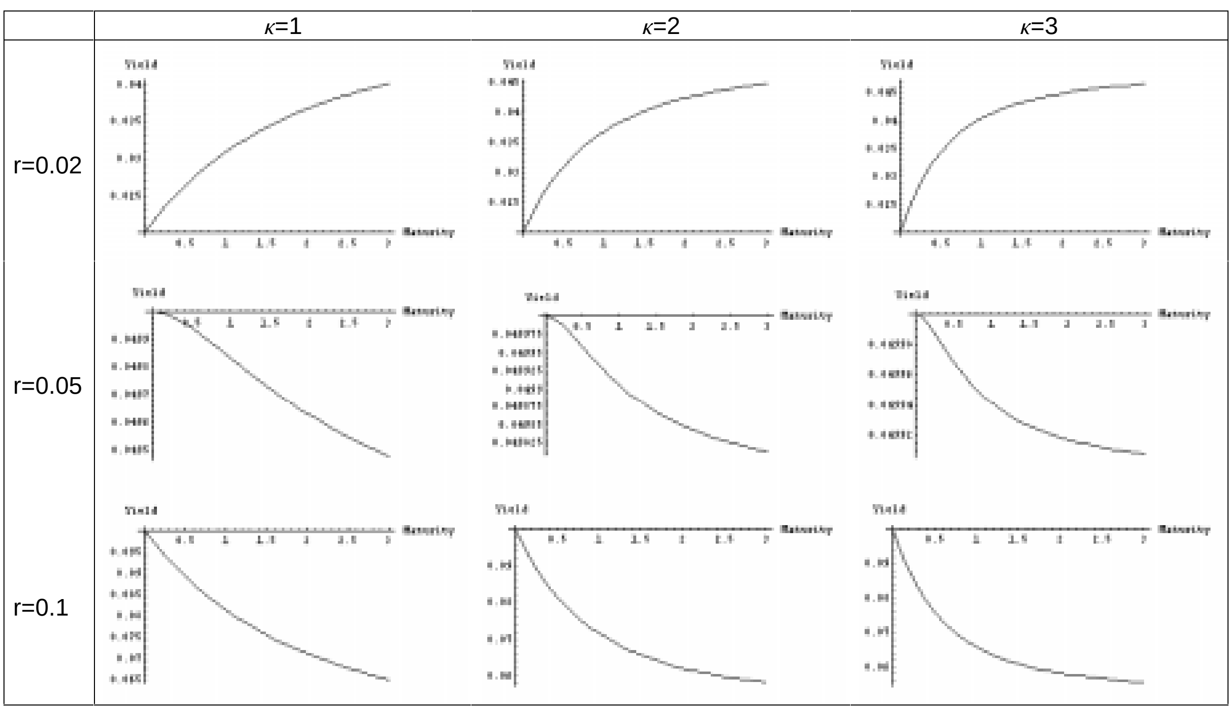


Figure 3.2 Possible shapes of the yield curve when CIR model is used (σ=0.2, θ=0.05, λ=0 and τ=(T-t)=3)

It is important to notice that the volatility parameter σ^2 of the short-rate is linked to the parameters of the discount bond price directly by the equation

$$\sigma^2 = \tfrac{1}{2}(\gamma - \kappa - \lambda)(\gamma + \kappa + \lambda)\,, \tag{3.2.37}$$

which can be obtained by straightforward computation from equation (3.2.36). That means that the long-term yield to which yield curves of different spot-rates but same reversion levels converge is a function of the volatility of the short-rate process. Applying Ito's lemma to the equation (3.2.28) and using equation (3.1.3), the volatility of $R(t,T,r)$ is given by

$$\sigma_R = \frac{\sigma\sqrt{r}}{\kappa(T-t)}[1 - \exp(-\kappa(T-t))]\,. \tag{3.2.38}$$

Cox, Ingersoll and Ross considered the problem of determining the term structure as a problem in general equilibrium theory, and therefore their approach permits detailed predictions about how changes in a wide range of underlying variables effect the term structure. There are several constraints implied by the model on the yield curve dynamics. The term structure of interest rates is the function of one state variable - the short-rate r, and three parameters κ, θ and λ which are independent of time. As shown before, a suitable combination of these three parameters yields the volatility of the short-rate. But the long-rate is not allowed by this model to vary with time, it is strictly constant. From a practical point of view a feature which is the main drawback of the CIR model (as well as Vasicek's) is the fact that observed yield curves are often poorly reproduced because of the limited flexibility of the function which describes the discount factor.

3.3 The Ho and Lee model

This model was originally developed in the form of a binomial tree of bond prices relating future movements of the yield curve explicitly to the initial state. Algorithmically it was constructed in such a way as to price exactly any set of market discount bonds without requiring the explicit specification of investors' risk preferences.

Here it will be considered the continuous time version of the model derived by Dybvig (1988), which is characterised by the following short-rate process

$$dr = \theta(t)dt + \sigma dz \tag{3.3.1}$$

where σ, the instantaneous volatility of the short-rate, is constant and $\theta(t)$, the drift of the process, is the function of time to make the model consistent with the initial, i.e. observed, term structure of interest rates. The variable $\theta(t)$ defines the average direction in which r moves at time and is independent of the level of r. It can be calculated from the initial term structure and is given by

$$\theta(t) = \frac{\partial f}{\partial t}(0,t) + \sigma^2 t\,. \tag{3.3.2}$$

This shows that the drift term reflects the slope of initial forward curve and the volatility of the short-rate.

Within the framework used before (in Vasicek/CIR models), the term structure equation satisfies the general form

$$\frac{\partial P}{\partial t} + \phi(t)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{3.3.3}$$

where $\phi(t) = \theta(t) - \lambda(t)\sigma$, and is solved subject to the boundary condition $P(T,T,r) = 1$, according to the assumption that pure discount bond pays unity at maturity time T.

In order to calibrate the model to the current term structure, function $\phi(t)$ must be set equal to the term $\theta(t)$. This additional assumption implies that the risk preferences of investors are included in market prices of pure discount bonds which determine the drift term $\theta(t)$. The solution to (3.3.3) can be valued analytically and assume that is expressed by

$$P(t,T,r) = A(t,T)e^{-B(t,T)r(t)} \tag{3.3.4}$$

with

$$B(t,T) = T - t \tag{3.3.5}$$

$$\ln A(t,T) = \ln\frac{P(0,T,r)}{P(0,t,r)} - B(t,T)\frac{\partial \ln P(0,t,r)}{\partial t} - \frac{1}{2}\sigma^2 t B(t,T)^2 \tag{3.3.6}$$

where the set $\{P(0,T);T \geq 0\}$ is the initial exogenously given term structure.

To verify this, by taking the restrictive assumption about function $\phi(t)$ and the appropriate derivatives, which have the same general form like in the case of the Vasicek model and therefore they satisfy equations (3.2.22) – (3.2.24), by substituting into term structure equation (3.3.3) yields

$$P\left(\frac{1}{A(t,T)}\frac{\partial A(t,T)}{\partial t} - r\frac{\partial B(t,T)}{\partial t}\right) - \theta(t)B(t,T)P + \tfrac{1}{2}\sigma^2 B(t,T)^2 P - rP = 0 \tag{3.3.7}$$

$$\frac{1}{A(t,T)}\frac{\partial A(t,T)}{\partial t} + r - \theta(t)(T-t) + \tfrac{1}{2}\sigma^2 (T-t)^2 - r = 0 \tag{3.3.8}$$

$$\begin{aligned}&\frac{1}{A}A\left[-\frac{\partial \ln P(0,t)}{\partial t} + \frac{\partial \ln P(0,t)}{\partial t} - (T-t)\frac{\partial^2 \ln P(0,t)}{\partial t^2} - \tfrac{1}{2}\sigma^2 (T-t)^2 + \tfrac{1}{2}2\sigma^2 t(T-t)\right]\\ &-\theta(t)(T-t) + \tfrac{1}{2}\sigma^2 (T-t)^2 = 0\end{aligned} \tag{3.3.9}$$

$$-\frac{\partial^2 \ln P(0,t)}{\partial t^2} + \sigma^2 t - \theta(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t - \theta(t) = 0 \tag{3.3.10}$$

which is, by using the relation (3.3.2), an identity.

The term structure $R(t,T,r)$ of interest rates is then directly evaluated from the equation

$$R(t,T,r) = -\frac{1}{T-t}\ln A(t,T) + \frac{1}{T-t}B(t,T)r(t). \tag{3.2.28}$$

The single function of time (which is the drift) in the short-rate process allows the model to fit only the term structure of interest rates and the term structure of volatilities is determined within the model. Applying Ito's lemma to $R(t,T,r)$ seen as a function of r yields

$$\sigma_R(t,T) = \frac{\partial R(t,T)}{\partial r(t)}\sigma = \frac{B(t,T)}{T-t}\sigma = \sigma \tag{3.3.11}$$

which implies that all spot and forward rates have the same instantaneous volatility. A further difficulty is that it does not incorporate the mean reversion, and as a result there is a positive probability that interest rate can become negative.

IV. ECONOMETRIC ESTIMATION OF ONE-FACTOR MODELS

4.1 Concept and notation

All term structure models discussed before are set in continuous time, which simplifies some of the theoretical analysis but complicates the empirical estimation. In order to provide the practical econometric testing of the models and their empirical implications, in next the discrete time will be adopted, since it makes fewer technical demands on users. The state variables, which are interest rates, will be assumed to take on a continuous range of real values.

The starting point for the analysis is the result of asset pricing theory, which claims that in any arbitrage-free environment there exists a positive random variable M, which will be referred to as *stochastic discount factor* or *pricing kernel*, since the prices of assets grow from it. The random variable M has different names in the literature – it is also known as an equivalent martingale measure, or a Radon-Nicodym derivative.

Consider the choice problem of the investor who can trade freely and who maximises the expectation of an additive and time-separable utility function

$$\underset{c_t}{Max}\, E_t\left[\sum_{t=0}^{\infty}\delta^t U(c_t)\right] \tag{4.1.1}$$

subject to the budget constraint

$$c_t = f(k_t) - (\beta + n)k_t - (k_{t+1} - k_t), \tag{4.1.2}$$

where δ is the time discount factor, c_t is the investor's consumption in time period t and $U(c_t)$ is the utility of consumption in time period t. $f(k_t)$ is the output per unit of labour and

k_t is the capital intensity at time t. n denotes the population growth and $0 \le \beta < 1$ is the depreciation rate of the capital. The expectation is taken conditional on the information available at time t.

The Euler equation describing the optimal consumption of the investor gives the following first-order condition

$$U'(c_t) = \delta E_t[(1+R_{t+1})U'(c_{t+1})] \tag{4.1.3}$$

where $R_{t+1} = f'(k_{t+1}) - (\beta + n)$ is by definition the real return on the asset. The left side of the above expression is the marginal utility cost of consuming one dollar less at time t. The right side expresses the expected marginal utility benefit from investing the dollar in the asset at time t, selling it at time $t+1$ for $1+R_{t+1}$ dollars and consuming what proceeds. Because marginal cost and marginal benefit is equated, (4.1.3) describes the optimum.

By dividing both sides of the equation (4.1.3) by $U'(c_t)$ yields

$$1 = E_t[(1+R_{t+1})M_{t+1}], \tag{4.1.4}$$

which is the fundamental relation for bond pricing as will be shown later.

$M_{t+1} = {}^{\delta U'(c_{t+1})}\!/_{U'(c_t)}$ denotes the pricing kernel and in the presented model it is equivalent to the discounted ratio of marginal utilities ${}^{\delta U'(c_{t+1})}\!/_{U'(c_t)}$, which is called the intertemporal marginal rate of substitution. Since marginal utilities are always positive, also the intertemporal marginal rate of substitution and then the pricing kernel are positive.

Condition (4.1.4) implies that the expected return on any asset is negatively related to its covariance with the stochastic discount factor: by taking unconditional expectations of the left and right sides of (4.1.4) and lagging one period to simplify the notation, the unconditional version of (4.1.4) is obtained

$$1 = E[(1+R_t)M_t]. \tag{4.1.5}$$

This relationship can be rearranged so that it explicitly determines expected asset returns:

$E[(1+R_t)M_t] = E[1+R_t]E[M_t] + Cov[R_t, M_t]$ and after rearrangement yields

$$E[1+R_t] = \frac{1}{E[M_t]}(1 - Cov[R_t, M_t]). \tag{4.1.6}$$

Fixed-income securities are particularly easy to price using the outlined framework – cash flows are deterministic so the covariance with the stochastic discount factor is presented only because there is time-variation in interest rates. This variation in interest rates is driven by the time series behaviour of the stochastic discount factor, which means that term structure models are equivalent to time series models for the stochastic discount factor.

Next the following notation will be used: P_t^n is the price of the pure discount bond at time t maturing at $t+n$, and Y_t^n is the bond's yield to maturity, so the yield can be found from the price by using the relation (2.2.1) as

$$(1+Y_t^n) = (P_t^n)^{-\frac{1}{n}}. \tag{4.1.7}$$

Using the notation $p_t^n \equiv \ln P_t^n$ and $y_t^n \equiv \ln Y_t^n$ the equation (4.1.7) becomes

$$y_t^n = -\tfrac{1}{n} p_t^n. \tag{4.1.8}$$

One-period forward rates are denoted (using equation (2.2.3)) as

$$f_t^n = \ln(P_t^n / P_t^{n+1}) = p_t^n - p_t^{n+1} \tag{4.1.9}$$

and the short-rate is labelled $r_t = y_t^1 = f_t^0$.

The holding-period return on a bond is the return over some holding period less than the bond maturity. Define R_{t+1}^n as the one-period return on an n- period bond purchased at time t and sold at time $t+1$. Since the bond will be then an $(n-1)$- period bond, the sale price is P_{t+1}^{n-1} and the holding-period return is

$$(1+R_{t+1}^n) = \frac{P_{t+1}^{n-1}}{P_t^n}. \tag{4.1.10}$$

Bond pricing is a straightforward application of equation (4.1.4). Substituting equation (4.1.10) into equation (4.1.4), the price of an n- period bond, P_t^n, satisfies

$$P_t^n = E_t[P_{t+1}^{n-1} M_{t+1}]. \tag{4.1.11}$$

From equation (4.1.11) bond prices can be computed recursively, starting with the initial condition $P_t^0 = 1$. On the other hand, equation (4.1.11) can be solved also as the expected product of n stochastic discount factors

$$P_t^n = E_t[M_{t+1} ... M_{t+n}]. \tag{4.1.12}$$

In following sections it will be assumed that the distribution of the pricing kernel M_{t+1} is conditionally lognormal and the bond prices P_t^n are jointly lognormal with M_{t+1}. Taking the logarithm of equation (4.1.12) and using the properties of lognormal distribution yields to the pricing equation

$$p_t^n = E_t[m_{t+1} + p_{t+1}^{n-1}] + \tfrac{1}{2} Var_t[m_{t+1} + p_{t+1}^{n-1}], \tag{4.1.13}$$

where $m_{t+1} \equiv \ln M_{t+1}$.

Models analysed in the proceeding are governed by a single factor, which means that the prices depend on a single state variable associated (here) to the short-rate. The first two models are also similar in having four parameters: three governing the dynamic behaviour of the state variable and one associated with the market price of risk. Each of these models has the property that log bond prices, and hence log bond yields, are linear or affine in the state variables. This ensures demanded joint lognormality of bond prices with the stochastic discount factor.

4.2 Discrete-time version of the Vasicek model

The discrete-time version of the model would be designed in two steps. The first step involves the construction of the pricing kernel m, which without loss of generality can be expressed as the sum of its conditional expectation x_t and the innovation ξ_{t+1} and therefore satisfies the equation

$$-m_{t+1} = \delta + x_t + \xi_{t+1}, \tag{4.2.1}$$

where δ is constant, and is set to $\frac{1}{2}(\lambda\sigma)^2$ which will be explained later. Assume that ξ_{t+1} is distributed normally and independently with mean zero and variance σ^2.

In the next step it is assumed that an abstract single state variable x whose dynamics follows a first-order autoregression process AR (1) can be expressed as

$$x_{t+1} = (1-\phi)\theta + \phi x_t + \varepsilon_{t+1}, \tag{4.2.2}$$

where ε_{t+1} is distributed normally and independently with mean zero and variance σ^2. The parameter ϕ controls the mean-reversion: if $\phi = 1$ then (4.2.2) is a random walk but when $0 < \phi < 1$, x is expected to return to its mean θ at rate $1-\phi$ as can be seen from the rearrangement of (4.2.2)

$$x_{t+1} = x_t + (1-\phi)(\theta - x_t) + \varepsilon_{t+1}.$$

Note that equation (4.2.2) presents the discrete-time interpretation of the process for the short-rate r given by the stochastic differential equation $dr = \kappa(\theta - r)dt + \sigma dz$ in continuous-time version of Vasicek's model.

Shocks ε_{t+1} and ξ_{t+1} can be correlated and to recover this, assume that ξ_{t+1} may be expressed as

$$\xi_{t+1} = \lambda\varepsilon_{t+1} + \eta_{t+1}, \tag{4.2.3}$$

where η_{t+1} is distributed normally with constant variance and is uncorrelated to ε_{t+1}. Variable η_{t+1} affects only the average level of the yield curve and not its slope or time-series behaviour so it will be dropped out in the next and it will be assumed that $\xi_{t+1} = \lambda\varepsilon_{t+1}$. Equation (4.2.1) can be then rewritten as

$$-m_{t+1} = \delta + x_t + \lambda\varepsilon_{t+1} \tag{4.2.4}$$

and ε_{t+1} is after transformation the only shock to the system.

Bond prices can be computed in two ways by using the theory outlined in previous section. The first concept uses the pricing equation (4.1.13) and prices are obtained recursively. Starting with the case $n=1$, it follows that $p_{t+1}^{n-1} = p_{t+1}^{0} = 0$. Conditional mean of m_{t+1} is $-(\delta + x_t)$ and conditional variance $(\lambda\sigma)^2$, which gives after substituting into equation (4.1.13), the price of a one-period bond

$$p_t^1 = E_t[m_{t+1}] + \tfrac{1}{2}Var_t[m_{t+1}] = -(\delta + x_t) + \tfrac{1}{2}(\lambda\sigma)^2 \tag{4.2.5}$$

and by setting δ equal to $\tfrac{1}{2}(\lambda\sigma)^2$, the short-rate r defined as $r_t = -p_t^1$ is equal to the state variable x_t.

Prices of longer bonds follow by induction. Assume that price of n- period bond can be expressed as

$$-p_t^n = A_n + B_n x_t \tag{4.2.6}$$

for some choice of coefficients $\{A_n, B_n\}$. For $n=0$, $A_0 = B_0 = 0$ and for $n=1$, $A_1 = 0$ and $B_1 = 1$, so bond prices satisfy the equation (4.2.6).

In order to determine the price of n- period bond, it holds that

$$m_{t+1} + p_{t+1}^{n-1} = -[A_{n-1} + \delta + (1-\phi)\theta B_{n-1}] - (1+\phi B_{n-1})x_t - (\lambda + B_{n-1})\varepsilon_{t+1} \tag{4.2.7}$$

where conditional moments are

$$E_t[m_{t+1} + p_{t+1}^{n-1}] = -[A_{n-1} + \delta + (1-\phi)\theta B_{n-1}] - (1+\phi B_{n-1})x_t \tag{4.2.8}$$

$$Var_t[m_{t+1} + p_{t+1}^{n-1}] = (\lambda + B_{n-1})^2\sigma^2. \tag{4.2.9}$$

Both conditional mean and conditional variance are linear functions of x, and as a result the bond prices are log-linear functions of x as in equation (4.2.6). The implied bond price satisfies therefore (substituting the above expressions into equation (4.1.13))

$$p_t^n = A_{n-1} + \delta + (1-\phi)\theta B_{n-1} - \tfrac{1}{2}(\lambda + B_{n-1})^2\sigma^2 + (1+\phi B_{n-1})x_t. \tag{4.2.10}$$

Comparing the coefficients in equation (4.2.6) and (4.2.10) gives the final recursions

$$A_n = A_{n-1} + \delta + (1-\phi)\theta B_{n-1} - \tfrac{1}{2}(\lambda + B_{n-1})^2\sigma^2 \tag{4.2.11}$$

$$B_n = 1 + \phi B_{n-1}. \tag{4.2.12}$$

The assumption (4.2.6) is now verified since with the coefficients (4.2.11) and (4.2.12) the price function (4.2.6) satisfies the pricing equation (4.1.13) and the assumption that bond prices are conditionally lognormal.

These equations provide a closed-form solution to the model, in the sense of being computable with a finite number of elementary operations, given the values of four parameters $(\phi, \lambda, \theta, \sigma)$.

There are several implications of this homoskedastic bond-pricing model. Considering the equation (4.2.12), coefficient B_n measures the fall in the logarithm price of an n- period bond when there is an increase in the state variable x or equivalently in the one-period interest rate r, and therefore it measures the sensitivity of the n- period bond return to the one-period interest rate. From equation (4.2.12) it results that B_n follows a linear difference equation in n with the solution

$$B_n = \frac{1-\phi^n}{1-\phi}. \tag{4.2.13}$$

As n increases, B_n approaches a limit

$$B = \lim_{n\to\infty} B_n = \frac{1}{1-\phi}. \tag{4.2.14}$$

Thus bond prices fall when the short-rate rise, and the sensitivity of bond returns to short-rates increases with maturity.

However, the disadvantage is that these formulas for recursions provide just the implicit solutions and therefore also the alternative approach, which leads to the explicit terms, would be presented in the next.

The starting point is the equation (4.1.12) - taking the logarithm of it yields to the n-period log-linear equation

$$p_t^n = E_t[m_{t+1} + ... + m_{t+n}] + \tfrac{1}{2} Var_t[m_{t+1} + ... + m_{t+n}]. \tag{4.2.15}$$

Consider again the price of a one-period bond given by (4.2.5). Because δ was set to $\frac{1}{2}(\lambda\sigma)^2$, the short-rate r is equal to the state variable x_t. This equality also implies behaviour of the stochastic process for the short-rate

$$r_{t+1} = r_t + (1-\phi)(\theta - r_t) + \varepsilon_{t+1} \tag{4.2.16}$$

and the future values of the short-rate are therefore determined by

$$r_{t+n} = r_t + (1-\phi^n)(\theta - r_t) + \sum_{i=1}^{n} \phi^{n-i} \varepsilon_{t+i} \tag{4.2.17}$$

for $n \geq 1$. Conditional mean and variance are given by, respectively

$$E_t[r_{t+n}] = r_t + (1-\phi^n)(\theta - r_t) \tag{4.2.18}$$

$$Var_t[r_{t+n}] = \left(\frac{1-\phi^{2n}}{1-\phi}\right)\sigma^2. \tag{4.2.19}$$

Denoting

$$\widetilde{m}_t^n \equiv \sum_{i=1}^{n} m_{t+i} \tag{4.2.20}$$

and applying the pricing kernel of this model (expression (4.2.4)) into (4.2.20) yields

$$-\widetilde{m}_t^n = -\sum_{i=1}^{n} m_{t+i} = n\delta + n\theta + (\frac{1-\phi^n}{1-\phi})(x_t - \theta) + \sum_{i=1}^{n}(\lambda + \frac{1-\phi^{n-i}}{1-\phi})\varepsilon_{t+i} \, . \tag{4.2.21}$$

By substituting the above equation into (4.2.15) gives the formulae for the price of n- period bond

$$p_t^n = -n\delta - n\theta - (\frac{1-\phi^n}{1-\phi})(x_t - \theta) + \tfrac{1}{2}\sum_{i=1}^{n}(\lambda + \frac{1-\phi^{n-i}}{1-\phi})^2 \sigma^2 \, . \tag{4.2.22}$$

Properties of bond prices can be summarised in terms of forward-rates, which are linear functions of the short-rate

$$f_t^n = p_t^n - p_t^{n+1} = (1-\phi^n)\theta + \phi^n x_t + \tfrac{1}{2}\left[\lambda^2 - (\lambda + \frac{1-\phi^n}{1-\phi})^2\right]\sigma^2 \tag{4.2.23}$$

for all $n \geq 0$. Given forward-rates, bond prices and yields can be computed from their definitions. By comparing the right side of the above equation to (4.2.18) note that the two first terms in (4.2.23) are the expected short-rate n-periods in the future. This means that forward rate movements reflect movements in the expected future short-rate, which are given by ϕ^n times movements in the current short-rate. The last term is the risk premium which consists of three factors: the magnitude of risk (σ), the price of risk (λ) and the mean-reversion (ϕ).

4.3 Discrete-time version of CIR model

CIR discrete-time model has similar structure like Vasicek's model. The main difference is that in Vasicek's model conditional variance is constant, while in CIR model varies with the state. The state variable x follows a "square-root process" given by

$$x_{t+1} = (1-\phi)\theta + \phi x_t + \sqrt{x_t}\,\varepsilon_{t+1} \, , \tag{4.3.1}$$

which is the discrete-time version of continuous-time CIR model and was derived by Sun (1992). ε_{t+1} is distributed normally and independently with mean zero and variance σ^2, $\phi \in (0,1)$ and $\theta > 0$. Despite the unusual form of the shock, equation (4.3.1) obeys an AR (1) process.

In continuous-time version, the state variable is strictly positive. With the square-root process conditional variance gets smaller as x approaches zero, i.e. it reduces the chance to get a negative value. As ε's are distributed normally, there is still a positive probability that x_{t+1} is negative, but the probability goes to zero as time interval shrinks. This means that the nonnegativity is guaranteed when the time interval is small.

The pricing kernel for a discrete-time version of CIR model is given by

$$-m_{t+1} = (1+\tfrac{1}{2}(\lambda\sigma)^2)x_t + \lambda\sqrt{x_t}\,\varepsilon_{t+1}. \tag{4.3.2}$$

The kernel is conditionally lognormal, and the coefficient of x was chosen in such a way to equal it to a short-rate.

Comparing the square-root process where the state variable x follows conditionally lognormal but heteroskedastic process to the homoskedastic model of Vasicek, the new element here is that the innovation ε_{t+1} is multiplied by $\sqrt{x_t}$. It means that x_{t+1} and m_{t+1} are normal conditional on x_t (as in Vasicek's model), but x_{t+i} and m_{t+i} are not normal conditional on x_t for all $i > 1$, whereas in the previous model x_{t+i} and m_{t+i} are normal conditional on x_t for all $i \geq 1$. This means that square-root model can be analysed by using the recursive approach of equation (4.1.13), but the n-period log-linear relation (4.2.15) does not hold in this case.

Proceeding by using the recursive analysis similar to that one used in the case of Vasicek's model, the price of a one-period bond can be determined by substituting (4.3.2) into (4.1.13), which yields

$$p_t^1 = E_t[m_{t+1}] + \tfrac{1}{2}Var_t[m_{t+1}] = -(1+\tfrac{1}{2}(\lambda\sigma)^2)x_t + \tfrac{1}{2}(\lambda\sigma)^2 x_t = -x_t. \tag{4.3.3}$$

Again, the short-rate r, defined as $r_t = -p_t^1$, is equal to the state variable x_t.

Next it is assumed that the price function has the same linear form $-p_t^n = A_n + B_n x_t$. In this model $A_0 = B_0 = 0$ and $A_1 = 0$ and $B_1 = 1$. Using the same method like in the previous model, conditional moments satisfy

$$E_t[m_{t+1} + p_{t+1}^{n-1}] = -[A_{n-1} + (1-\phi)\theta B_{n-1}] - [1+\tfrac{1}{2}(\lambda\sigma)^2 + \phi B_{n-1}]x_t \tag{4.3.4}$$

$$Var_t[m_{t+1} + p_{t+1}^{n-1}] = (\lambda + B_{n-1})^2\sigma^2 x_t, \tag{4.3.5}$$

which yields for the implied bond price

$$-p_t^n = -[A_{n-1} + (1-\phi)\theta B_{n-1}] - [1+\tfrac{1}{2}(\lambda\sigma)^2 + \phi B_{n-1}]x_t + \tfrac{1}{2}(\lambda + B_{n-1})^2\sigma^2 x_t \tag{4.3.6}$$

and therefore coefficients of the bond price formulas satisfy following recursions

$$A_n = A_{n-1} + (1-\phi)\theta B_{n-1} \tag{4.3.7}$$

$$B_n = 1 + \tfrac{1}{2}(\lambda\sigma)^2 + \phi B_{n-1} - \tfrac{1}{2}(\lambda + B_{n-1})^2\sigma^2. \tag{4.3.8}$$

Aligning these recursions to the solution obtained in Vasicek's model, it is apparent that the term in σ^2 has been moved from the equation describing A_n to the equation describing B_n.

This is because variance is now proportional to the state variable, so it affects the slope coefficient rather than intercept coefficient for the bond price.
The implied formulae (since it is not possible to derive the explicit representation) for the forward rates takes the form

$$f_t^n = x_t - B_n[(1-\phi)(x_t - \theta) + \lambda\sigma^2 x_t] - \tfrac{1}{2} B_n^2 \sigma^2 x_t \qquad (4.3.9)$$

and the derivation is provided in Campbell, Lo and MacKinley (1997). Comparing this result to analogous expression (equation (4.2.22)) in Vasicek model, which can be transformed by substituting equation (4.2.13) into

$$f_t^n = x_t - B_n[(1-\phi^n)(x_t - \theta) + \lambda\sigma^2] - \tfrac{1}{2} B_n \sigma^2, \qquad (4.3.10)$$

it can be seen that the difference between the two is that the variance terms in CIR model are multiplied by x_t where appropriate.

The logic behind the approach presented before is to choose the parameters that approximate average behaviour of bond yields. But for practical usage this kind of approximation is inadequate - parameters in Vasicek and CIR models can be chosen to match the five points on the yield curve (four parameters plus the short-rate) but cannot approximate the whole yield curve to the degree of accuracy needed by potential users. Therefore the more recent approach known as “no-arbitrage”, presented in the next by Ho and Lee model, suggests to add the additional time-dependent adjustment parameters to the model to suit it to observed prices. Usually, and in most of the applications, adjustment factors are used to allow the model to match the current yield curve exactly.

4.4 Discrete-time version of the Ho and Lee model

The state variable x_t obeys in this model the following process

$$x_{t+1} = x_t + \alpha_{t+1} + \varepsilon_{t+1}, \qquad (4.4.1)$$

where α_{t+1} is time dependent parameter and the innovation ε_{t+1} is distributed normally and independently with mean zero and variance σ^2. This differs from previous models in two ways: first, the process does not incorporate mean-reversion ($\phi = 1$ in equation (4.4.1)) and second, the state equation includes time-dependent drift α_{t+1}.
The pricing kernel is defined by

$$-m_{t+1} = \delta + x_t + \lambda\varepsilon_{t+1}. \qquad (4.4.2)$$

Given the equations (4.4.1), (4.4.2) and using the pricing relation $r_t = -p_t^1$ implies that a short-rate is equal to

$$p_t^1 = E_t[m_{t+1}] + \tfrac{1}{2}Var_t[m_{t+1}] = -(\delta + x_t) + \tfrac{1}{2}(\lambda\sigma)^2 . \tag{4.4.3}$$

If δ is set to $\frac{1}{2}(\lambda\sigma)^2$, the short-rate r is equal to the state variable x_t.

Applying the recursive analysis again, prices are log-linear functions of x, but in this case they are time dependent

$$-p_t^n = A_{nt} + B_{nt}x_t . \tag{4.4.4}$$

Conditional moments are given by

$$E_t[m_{t+1} + p_{t+1}^{n-1}] = -[A_{n-1,t+1} + \delta + B_{n-1,t+1}\alpha_{t+1}] - (1 + B_{n-1,t+1})x_t \tag{4.4.5}$$

$$Var_t[m_{t+1} + p_{t+1}^{n-1}] = (\lambda + B_{n-1,t+1})^2\sigma^2 , \tag{4.4.6}$$

which yields for the implied bond price

$$-p_t^n = \delta + A_{n-1,t+1} + B_{n-1,t+1}\alpha_{t+1} + (1 + B_{n-1,t+1})x_t - \tfrac{1}{2}(\lambda + B_{n-1,t+1})^2\sigma^2 \tag{4.4.7}$$

and leads to the final recursions

$$A_{nt} = \delta + A_{n-1,t+1} + B_{n-1,t+1}\alpha_{t+1} - \tfrac{1}{2}(\lambda + B_{n-1,t+1})^2\sigma^2 \tag{4.4.8}$$

$$B_{nt} = 1 + B_{n-1,t+1} . \tag{4.4.9}$$

In this homoskedastic model the n-period log-linear relation (4.2.15) holds and therefore it can be analysed by applying the similar approach like in the case of Vasicek's model. Substituting the given pricing kernel (formulae (4.4.2)) into the equation (4.2.20) yields

$$-\tilde{m}_t^n = nx_t + n\delta + \sum_{i=1}^{n}(n-i)\alpha_{t+i} + \sum_{i=1}^{n}(\lambda + n - i)\varepsilon_{t+i} . \tag{4.4.10}$$

Bond prices can be computed by using the equation (4.2.15) and substituting into relation (4.4.10)

$$p_t^n = -nx_t - n\delta - \sum_{i=1}^{n}(n-i)\alpha_{t+i} + \tfrac{1}{2}\sum_{i=1}^{n}(\lambda + n - i)^2\sigma^2 . \tag{4.4.11}$$

And for forward rates the following relationship holds

$$f_t^{\,n} = r_t + \sum_{i=1}^{n}\alpha_{t+i} + \tfrac{1}{2}(\lambda^2 - (\lambda + n)^2)\sigma^2 . \tag{4.4.12}$$

This differs from the solution obtained in Vasicek in two ways: first, a unit increase in r is associated with unit increase in $f^{\,n}$, whereas in Vasicek this increase is $(1-\phi^n)$, which is less than unity. The second difference is the risk premium, the final term in (4.4.12).

Despite these differences, Backus, Foresi and Zin (1998) show that time-dependent drift allows Ho and Lee model to reproduce some of the features of Vasicek's model, and among them the conditional mean of future short-rate. Future short-rates take the following form

$$r_{t+n} = r_t + \sum_{i=1}^{n} (\alpha_{t+i} + \varepsilon_{t+i}) \tag{4.4.13}$$

with the conditional mean

$$E_t[r_{t+n}] = r_t + \sum_{i=1}^{n} \alpha_{t+i} \tag{4.4.14}$$

for $n \geq 1$. Comparing this to the analogous expression in Vasicek (equation (4.2.18)), the two are equal if

$$\sum_{i=1}^{n} \alpha_{t+i} = (1-\phi^n)(\theta - r_t). \tag{4.4.15}$$

Thus the time-dependent drift can be set to imitate this consequence of mean-reversion. However, in practice it is more common to use the drift parameter to match the model to the current yield curve. To fit forward rates generated by Vasicek model, the drift parameter must satisfy (compare (4.2.23) and (4.4.12))

$$\sum_{i=1}^{n} \alpha_{t+i} = (1-\phi^n)(\theta - r_t) + \tfrac{1}{2}\left[\lambda_1^2 - (\lambda_1^2 + \frac{1-\phi^n}{1-\phi})^2\right]\sigma_1^2 - \tfrac{1}{2}\left[\lambda_2^2 - (\lambda_2^2 + n)^2\right]\sigma_2^2. \tag{4.4.16}$$

The drift parameters implied by (4.4.15) and (4.4.16) are in general different. Since parameter values possibly may take on different values in the respective models; λ_1 and σ_1 denote the parameters of the Vasicek and λ_2 and σ_2 of the Ho and Lee model. Because $\lim_{\phi \to 1} \frac{1-\phi^n}{1-\phi} = n$, the two expressions above may be equated when $\phi = 1$ by setting $\lambda_1 = \lambda_2$ and $\sigma_1 = \sigma_2$. If $\phi \in (0,1)$ it is not possible to equate the expressions.

This property points to the fact that the drift-parameter does not adequately capture the effects of mean-reversion. The reason for that is the non-linear relation in risk premium between mean-reversion (ϕ) and the price of risk (λ). The risk premium on the n-period forward-rate equals in Vasicek model to

$$\tfrac{1}{2}\left[\lambda_1^2 - (\lambda_1^2 + \frac{1-\phi^n}{1-\phi})^2\right]\sigma_1^2 \tag{4.4.17}$$

and the analogous expression in Ho and Lee model is

$$\tfrac{1}{2}[\lambda_2^2 - (\lambda_2^2 + n)^2]\sigma_2^2. \tag{4.4.18}$$

If $\lambda_1 = \lambda_2$ and $\sigma_1 = \sigma_2$, (4.4.17) and (4.4.18) are equal for $n = 1$. As n grows the difference between the two is also growing.

Another discrepancy is that Ho and Lee model does not capture the conditional variances of future short-rates since it takes the form

$$Var_t[r_{t+n}] = n\sigma^2 \,. \tag{4.4.19}$$

Comparing this expression to the analogous in the Vasicek's model (equation (4.2.19)) conditional variances are in general different and for longer time horizons they are greater in Ho and Lee model.

In summary, the parameters of the Ho and Lee model can be chosen in such a way that they fit the conditional mean of future short-rates or the current yield curve, but in general it is not possible for them to do both simultaneously. The parameters cannot be chosen to reproduce the conditional variances of future short-rates and this results to the mispricing options on bonds.

V. FITTING MODELS TO THE DATA

Vasicek and CIR models need additional error terms to fit them to the market data. These error terms can be seen as a measurement of errors in bond prices, errors in calculating implied pure discount bond prices from an observed coupon-bearing bond prices, or specification errors arising from tax effects or transaction costs. To keep the models testable, it is necessary to accept supplementary assumptions.

One common assumption is that bond price errors are serially uncorrelated. With this assumption the time series implications of term structure models can be examined. Under specific assumptions about the additional bond price errors Campbell, Lo and MacKinley (1997) derived what they call "latent-variable" models for bond returns. They are expressed in the form of the system of regression equations and can be estimated by Generalised Method of Moments.

Different approach assumes that bond price errors are uncorrelated across bonds, although they may be correlated over time. This method was introduced in 1986 and after then used by a number of authors. They ignore the models' time series implications and estimate all the parameters from the term structure observed at a point in time. By repeating the procedure over many time periods, it generates a sequence of parameter estimates, which should in theory be identical for all time periods, but in practice it does not hold. It is true that bond price errors might cause estimated parameters to vary over time even if true underlying parameters are constant. But in simple one-factor models analysed before there also appear to be some systematic differences between the parameter values implied by the respective approaches, which indicates the misspecification in the models.

In the discrete time version of the Vasicek model, the four parameters $(\phi, \sigma, \theta, \lambda)$ of the model can be identified (following the procedure used by Backus, Foresi and Telmer (1998)) by considering the following moments of the data

$$Corr[y_t^1, y_{t-1}^1] = \phi \tag{5.1}$$

$$E[y_t^1] = \theta \tag{5.2}$$

$$Var[y_t^1] = \frac{\sigma^2}{1-\phi^2}\ . \tag{5.3}$$

The mean-reversion parameter ϕ is the first-order autocorrelation of the short-rate process. Since θ is the unconditional mean of the short-rate process, it is set equal to the sample mean. Given ϕ, the unconditional variance of the short-rate process then identifies the innovation variance σ^2. The final parameter, the price of risk λ, is chosen to approximate the average slope of the yield curve. It is identified from the average excess return on a long-term bond, or equivalently from the average difference between a very long-term forward-rate and the short-rate

$$\lim_{n\to\infty} E[r_{t+1}^n - y_t^1] = \lim_{n\to\infty} E[f_t^n - y_t^1] = -\frac{\lambda\sigma^2}{1-\phi} - \frac{\sigma^2}{2(1-\phi)^2}\ . \tag{5.4}$$

Thus those four parameters are required to govern the dynamics of interest rates and the average slope of the yield curve.

As pointed out by Gibbons and Ramaswamy (1993), Vasicek model generates a mean yield curve with significantly less curvature than can be usually seen in the data. The problem arises because the time series behaviour of the short-rate implies a value of ϕ close to one (it is highly persistent), but a smaller value is required to generate the desired concavity of the yield curve.

Parameter identification is very similar in the discrete time version of Cox, Ingersoll and Ross model. Here the moments become

$$Corr[y_t^1, y_{t-1}^1] = \phi \tag{5.5}$$

$$E[y_t^1] = \theta \tag{5.6}$$

$$Var[y_t^1] = \frac{\theta\sigma^2}{1-\phi^2}\ . \tag{5.7}$$

$$\lim_{n\to\infty} E[r_{t+1}^n - y_t^1] = \lim_{n\to\infty} E[f_t^n - y_t^1] = -\theta\sigma^2[B\lambda + \tfrac{1}{2}B^2] \tag{5.8}$$

where B is the limiting value of B_n given by the equation (4.3.8). As before, ϕ can be identified from the estimated first-order autocorrelation of the short-rate, θ is the sample mean and σ^2 identifies the innovation variance.

CIR model produces also an average yield curve with substantially less curvature when the short-rate is highly persistent and therefore it shares many of the empirical limitations of the Vasicek model.

In summary, there are several discrepancies between these two one-factor models and the real state of matters. Those models can serve as an archetype, but for practical usage they are not adequate, which point toward the construction of more complex models with more parameters. Additional parameters are needed in applied work, where a model that is not able to reproduce the current yield curve can hardly price correctly more complex securities. Ho and Lee pioneered this alternative approach in the form of a binomial tree, but the idea is more general: to add time dependent parameters to the model to match it to observed prices. As it was demonstrated in the case of Ho and Lee model, although it fits one set of asset prices exactly to the current term structure, it does not automatically guarantee to fit other asset prices accurately. This called for more sophisticated approach with more additional parameters. Black, Derman and Toy (1990) allowed the volatility parameter, σ, to vary with time. This extension was fundamental to the pricing of interest rate dependent options for which is the volatility critical parameter. Hull and White (1990) further refined the approach by assuming that not only σ, but also analogues of θ and ϕ can vary with time.

VI. CONCLUSION

In this thesis, different models of the term structure of interest rates were analysed. Each approach has advantages as well as disadvantages when compared on the basis of the analytical tractability of the model solution, the number and the estimatability of its parameters and the amount of the market information used.

First were analysed two models (Vasicek and Cox, Ingersoll and Ross), which are based on equilibrium characteristics of the term structure and a single source of the uncertainty – the short-rate. Models belonging to that category can be characterised by their tractability and ease of use, but with the resulting disadvantage of the unrealistic behaviour of the short-rate and the limitation of the possible shapes of the term structure.

The second approach uses the idea to model the dynamics of the whole term structure from an initially exogenously given set. It is represented by the simplest one-factor model in this category, which is the Ho and Lee model. This has the advantage that it is analytically tractable, but its main drawback is that it implies that all rates are equally variable. In general, these models have the property that they can be fitted to the current term structure data, and their main use is therefore for pricing derivative instruments written on discount bonds.

As it was discussed, there is accordingly great interest in developing more flexible models that fully exploit the information in the yield curve. Such models are complicated what is undoubtedly given by the complexity of modern financial markets.

VII. RESUME

Skúmanie časovej štruktúry úrokových mier je dôležité pri analýze cenných papierov závislých od úrokovej miery, pričom možnosti použitia sú veľmi rozsiahle; medzi inými napr.: analýza výnosov kontraktov s fixnými platbami, ktoré majú rôzne časy splatnosti; predpovedanie budúcich úrokových mier; oceňovanie dlhopisov a iných kontraktov s fixnými platbami; oceňovanie opcií na kontrakty s fixnými platbami a tiež formovanie očakávaní o celkovom ekonomickom vývoji.

Existuje niekoľko teórií a hypotéz vysvetľujúcich dynamiku časovej štruktúry úrokových mier, pričom jedným z najnovších prístupov, ktorý je prezentovaný v tejto práci, je modelovanie pomocou stochastických procesov. Teória vyžaduje niekoľko predpokladov - časová štruktúra a ceny dlhopisov sú priradené k určitým stochastickým faktorom, pričom sa predpokladá, že tieto faktory sa v čase vyvíjajú podľa určitého stochastického procesu a výsledné úrokové miery a ceny dlhopisov musia spĺňať podmienky arbitráže. Podľa počtu (jeden alebo viac) použitých stochastických faktorov sú definované jedno- alebo viac-faktorové modely, pričom v prípade jedno-faktorových modelov sa zvyčajne predpokladá, že jediným zdrojom neistoty je okamžitý úrok (short-rate).

V prvej časti práce je popísaný tradičný prístup k modelovaniu časovej štruktúry úrokových mier v spojitom čase, ktorý je reprezentovaný jedno-faktorovými modelmi Vašíčka a Cox, Ingersoll a Rossa (CIR). Modely patriace do tejto kategórie sú charakteristické jednoduchým použitím a ich riešenia sa dajú vyjadriť v analytickom tvare. Ich nedostatkom však je, že nedokážu dostatočne aproximovať celú výnosovú krivku do stupňa presnosti požadovaného užívateľmi. Alternatívny prístup vychádza z myšlienky modelovať dynamiku časovej štruktúry pridaním časovo závislých parametrov, ktoré sú volené tak, aby presne modelovali súčasnú (pozorovanú) výnosovú krivku. Tento prístup je v práci reprezentovaný najjednoduchším jedno-faktorovým modelom v tejto kategórii, ktorým je Ho a Lee model. Jeho hlavnou výhodou je, že poskytuje riešenie v analytickej forme, medzi jeho nedostatky patrí, že implikuje rovnakú volatilitu pre všetky úrokové miery.

V ďalšej časti práce sú tieto tri teoretické modely reformulované do diskrétneho času, čo umožňuje ich jednoduchšiu ekonometrickú analýzu. Zosumarizovaním teórie, ktorá priraďuje tento prístup k teórii oceňovania cenných papierov zavedením tzv. „stochastického diskontného faktora", sú jednotlivé modely v diskrétnom čase prezentované ako modely časových radov pre stochastický diskontný faktor. Dôraz je kladený na praktické vlastnosti výnosov dlhopisov implikovaných štruktúrou jednotlivých modelov. Pre Vašíčkov a Ho a Lee model sú odvodené explicitné riešenia pre výpočet forward mier, v prípade modelu CIR to nebolo možné, a preto je uvedený rekurzívny vzťah. Nasledujúca kapitola je venovaná voľbe hodnôt parametrov, ktoré plnia rozhodujúcu úlohu pri tom, aké výsledky modely dosiahnu, a preto si zaslúžia nie menšiu pozornosť ako samotné modely.

Jedno-faktorové modely sa dajú vyjadriť v jednoduchej forme a ich použitie je taktiež jednoduché, ale existuje príliš veľa rozporov medzi nimi a realitou. Tieto rozpory vedú ku konštrukcii flexibilnejších modelov s väčším počtom parametrov, ktoré plne využívajú informácie obsiahnuté vo výnosovej krivke. Takéto modely sú komplikované, čo je bezpochyby dané komplexnosťou moderných finančných trhov.

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