Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

mathematics of economy and finance

OSCILLATIONS

OF THE

FOREIGN EXCHANGE RATE

AND THE

DEVIL'S STAIRCASE

BRATISLAVA, 2004 BOĎOVÁ KATARÍNA

Fakulta matematiky, fyziky a informatiky
Univerzita Komenského, Bratislava

ekonomická a finančná matematika

D I P L O M O V Á   P R Á C A

Diplomant: Katarína Boďová

Vedúci diplomovej práce: Prof. RNDr. Pavol Brunovský, DrSc.

Prehlasujem, že som diplomovú prácu vypracovala samostatne, iba s pomocou literatúry, uvedenej v zozname a konzultácií s vedúcim diplomovej práce.

Týmto sa chcem poďakovať Prof. RNDr. Pavlovi Brunovskému, DrSc. za odbornú pomoc, množstvo cenných pripomienok a rád, ako aj za ochotu, prejavenú pri vedení práce.

V Bratislave, 5. apríla, 2004

Contents

1 INTRODUCTION

The question of estimating and explaining movements of a real foreign exchange rate under the floating regime is crucial in open economies. Economic theory already presents various models to explain these movements: the purchasing power parity, the uncovered interest rate parity, the monetary model, the portfolio balanced model or the Dornbusch model. Although these models vary in the approach used, they have one common property, that they explain movements of the real foreign exchange rate only by slowly changing economic fundamentals.

A critique of the above mentioned models first appeared in *Baxter and Stockmann* (1989) [2]. The evidence was found that the exchange rate has a tendency to fluctuate considerably more than financial market analysts think is justified by changes in the economic fundamentals. *Azariadis* (1993) [1] demonstrates this phenomenon, called "exchange–rate overshooting" in the Dornbusch model. Also the fact that abrupt changes in economical fundamentals do not generally reflect strongly on the exchange rate movement (and reversely) supports the idea that the exchange rate movements are not fully explained by economical fundamentals.

Distinction between two types of agents on the market is the idea, occurred in *Jeanne, O. and Rose, A.K.* (2002) [11] ("informed" and "noise" traders) and *De-Grauwe and Grimaldi* (2002) [8] ("chartists" and "fundamentalists"). Considering only one type of agents on the market adapting with their strategy to the situation, *Erdélyi, A.* (2003) [7] and *Brunovský, P., Erdélyi, A. and Walther H.O.* (2004) [3] analyzed the model of the deviation of the real exchange rate from its equilibrium.

Despite the fact that economical fundamentals do not explain the real exchange rate properly, their presence in the model is a contribution. The typical frame for a model of exchange rate, denoted by S_n is

$$S_{n+1} = f(\text{economic fundamentals}) + x_n \tag{1.1}$$

This paper presents and analyses the model of x_n, inspired by the model of *Brunovský, P., Erdélyi, A. and Walther H.O.* [3, 7].

1.1 Model description

The purpose of this diploma thesis is to find an acceptable model of the deviation x_n from the "natural" exchange rate which both corresponds with the behaviour of agents on the market and is simple enough for analytical and numerical explorations. There are many possible ways how to simulate such a process, within the framework of the previous section but only certain types of approach satisfy all the important requirements needed.

There are at least three different ways how one can set up such a model. The first way, analysed in the recent diploma work of *Erdélyi, A.* [7], and a paper of *Brunovský, P., Erdélyi, A. and Walther H.O.* [3] is to consider a delay differential equation of the form

$$\dot{x}(t) = A\,(x(t) - x(t-1)) - Bx(t)|x(t)| \tag{1.2}$$

with $A, B \in \mathbb{R}^+$. This model captures most of the important features of the behavior of agents on a financial market although it is too simple to explain the observed reality. In this approach, decisions of agents are distributed in time in order to profit from transactions made on the market. There is also a parallel approach, in which agents make their decisions in one moment. The form of such difference model is

$$x_{n+t} = x_n + A(x_n - x_{n-t}) - Bx_n|x_n| \tag{1.3}$$

and is equivalent to the model of *De Grauwe and Grimaldi* (2002) [8]. The value of the time step t can be chosen arbitrarily, for simplicity we set $t = 1$. The second term in this model causes the trajectories to converge to the zero fixed point ($A < 1$) or to diverge to the infinity ($A > 1$), as indicated by our numerical simulations. This property is quite unfortunate since it implies that either the real exchange rate loses completely the influence on the psychology of agents as the time flows or it is prevailed by that influence and tends to the infinity (since A is constant). Hence we do not explore this form of the model further.

To set up a reasonable model of x_n we substituted for the constant A in (1.3) the function $\bar{A}(x_n)$ of x_n. This change implies the improvement of (1.3). In the original equation (1.3), expectations of people on the market were the same during the whole time period, whereas with A taken as a function of x_n, agents adapt their expectations to the value of x_n. The function $\bar{A}(x_n)$ is set as

$$\bar{A}(x_n) = A(M - |x_n|)^+\,, \tag{1.4}$$

where $A > 0$ and $M > 0$ are constant and $(x)^+ = \max\{x, 0\}$. The interpretation of this particular form of $\bar{A}(x_n)$ is the following: The higher the value $|x_n|$ is, the lower is the number of agents, making their decisions due to the increase (resp. decrease) of the real exchange rate. The proposed model has the form

$$x_{n+1} = x_n + A(x_n - x_{n-1})(M - |x_n|)^+ - Bx_n|x_n|\,, \tag{1.5}$$

where A, B and M are positive.

The precise form of the difference equation (1.5) is derived from our assumptions on the behavior of agents on the market. Here we present the reasoning behind the presence of the terms of (1.5).

• The best approximation for x_{n+t} in the near future with the minimum information available (only the current value of variable) is x_n. Therefore the first term in (1.5) is x_n.

• For t sufficiently small we have $(x_n - x_{n-t}) \approx (S_n - S_{n-t})$. As long as S_n depreciates (appreciates), investors tend to buy (sell) a foreign currency to profit from this depreciation (appreciation). This tendency causes a rise (fall) of the demand for the foreign currency and consequently S_n continues in the depreciation (appreciation). The factor of influence of the rise (fall) of x_n, denoted by A represents essentially sensitivity (elasticity) of potential investors caused by the change of the exchange rate. Therefore the second term in (1.5) includes $A(x_n - x_{n-t})$.

• When the value of x_n becomes too high (low), one can reason that the real exchange rate is higher (lower) then its "natural" value and that such a situation can not persist any longer, what means that it will fall down (rise up). Agents start to sell (buy) the foreign currency to prevent themselves from a loss of profit and the real exchange rate consequently falls down (rises up). The higher $|x_n|$ is, the more people think this way. Therefore this effect is proportionate to the value of x_n. The third term in the equation (1.5) express this contribution.

The model (1.5) consists of the information about x_n and a weighted average of the two mentioned effects, with weights $M - |x_n|$ and $|x_n|$ (for $|x_n| \leq M$). As was already stated, the number of agents, taking the "precautious" strategy is proportionate to $|x_n|$. Therefore if we have M agents (resp. N), buying and selling the foreign currency, $|x_n|$ (resp. $\frac{N}{M}|x_n|$) represents number of agents, behaving "precautious" in this situation and $M - |x_n|$ (resp. $\frac{N}{M}(1 - |x_n|)$) is the number of agents, trying to profit from the change of the foreign currency. In the case when $|x_n| > M$, everybody waits for the change in the trend of the foreign currency and the second term vanishes.

The closer look to the equation (1.5) reveals that there are only two constant parameters needed. After a substitution $x_n := Mx_n$ and $a = MA$, $b = MB$, we obtain

$$x_{n+1} = x_n + a(x_n - x_{n-1})(1 - |x_n|)^+ - bx_n|x_n|. \tag{1.6}$$

Below, we will deal with the normalized equation (1.6).

2 STABILITY OF THE ZERO SOLUTION

In this section, we look closer at the difference equation (1.6) introduced in the previous chapter. Generally, a second-order difference equation can be rewritten in the form of a system of two first-order difference equations. We deal with two systems corresponding to (1.6). The first one takes the form

$$x_{n+1} = F_1(x_n, y_n) = x_n + a(x_n - y_n)(1 - |x_n|)^+ - bx_n|x_n|, \tag{2.1}$$

$$y_{n+1} = F_2(x_n, y_n) = x_n. \tag{2.2}$$

and is obtained by the linear transformation $x_n \to x_n$, $x_{n-1} \to y_n$. The second one is obtained by $x_n \to x_n$, $x_n - y_n \to y_n$ and takes a form

$$x_{n+1} = G_1(x_n, y_n) = x_n + ay_n(1 - |x_n|)^+ - bx_n|x_n|, \tag{2.3}$$

$$y_{n+1} = G_2(x_n, y_n) = ay_n(1 - |x_n|)^+ - bx_n|x_n|. \tag{2.4}$$

From the mathematical point of view, a stability analysis of fixed points which are periodic points of period one, tells us important facts about the evolution of trajectories of the recurrence system. As we can easily verify, our system has only one fixed point, which is the origin in the two-dimensional real space. This point has also a significant meaning from the economical point of view. It represents the equilibrium, determined by economical fundamentals. The result, stability or unstability of the zero fixed point (depending on values of parameters a and b), can tell us whether the real state converges to the equilibrium or not.

2.1 Numerical simulations

In this section, we present graphs of trajectories, satisfying the system (2.1), (2.2), obtained by numerical simulations. [1] These graphs indicate that for $a < 1$, the origin is asymptotically stable and for $a > 1$, the origin is unstable. As we will see later, these simulations are in agreement with our theoretical results on stability of the origin. The case $a = 1$ can be also simulated, despite the fact that the theoretical result remains unknown.

[1] Numerical simulations were performed in MATLAB.

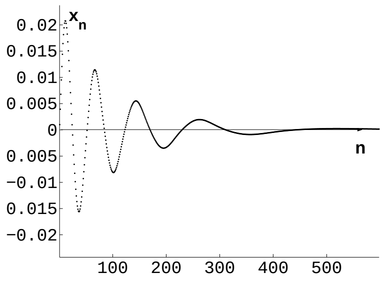


Figure 1: Simulations of trajectories for $a = 0.98$, $b = 1$

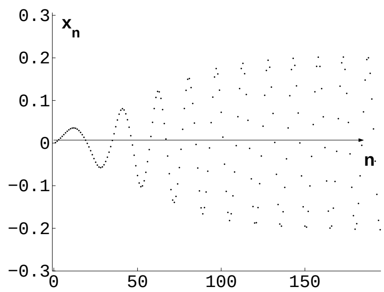


Figure 2: Simulations of trajectories for $a = 1.1$, $b = 1$

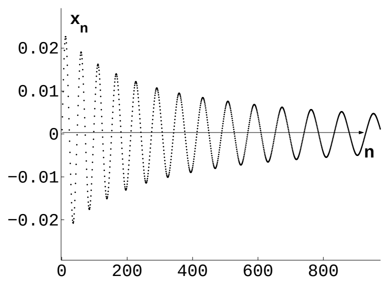


Figure 3: Simulations of trajectories for $a = 1$, $b = 1$

2.2 Center manifold theory

The first thing to do when investigating the linear stability of a fixed point is to look at eigenvalues of the matrix representing the linearized system about that fixed point. For $|x_n| < 1$ the recurrence equations (2.1), (2.2) can be written as

$$x_{n+1} = x_n + a(x_n - y_n)(1 - |x_n|) - bx_n|x_n|, \tag{2.5}$$
$$y_{n+1} = x_n. \tag{2.6}$$

The linearization about a trivial solution $(x_n, y_n) = (0,0)$ reduces the system (2.5)–(2.6) to

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = L \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \text{where} \quad L = \begin{pmatrix} 1+a & -a \\ 1 & 0 \end{pmatrix}. \tag{2.7}$$

It is tempting to say that stability (unstability) of the origin depends only on the parameter a. But as we will see later, this is not so obvious. When we denote eigenvalues of L by λ_1 and λ_2, we obtain that $\lambda_1 = 1$ and $\lambda_2 = a$. If $a > 1$, the origin is unstable (one eigenvalue lies outside the unit circle). Since one eigenvalue is always equal to 1, it is impossible to decide at this point for which values of a is the origin asymptotically stable. The stability is therefore depended also on nonlinearity in (2.5), where the parameter b seems to play an important role.

To solve the problem of stability we apply the technique of center manifold reduction which was also used in the diploma thesis of *A. Erdélyi* (2003) [7] but in the case of a differential equation. This approach allows us to study in details the critical case when m eigenvalues ($m \in \mathbb{N}$, $m \geq 1$) of a matrix representing the linearization of a mapping $\mathbb{R}^k \to \mathbb{R}^k$ ($k \in \mathbb{N}$), lie on the unit circle. The method is based on the projection of the original system to the recurrent system, defined on $\mathbb{R}^{k-m}$. The inspection of stability is then provided on a reduced system. The details of this approach can be found in *Carr, J.* (1981) [5] for C^2 mapping and in *Iooss, G.* (1979) [10] in the generalized version for C^n mapping, where $n \geq 1$.

2.3 Reduction to a center manifold

Before the center manifold theory can be applied to the system (2.5)–(2.6), the transformation into a form, in which the matrix, corresponding to the linear part of the system is diagonal, is needed. After the linear transformation

$$\begin{aligned} x_n &= u_n + a v_n, & (2.8)\\ y_n &= u_n + v_n, & (2.9) \end{aligned}$$

the system (2.5)–(2.6) becomes

$$\begin{aligned} u_{n+1} &= H_1(u_n, v_n) = u_n - g(u_n, v_n), & (2.10)\\ v_{n+1} &= H_2(u_n, v_n) = a v_n + g(u_n, v_n), & (2.11) \end{aligned}$$

where

$$g(u,v) = -av|u+av| + \frac{b}{1-a}(u+av)|u+av|. \tag{2.12}$$

Theorem 1. *If $b > 0$, the fixed point $\bar{x} = (0, 0)$ of (2.10), (2.11) is*

- *asymptotically stable if $a < 1$;*
- *unstable if $a > 1$.*

Proof. Let $a < 1$. If the map (2.5)–(2.6) is C^r the center manifold theory guarantees the existence of the center manifold of the same smoothness. Although our map is only C^1, it is C^∞ in the half-planes $\pm v \geq 0$. Therefore the center manifold $v_n = h(u_n)$ can be written separately in the two subspaces in the form of the Taylor series. By the center manifold properties $h(0) = h'(0) = 0$ and the Taylor series representation of h is

$$h(u) = k_2^\pm u^2 + k_3^\pm u^3 + k_4^\pm u^4 + \cdots = O(u^2)\,, \tag{2.13}$$

where $k_i^\pm$ applies in the half-plane $\pm v \geq 0$ respectively. By substituting (2.13) into (2.10) we get a one dimensional recurrence equation for u_n on the center manifold $v_n = h(u_n)$ in the form

$$u_{n+1} = u_n + ah(u_n)|u_n + ah(u_n)| - \frac{b}{1-a}(u_n + ah(u_n))|u_n + ah(u_n)|. \tag{2.14}$$

The second term in (2.14) has order $O(u_n^3)$, while the third term has order $O(u_n^2)$. In addition, there exists $\delta > 0$ such that for $|u_n| < \delta$ we have $|ah(u_n)| < |u_n|$ (because of (2.13)). This inequality implies that

$$|u_n + ah(u_n)| = (u_n + ah(u_n))\,\mathrm{sgn}(u_n)\,. \tag{2.15}$$

For $|u_n| < \delta$ we obtain

$$\begin{aligned} u_{n+1} &= u_n + ah(u_n)|u_n + ah(u_n)| - \frac{b}{1-a}(u_n + ah(u_n))^2\,\mathrm{sgn}(u_n) \\ &= u_n - \frac{b}{1-a}u_n^2\,\mathrm{sgn}(u_n) + O(u_n^3) \\ &= u_n(1 - u_n\frac{b}{1-a}\,\mathrm{sgn}(u_n)) + O(u_n^3) \\ &= u_n(1 - \frac{b}{1-a}|u_n|) + O(u_n^3)\,. \end{aligned} \tag{2.16}$$

Now we inspect two cases. The first one, when $u_n > 0$ and the second one, when $u_n < 0$ ($u_n = 0$ is a fixed point of (2.16)). In both cases there exists $\delta_1 > 0$ such that for $|u_n| < \delta_1$ the third term in (2.16) satisfies

$$O(|u_n^3|) < \frac{b}{2(1-a)}u_n^2. \tag{2.17}$$

If $0 < u_n < \min\{\delta, \delta_1\}$ then

$$u_{n+1} \leq u_n - \frac{b}{1-a}u_n^2 + \frac{b}{2(1-a)}u_n^2 = u_n\left(1 - \frac{b}{2(1-a)}|u_n|\right). \tag{2.18}$$

Since $\frac{b}{2(1-a)}$ is positive ($a < 1$), we obtain for $u_n < \min\{\delta, \delta_1, 2(1-a)/b\}$ (which is positive)

$$0 < u_{n+1} < u_n. \tag{2.19}$$

On the other hand $0 < -u_n < \min\{\delta, \delta_1\}$ implies that

$$\begin{aligned} u_{n+1} &\geq u_n + \frac{b}{1-a}u_n^2 - \frac{b}{2(1-a)}u_n^2 = u_n + \frac{b}{2(1-a)}u_n^2 \\ &= u_n\left(1 - \frac{b}{2(1-a)}|u_n|\right), \end{aligned} \tag{2.20}$$

and for $-u_n < \min\{\delta, \delta_1, 2(1-a)/b\} > 0$ we obtain

$$u_n < u_{n+1} < 0. \tag{2.21}$$

From (2.19), (2.21) follows that if $|u_n| < \min\{\delta, \delta_1, 2(1-a)/b\} > 0$ and $u_n \neq 0$ then $|u_{n+1}| < |u_n|$. Since the origin is the only fixed point of (2.10)-(2.11), it is also the only fixed point of (2.16). Therefore $u_n \to 0$ for $n \to \infty$, i.e. the origin is an assymptotically stable fixed point of (2.16) for $a < 1$. The center manifold theory consequently implies also the asymptotical stability of the origin, as a fixed point of (2.10)–(2.11) for $a < 1$.

On the other hand if $a > 1$ there are two linearly independent repelling directions from the origin in $\mathbb{R}^2$, and therefore the origin is repellor. Unfortunately, we were unable to analyze the case $a = 1$ in this way and thus we will not discuss it any further in this thesis. □

3 AN INVARIANT SET, THE CASE a > 1

In this chapter we show that the function $G = (G_1, G_2) : X \to Y$, $X, Y \subset \mathbb{R}^2$ which represents the difference system

$$x_{n+1} = G_1(x_n, y_n) = x_n + ay_n(1 - |x_n|)^+ - bx_n|x_n|, \quad (3.1)$$
$$y_{n+1} = G_2(x_n, y_n) = ay_n(1 - |x_n|)^+ - bx_n|x_n|, \quad (3.2)$$

is a homeomorphism in a certain region. This feature is crucial in identifying limit properties of trajectories in the case when the value of parameter a is greater then 1. As we show, for special values of parameters a, b, there exists a monotone sequence of closed curves, homeomorphic to circle with a bounded "limit" (understood as a boundary).

3.1 Numerical simulations of an invariant set

First we present a figure of trajectories "winding up" to the invariant set. When choosing the initial point from the interior of the area, bounded by the invariant set, the trajectory winds up from the interior, whereas the trajectory of the exterior point, winds up to the same invariant set from the exterior.

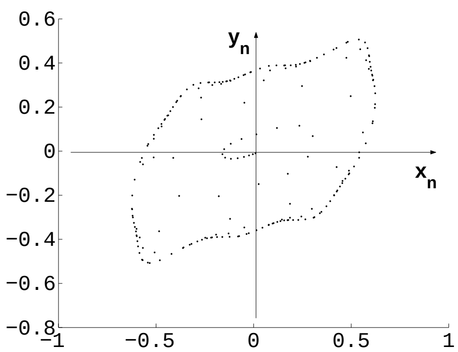


Figure 4: Convergence to the invariant set from the interior

According to our numerical investigations (see Fig. 5) for fixed b the size of the invariant set grows with an increasing parameter a. The problem appears for values of a, close to 1.5, where the computer no longer gives a result.

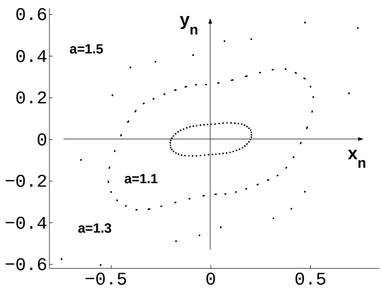


Figure 5: Invariant set for $a = 1.1$, $a = 1.3$ and $a = 1.5$ $(b = 1)$

One can observe that the invariant set is not necessarily symmetric, although the recurrence system is symmetric. The Fig. 6 shows the two possible nonsymmetric invariant sets, according to the initial values chosen. Taking the two sets together, one obtains a set, which is symmetric around the origin.

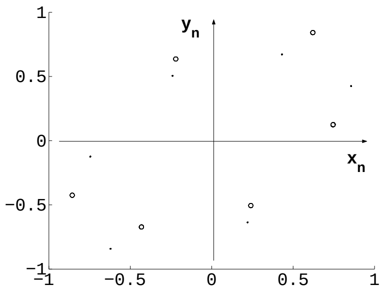
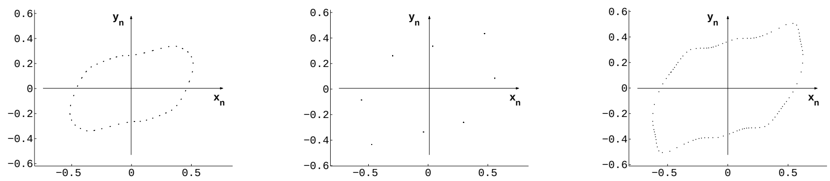


Figure 6: Invariant sets for different initial values

Another interesting phenomena can be observed on Fig. 7. For certain special values of parameter a, the computed limit points of the trajectory indicate a periodic trajectory with a small period, in other cases they fill an invariant set rather densely. The number of these points depends also on the value of the parameter a and it can attain different values (6, 8, 12,...).



$a = 1.3,\ b = 1$ $a = 1.35,\ b = 1$ $a = 1.4,\ b = 1$

Figure 7: Invariant curve

These results convinced us to compute a rotation number of our mapping $G(a, b)$ on the invariant set. Since there is an evidence that the mapping $G(a, b)$ is an orientation preserving homeomorphism on the invariant set, this counting has a rational aspect. The result is a graph of the average rotation of the arbitrarily chosen point on the invariant set (simulated with a very large number of iterations), as a function of a parameter a with b fixed.

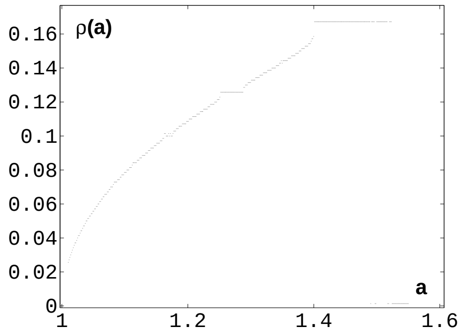


Figure 8: Rotation number - Devil's Stairs

There is a strong similarity between the graph on Fig. 8 and the standard graphs of rotation number, as a function of some parameter of a corresponding system. The similarities are in the following facts:

- $\rho(a, \bar{b})$ is a nondecreasing function of a until numerical obstructions occurs;
- for rational numbers $1/q$ there is an interval $I_{1/q}$ with nonempty interior such that for all $a \in I_{1/q}$ from a discretization of a we have $\rho(a, \bar{b}) = 1/q$;
- the more discretization points of a, the more a number of steps on the graph of $\rho(a, \bar{b})$ is increasing with refining of discretization of a.

Numerical obstructions mentioned, occur for the value of $a \approx 1.5$ for $b = 1$. This is approximately the value, for which the invariant set almost touches the lines $x = \pm 1$. Therefore the problem, we encounter while making numerical observations may be caused by the fact that the invariant set no longer lies in the area, where G is homeomorphic (in the next section we show the homeomorphism property of G).

3.2 Homeomorphism property of the mapping G

First, we give a definition of homeomorphism for planar maps.

Definition 1. *A mapping $G_\delta : X \to Y$ with $\delta > 0$ fixed, where $X \subset \mathbb{R}^2$, $Y \subset \mathbb{R}^2$ is said to be a homeomorphism, if (i) G_δ is invertible and (ii) both G_δ and G_δ^{-1} are continuous.*

Define the mapping $G_\delta = (G_{1,\delta}, G_{2,\delta}) : \mathbb{R}^2 \to \mathbb{R}^2$ defined for $1 > \delta > 0$ by

$$G_{1,\delta}(x,y) = x + ay\max\{1-|x|,\delta\} - bx|x|, \tag{3.3}$$
$$G_{2,\delta}(x,y) = ay\max\{1-|x|,\delta\} - bx|x|. \tag{3.4}$$

This map is equivalent to the original map (3.1)–(3.2) for x such that $\max\{1-|x|,\delta\} = \max\{1-|x|,0\}$, i.e. when $|x| < 1-\delta$. We can now formulate a homeomorphism theorem.

Theorem 2. *Let G_δ, $G_\delta = (G_{1,\delta}, G_{2,\delta}) : \mathbb{R}^2 \to \mathbb{R}^2$, $\delta > 0$ be the function defined by (3.3)–(3.4). Then G_δ is a homeomorphism.*

Proof. Definition 1 lists three requirements for G_δ to be a homeomorphism. One of the requirements is obviously fulfilled because G_δ is continuous. We show the existence of an inverse mapping to G_δ directly by finding its inverse. Denote by x' and y' the images of x and y in G_δ. First, we can observe that

$$x = x' - y'. \tag{3.5}$$

Since (3.5) is already the equation for x as a function of x', y' and (3.3) is linear in y, we can simply replace x by $x' - y'$ in (3.3) and derive an equation for y as a function of x', y'. Starting with

$$x' = (x'-y') + ay\max\{1-|x'-y'|,\delta\} - b(x'-y')|x'-y'|, \tag{3.6}$$

we obtain

$$y = \frac{y' + b(x'-y')|x'-y'|}{a\max\{1-|x'-y'|,\delta\}}. \tag{3.7}$$

The inverse mapping to a mapping G_δ then has the form

$$x = G^{-1}_{1,\delta}(x',y') = x' - y', \tag{3.8}$$
$$y = G^{-1}_{2,\delta}(x',y') = \frac{y' + b(x'-y')|x'-y'|}{a\max\{1-|x'-y'|,\delta\}}. \tag{3.9}$$

The last thing to prove is that $G^{-1}_\delta(x',y')$ is continuous. The function $G^{-1}_{1,\delta}(x',y')$ is linear, therefore it is continuous. Also the function $G^{-1}_{2,\delta}(x',y')$ is continuous because both the numerator and the denominator are continuous functions and the denominator is positive (greater than $a\delta$). □

We proved that G_δ is a planar homeomorphism. Since G is the same function as G_δ on the set $H = \{(x,y) \in \mathbb{R}^2 : |x| < 1-\delta\}$ also G is a homeomorphism from H to $G(H)$.

3.3 Basic properties of homeomorphism

In this section we present some characteristics of homeomorphisms with their proofs. We start with the definition of a disconnected metric space and Jordan Theorem to which we will refer later in the work.

Definition 2. *A metric space X is said to be disconnected if there exist open subsets A and B such that $A, B \neq \emptyset$, $A \cap B = \emptyset$, and $X = A \cup B$. A metric space is said to be connected if it is not disconnected.*

Theorem 3 (Jordan Theorem). *If $\mathbf{D} \subset \mathbb{R}^2$ is a topological circle, the set $E \setminus \mathbf{D}$ has exactly two different components where $\mathbf{D}$ is their common border. In addition, one of these two components is bounded and the other one is unbounded.*

Both the formulation and the proof of the Jordan Theorem can be found in the book of *Černý, I.* (1983) [6] and we omit it here.

Theorem 4. *Let $f : X \to Y$ be a homeomorphism, where $X \subset \mathbb{R}^2$ and $Y \subset \mathbb{R}^2$ are both compact. The following statements hold true:*

(1) $O \subset X$ *is open* $\Leftrightarrow f(O) \subset Y$ *is open;*

(2) $K \subset X$ *is compact* $\Leftrightarrow f(K) \subset Y$ *is compact;*

(3) $O \subset X$ *is open and connected* $\Leftrightarrow f(O) \subset Y$*; is open and connected;*

(4) $x \in X \setminus \partial X \Leftrightarrow f(x) \in Y \setminus \partial Y$*;*

(5) $x \in \partial X \Leftrightarrow f(x) \in \partial Y$*;*

Proof. We prove the five statements in the same order as they are stated.

(1) "⇒" Using the fact that f is continuous, we get that an inverse image of any open subset O of X is also open. We can write a set O as $O = f^{-1}(f(O))$. That means that $f(O)$ is an inverse image of O (which is open) in f. Therefore $f(O)$ is also open.

"⇐" The proof of this implication is analogous using the fact that f^{-1} is continuous.

(2) "⇒" Let $\{y_n\}$ be a sequence in the range of $f(X) = Y$. Then there are corresponding points $\{x_n\}$ in X with $y_n = f(x_n)$. Since X is compact we can find a subsequence of x_n that converges in X, $x_{n_k} \to x$. Since f is continuous one has $f(x_{n_k}) \to f(x)$ in Y. Hence $\{y_n\}$ has a convergent subsequence and Y is compact.

"⇐" The same argument can be used to prove this implication.

(3) "⇒" Assume that X is connected but Y is not. From the definition of a disconnected space there exist $A, B \neq \emptyset$, $A, B \subset Y$ open such that $A \cap B = \emptyset$ and $A \cup B = Y$. Because f is a homeomorphism, it is invertible with the inverse f^{-1}. Let us denote $U = f^{-1}(A)$, $V = f^{-1}(B)$. U and V are also open (according to (*1*) which we have already proven). In addition $U \cup V = X$ (because f^{-1} is one to one). The set X is connected and we have $U \cap V \neq \emptyset$. Let us denote $P = U \cap V$. Also $P \subset U$ implies $f(P) \subset f(U)$ and similarly $P \subset V$ implies $f(P) \subset f(V)$. Then we have $f(P) \in f(U) \cap f(V) \neq \emptyset$ which leads to contradiction.

"⇐" The proof of this implication can be carried out as in the previous case.

(4) "⇒" If $x \in X \setminus \partial X$ then there has to exist $O_x \subset X \setminus \partial X$ open such that $x \in O_x$. Then we can write

$$f(O_x) \subset f(X) = Y. \tag{3.10}$$

Since f is a homeomorphism and O_x is open $f(O_x)$ is also open. In addition we have $f(x) \in f(O_x)$ therefore $f(x) \in Y \setminus \partial Y$.

"⇐" The proof of this implication can be done similarly as it was in the previous one, with f^{-1} taken instead of f.

(5) This equivalence may be proven using (4). Since the image (and also the inverse image) of any point taken from an interior of X (or Y) lies also in an interior of the image of X (or an inverse image of Y) we can say the following. There does not exist a point $x \in X \setminus \partial X$ such that $f(x) \in \partial Y$ nor a point $y \in Y \setminus \partial Y$ such that $f^{-1}(y) \in \partial X$. Since the mapping f is one to one there exists a bijection between ∂X and ∂Y.

□

3.4 Boundeness of trajectories

In the section 1.1 of this paper, we also considered the simple model of a deviation of a foreign exchange rate from its equilibrium value (1.3). We did not investigated it further because of the dynamical properties of the model for $a > 1$. Numerical simulations indicate that trajectories of the simple model diverge to infinity when the parameter $a > 1$ is close to 1. On the other hand, a completely different dynamics occurs in the model proposed here. The observation is that trajectories are bounded when the initial values lie in a certain region around the origin and a, b are not very large.

To prove the boundeness of trajectories we try to find a symmetric area around the origin, which is mapped into itself and is a subset of $\{(x, y) : |x| < 1\}$ (where

G is a homeomorphism). The symmetry of the area around the origin is a natural requirement, because the system

$$\begin{aligned} x_{n+1} &= x_n + a y_n (1 - |x_n|)^+ - b x_n |x_n|, && (3.11)\\ y_{n+1} &= a y_n (1 - |x_n|)^+ - b x_n |x_n|, && (3.12) \end{aligned}$$

is also symmetric. Since we inspect only a subset of $\{(x, y) : |x| < 1\}$, we can use a simplification of (3.11), (3.12), denoted by $f = (f_1, f_2)$

$$\begin{aligned} x_{n+1} &= f_1(x_n, y_n) = x_n + a y_n (1 - |x_n|) - b x_n |x_n|, && (3.13)\\ y_{n+1} &= f_2(x_n, y_n) = a y_n (1 - |x_n|) - b x_n |x_n|. && (3.14) \end{aligned}$$

3.4.1 Lipschitz property of the mapping f

Let the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy (3.13)–(3.14). Consider a metric ρ in $\mathbb{R}^2$ defined as

$$\rho(X, U) = |x - u| + |y - v|, \tag{3.15}$$

where $X = (x, y) \in \mathbb{R}^2$ and $U = (u, v) \in \mathbb{R}^2$. In this subsection, we present a proof of Lipschitz property of the mapping f with respect to the metric ρ.

Theorem 5. *The mapping $f = (f_1, f_2)$, given by (3.13)–(3.14), has a Lipschitz property on $H = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}$, i.e. there exists $k \in \mathbb{R}^+$ satisfying:*

If $\varepsilon > 0$ and $X, U \in H$ such that $\rho(X, U) < \varepsilon$, then $\rho(f(X), f(U)) < k\varepsilon$,

with the Lipschitz constant $k = \max\{1 + 2a + 4b, 2a\}$.

Proof. Let us denote $\bar{X} = f(X)$ and $\bar{U} = f(U)$, where $X = (x_1, x_2)$ and $U = (u_1, u_2)$. In the first step we compute the upper bounds for the matrix elements of the gradient of f:

$$F_{11} = \frac{\partial f_1(x_1, x_2)}{\partial x_1}, \quad F_{12} = \frac{\partial f_1(x_1, x_2)}{\partial x_2}, \quad F_{21} = \frac{\partial f_2(x_1, x_2)}{\partial x_1}, \quad F_{22} = \frac{\partial f_2(x_1, x_2)}{\partial x_2}.$$

Then

$$|F_{11}| = |1 - a x_2 \operatorname{sgn}(x_1) - 2b|x_1|| \le 1 + a + 2b\,, \tag{3.16}$$

$$|F_{12}| = |a(1 - |x_1|)| \le a\,, \tag{3.17}$$

$$|F_{21}| = |-a x_2 \operatorname{sgn}(x_1) - 2b|x_1|| \le a + 2b\,, \tag{3.18}$$

$$|F_{22}| = |a(1 - |x_1|)| \le a\,. \tag{3.19}$$

Using the mean value theorem, $\rho(\bar{X}, \bar{U})$ can be computed directly:

$$\begin{aligned}
\rho(\bar{X}, \bar{U}) &= |f_1(x_1, x_2) - f_1(u_1, u_2)| + |f_2(x_1, x_2) - f_2(u_1, u_2)| \\
&= |f_1(x_1, x_2) - f_1(x_1, u_2) + f_1(x_1, u_2) - f_1(u_1, u_2)| \\
&\quad + |f_2(x_1, x_2) - f_2(x_1, u_2) + f_2(x_1, u_2) - f_2(u_1, u_2)| \\
&\leq |f_1(x_1, x_2) - f_1(x_1, u_2)| + |f_1(x_1, u_2) - f_1(u_1, u_2)| \\
&\quad + |f_2(x_1, x_2) - f_2(x_1, u_2)| + |f_2(x_1, u_2) - f_2(u_1, u_2)| \\
&= \left| \frac{\partial f_1(x,y)}{\partial y}|_{(x_1,c_1)} \right| |x_2 - u_2| + \left| \frac{\partial f_1(x,y)}{\partial x}|_{(c_2,u_2)} \right| |x_1 - u_1| \\
&\quad + \left| \frac{\partial f_2(x,y)}{\partial y}|_{(x_1,c_3)} \right| |x_2 - u_2| + \left| \frac{\partial f_2(x,y)}{\partial x}|_{(c_4,u_2)} \right| |x_1 - u_1|, \quad (3.20)
\end{aligned}$$

where $c_1, c_3 \in [x_2, u_2]$ and $c_2, c_4 \in [x_1, u_1]$. By substituting inequalities (3.16)–(3.19) into (3.20) we obtain

$$\rho(\bar{X}, \bar{U}) \leq 2a|x_2 - u_2| + (1 + 2a + 4b)|x_1 - u_1|. \quad (3.21)$$

Thus

$$\rho(\bar{X}, \bar{U}) \leq \max\{1 + 2a + 4b, 2a\}\rho(X, U). \quad (3.22)$$

Therefore, the function f has a Lipschitz property with a Lipschitz constant $k = \max\{1 + 2a + 4b, 2a\}$. □

3.4.2 The computer assisted proof of boundeness of trajectories

In our effort to find a region, mapped into itself we restricted ourselves to a class of regions with polygonal boundaries, symmetric around the origin. Naturally, the final shape depends on values of parameters a and b. We present a computer assisted proof of boundeness for special values of a, b.

Theorem 6 (Boundeness). *If $a = 1.1$ and $b = 1.3$, then there exists a region $P \subset \{(x,y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}$, containing the origin such that*

$$f(P) \subset P,$$

where f is the mapping, given by (3.13) and (3.14).

Lemma 1. *Let us denote by P the set of all $(x,y) \in \mathbb{R}^2$ bounded by the symmetric octagon defined by vertices $O_i = (O_i^1, O_i^2)$ where*

$$\begin{array}{ll} O_0 = (0.64, 0.43), & O_4 = (-0.64, -0.43), \\ O_1 = (0.55, 0.53), & O_5 = (-0.55, -0.53), \\ O_2 = (-0.29, 0.39), & O_6 = (0.29, -0.39), \\ O_3 = (-0.5, 0.1), & O_7 = (0.5, -0.1). \end{array}$$

If $a = 1.1$ *and* $b = 1.3$*, then* $f(P) \subset P$ *and* $f(O) \cap O = \emptyset$*, where* $O = \partial P$.

Before we start the proof of lemma, we present a figure of the octagon O and its image. The closed curve around the origin represents the numerically computed invariant set.

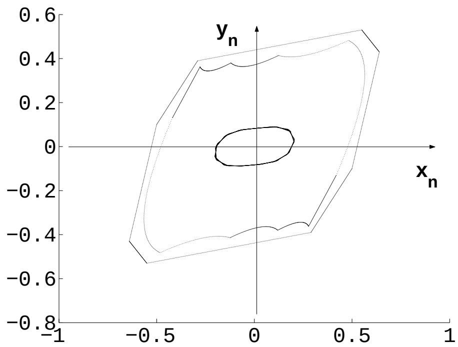


Figure 9: The Octagon and its Image

Computer Assisted Proof. The proof is carried out in three steps. The first and the second one consist of estimating errors produced during the computation and the third one summarizes the result. To obtain numerical results, a programming language MATLAB^{TM}, with a relative computing precision 10^{-15} of the operation of addition (includes also substraction and multiplication), is used. Constant b is fixed at 1.3 and a is fixed at 1.1.

The octagon O is represented by a discretization of its lateral sides O_n, where n is the number of equidistant points, which we use in place of every side of O. In our numerical implementation we used $n = 1000$.

In the first part, we compute the error, caused by the discretization. We denote by D the minimal distance

$$D = \min\{\rho(X, U) : X \in O, U \in f(O)\}. \quad (3.23)$$

The algorithm computes an estimate of D (based on a discretization), denoted by D_n, where

$$D_n = \min\{\rho(X_n, U_n) : X_n \in O_n, U_n \in f(O_n)\}. \quad (3.24)$$

Naturally, the inequality $D < D_n$ holds true. The Lipschitz property of f enables us to compute the error arising from the discretization of O. Let $X = (x, y) \in O$

and $U=(u,v)\in f(O)$ be arguments of the minimum of D and $X_n=(x_n,y_n)\in O_n$, $U_n=(u_n,v_n)\in f(O_n)$ be arguments of the minimum od D_n. Then, there has to exist $\bar{X}_n=(\bar{x}_n,\bar{y}_n)\in O_n$ and $\bar{U}_n=(\bar{u}_n,\bar{v}_n)\in f(O_n)$ such that $\rho(\bar{X}_n,X)<1/n$ (because the discretization of O is equidistant) and $\rho(U,\bar{U}_n)<k/n$, where $k=\max\{1+2a+4b,2a\}$ (the Lipschitz constant of f). Then

$$\begin{aligned} D_n &= \rho(X_n,U_n)\le\rho(\bar{X}_n,\bar{U}_n)=|\bar{x}_n-\bar{u}_n|+|\bar{y}_n-\bar{v}_n|,\\ &= |\bar{x}_n-x+x-u+u-\bar{u}_n|+|\bar{y}_n-y+y-v+v-\bar{v}_n|\\ &\le |\bar{x}_n-x|+|x-u|+|u-\bar{u}_n|+|\bar{y}_n-y|+|y-v|+|v-\bar{v}_n|\\ &= \rho(\bar{X}_n,X)+\rho(X,U)+\rho(U,\bar{U}_n)\\ &\le \frac{1}{n}+D+\frac{k}{n}\le D+\frac{1+k}{n}, \end{aligned} \tag{3.25}$$

and therefore, since $D_n\ge D$

$$0<D_n-D\le\frac{1+k}{n}. \tag{3.26}$$

In the second step, a precise absolute error of D_n, originating from the roundoff error of the computer is computed. Let us denote by $\hat{X}$ the computer estimate of X. Since D_n is the minimum of a finite set of numbers, then

$$|\hat{D}_n-D_n|\le\varepsilon, \tag{3.27}$$

with

$$\varepsilon=\max\{\rho(X_n,Y_n)-\hat{\rho}(\hat{X}_n,\hat{Y}_n)\}, \tag{3.28}$$

where $\hat{\rho}(\hat{X}_n,\hat{Y}_n)$ is the estimate of the distance $\rho(\hat{X}_n,\hat{Y}_n)$.

In order to compute the upper bound ε for $|\hat{D}_n-D_n|$ (which is in fact the absolute error of D_n), we inspect the whole algorithm computing D_n. We start by choosing an arbitrary point $X=(x_1,x_2)\in O$ such that

$$x_1 = tO_i^1+(1-t)O_j^1, \tag{3.29}$$

$$x_2 = tO_i^2+(1-t)O_j^2, \tag{3.30}$$

where $j=(i+1)\mod 8$ and $t\in[0,1]$. From the fact that the relative roundoff error is 10^{-15} it follows that

$$\left|\frac{\hat{t}-t}{t}\right|\le 10^{-15} \tag{3.31}$$

and consequently

$$\begin{aligned}
|\hat{t} - t| &\leq 10^{-15}|t| \leq 10^{-15}, && (3.32)\\
|\hat{x}_1 - x_1| &= |\hat{t}O_i^1 + (1-\hat{t})O_j^1 - tO_i^1 - (1-t)O_j^1| \\
&= |\hat{t} - t|.|O_i^1 - O_j^1| \leq |\hat{t} - t| \leq 10^{-15}, && (3.33)\\
|\hat{x}_2 - x_2| &= |\hat{t}O_i^2 + (1-\hat{t})O_j^2 - tO_i^2 - (1-t)O_j^2| \\
&= |\hat{t} - t|.|O_i^2 - O_j^2| \leq |\hat{t} - t| \leq 10^{-15}, && (3.34)\\
\rho(X, \hat{X}) &= |\hat{x}_1 - x_1| + |\hat{x}_2 - x_2| \leq 2.10^{-15}. && (3.35)
\end{aligned}$$

Since $f_1(x, y)$ and $f_2(x, y)$ satisfy (3.13), (3.14) and f has a Lipschitz property with Lipschitz constant $k = \max\{1 + 2a + 4b, 2a\}$, the relation

$$|\hat{f}_1(X) - f_1(X)| + |\hat{f}_2(X) - f_2(X)| \leq k\rho(X, \hat{X}) \leq 2k10^{-15} \tag{3.36}$$

holds true.

In this step of the algorithm, we already know the estimate of $f(X)$ and its distance from the octagon O has to be computed. Sides of O are lines, in the form

$$\left(O_i^1 - O_j^1\right) y - \left(O_i^2 - O_j^2\right) x - (O_i^1 O_j^2 + O_i^2 O_j^1) = 0. \tag{3.37}$$

We denote by $d_{x,i}$ the minimal distance of $f(X)$ from the line (3.37), by $\hat{d}_{x,i}$ the estimate of the minimal distance of $\hat{f}(X)$ from the same line and

$$d_X = \min_{i \in \{0,1,\ldots 7\}} d_{x,i}, \tag{3.38}$$

and similarly

$$\hat{d_X} = \min_{i \in \{0,1,\ldots 7\}} \hat{d}_{x,i}. \tag{3.39}$$

Then

$$\varepsilon = \max\{|\hat{d}_X - d_X| : X \in O\} \tag{3.40}$$

Now our task consist of the evaluation of ε. The formula for d_X (the distance of $f(X)$ from a line given by (3.37)) can be easily derived for ρ, defined by (3.15) as

$$d_X = \min_{i \in \{0,1,\ldots 7\}} \left\{ \frac{1}{|O_i^2 - O_j^2|}, \frac{1}{|O_i^1 - O_j^1|} \right\} |M(f(X))| \tag{3.41}$$

where

$$M(f(X)) = \left(O_i^1 - O_j^1\right) f_2(X) - \left(O_i^2 - O_j^2\right) f_1(X) - (O_i^1 O_j^2 + O_i^2 O_j^1). \tag{3.42}$$

Then

$$|\hat{d}_X - d_X| \leq \min\left\{\frac{1}{|O_i^2 - O_j^2|}, \frac{1}{|O_i^1 - O_j^1|}\right\} |\hat{M}(f(X)) - M(f(X))| \quad (3.43)$$

$$= \min\left\{\frac{1}{|O_i^2 - O_j^2|}, \frac{1}{|O_i^1 - O_j^1|}\right\} |M(\hat{f}(X)) - M(f(X))| \quad (3.44)$$

Finally, it is necessary to compute $|\hat{M}(f(X)) - M(f(X))|$.

$$\begin{aligned} |\hat{M}(f(X)) - M(f(X))| &= |(O_i^1 - O_j^1)\hat{f}_2(X) - (O_i^2 - O_j^2)\hat{f}_1(X) - (O_i^1 O_j^2 + O_i^2 O_j^1) \\ &\quad -(O_i^1 - O_j^1) f_2(X) + (O_i^2 - O_j^2) f_1(X) + (O_i^1 O_j^2 + O_i^2 O_j^1)| \\ &\leq |O_i^1 - O_j^1||\hat{f}_2(X) - f_2(X)| + |O_i^2 - O_j^2||\hat{f}_1(X) - f_1(X)| \\ &\leq |\hat{f}_2(X) - f_2(X)| + |\hat{f}_1(X) - f_1(X)| \\ &\leq 2k.10^{-15}\,. \quad (3.45) \end{aligned}$$

Using the coordinates of vertices of O, we obtain

$$|O_i^1 - O_j^1| \geq |0.64 - 0.55| \geq 2.10^{-1}\,, \quad (3.46)$$

$$|O_i^2 - O_j^2| \geq |0.53 - 0.43| \geq 10^{-1}\,. \quad (3.47)$$

The value of ε can be computed from (3.40) using (3.44) and (3.45) this way

$$\begin{aligned} \varepsilon &= \max\{|d_X - \hat{d}_X| : X \in O\} \\ &\leq \max\left\{\left[\min\{10, 5\}|\hat{M}(f(X)) - M(f(X))|\right] : X \in O\right\} \\ &\leq 5.\max\{2k.10^{-15}\} \\ &\leq k.10^{-14}\,. \quad (3.48) \end{aligned}$$

Finally, we can summarize the two errors, which arise from the discretization on one side and from the computer rounding on the other side. The crucial inequalities we obtained are (3.26) and (3.48). The final error equals

$$|\hat{D}_n - D| \leq |\hat{D}_n - D_n| + |D_n - D| = k.10^{-14} + \frac{1+k}{n}\,. \quad (3.49)$$

For $a = 1.1$ and $b = 1.3$ an easy computation gives us that $k < 9$, therefore for $n = 1000$ we obtain

$$|\hat{D}_n - D| < 9.10^{-14} + \frac{9}{n} \leq 10^{-2}\,. \quad (3.50)$$

Now we compare the result on $\hat{D}_n$, computed by the Matlab program, with the maximal error of $|\hat{D}_n - D|$ for $a = 1.1$, $b = 1.3$. On one side, we estimated that

$$|\hat{D}_n - D| \leq 10^{-2}. \quad (3.51)$$

On the other side,

$$\hat{D}_n = 2.9833.10^{-2}, \tag{3.52}$$

according to our program. It can be observed, that the result (3.52) is greater then the maximal possible absolute error (3.51). From this fact follows that the actual D (the distance between the octagon O and his image) is positive, e. g. the set P, with octagonal border, is mapped into itself (from Properties of Homeomorphism (4)) and $\partial P \cap \partial f(P) = \emptyset$ (from $D > 0$). Since $O = \partial P$ and $f(O) = \partial f(P)$, also $O \cap f(O) = \emptyset$.

□

Remark 1. It is already proved that for special values of a and b trajectories are bounded, but we can say more. One can observe that the function f is linear in parameter a. Moreover, $\hat{D}_n(a)$ is partly linear in a. The figure Fig. 10 of the function $\hat{D}_n(a)$ indicates, that for all parameters satisfying $1 < a < 1.19$ $b = 1.3$ $\hat{D}_n(a)$ is greater than the error $|\hat{D}_n - D|$, what implies that $f(P) \subset P$. The horizontal line in the graph represents the maximal error of $|\hat{D}_n - D|$ and the partly linear function represents $\hat{D}_n(a)$.

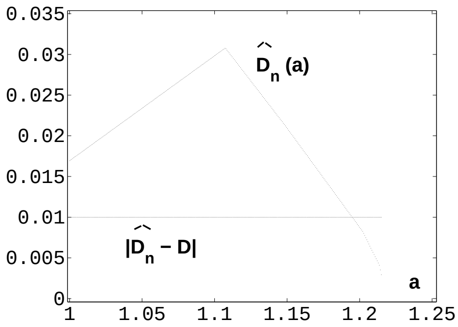


Figure 10: The comparison of $\hat{D}_n(a)$ and $|\hat{D}_n - D|$

Remark 2. The Boundeness Theorem is already proved by the Lemma 1.

Remark 3. We can formulate a trivial consequence of the Boundeness Theorem, which is: for $(x, y) \in P$ the trajectories, given by $(x_n, y_n) = f^n(x, y)$ lies in P.

3.5 Existence of a monotone sequence of closed curves

In the region where $|x_n| < 1$, the system given by (3.1), (3.2) is equivalent to

$$x_{n+1} = f_1(x_n, y_n) = x_n + a y_n(1 - |x_n|) - b x_n |x_n|, \tag{3.53}$$
$$y_{n+1} = f_2(x_n, y_n) = a y_n(1 - |x_n|) - b x_n |x_n|. \tag{3.54}$$

We transform (3.53)–(3.54) into

$$\begin{aligned} u_{n+1} &= H_1(u_n, v_n) = au_n - au_n|au_n + v_n| - \frac{b}{a-1}(au_n + v_n)|au_n + v_n|, \\ v_{n+1} &= H_2(u_n, v_n) = v_n + au_n|au_n + v_n| + \frac{b}{a-1}(au_n + v_n)|au_n + v_n|, \end{aligned} \tag{3.55}$$

by

$$x_n = au_n + v_n, \qquad y_n = u_n + v_n\,. \tag{3.56}$$

In our work we use the notation $H = (H_1, H_2)^T : \mathbb{R}^2 \to \mathbb{R}^2$. Note that since the function G is a homeomorphism in a neighborhood of the origin (what we proved in Section 3.2) the function obtained from G by the regular linear change of coordinates (3.56), will be a homeomorphism in a neighborhood of the origin. Therefore we can work with the diagonal system near the origin and then apply the result to the function G.

Theorem 7. *Let $a > 1$, $b > 0$. There exists some $\varepsilon > 0$ such that the following holds true: If $M(\varepsilon) = \{(u,v) \in \mathbb{R}^2 : \frac{u^2}{a^2} + v^2 = \varepsilon^2\}$ and $N(\varepsilon) = \{(u,v) \in \mathbb{R}^2 : \frac{u^2}{a^2} + v^2 \le \varepsilon^2\}$, then for every $(u,v) \in M(\varepsilon)$:*

$$\frac{H_1(u,v)^2}{a^2} + H_2(u,v)^2 > \varepsilon^2.$$

In addition, $N(\varepsilon) \subset H(N(\varepsilon))$.

Proof. Let (u_n, v_n) satisfies a difference system (3.55) and denote

$$R(u,v) = \frac{u^2}{a^2} + v^2. \tag{3.57}$$

To capture crucial features of our problem, let us transform (u,v) into the coordinates (ε, θ) by

$$u = a\varepsilon\cos\theta, \qquad v = \varepsilon\sin\theta\,.$$

Substituting these coordinates into the system (3.55) we obtain

$$\begin{aligned} H_1(u(\varepsilon,\theta), v(\varepsilon,\theta)) &= a^2\varepsilon\cos\theta - a^2\varepsilon^2\cos\theta|a^2\cos\theta + \sin\theta| \\ &\quad - \frac{b}{a-1}\varepsilon^2(a^2\cos\theta + \sin\theta)|a^2\cos\theta + \sin\theta|, \\ H_2(u(\varepsilon,\theta), v(\varepsilon,\theta)) &= \varepsilon\sin\theta + a^2\varepsilon^2\cos\theta|a^2\cos\theta + \sin\theta| \\ &\quad + \frac{b}{a-1}\varepsilon^2(a^2\cos\theta + \sin\theta)|a^2\cos\theta + \sin\theta|. \end{aligned} \tag{3.58}$$

In terms of ε and θ, the first statement of the theorem transforms into the following form: There exists some $\varepsilon \in \mathbb{R}^+$ such that if $R(u,v) = \varepsilon^2$ then $R(H(u,v)) > \varepsilon^2$ for every $\theta \in [0, 2\pi]$.

Now we determine the value of $R(H(u,v))$ as a function of ε, θ:

$$\begin{aligned} R(H(u,v)) &= \frac{H_1(u(\varepsilon,\theta), v(\varepsilon,\theta))^2}{a^2} + H_2(u(\varepsilon,\theta), v(\varepsilon,\theta))^2 \\ &= a^2\varepsilon^2\cos^2\theta + \varepsilon^2\sin^2\theta + h(\varepsilon,\theta) \\ &= \varepsilon^2 + (a^2-1)\varepsilon^2\cos^2\theta + h(\varepsilon,\theta)\,, \end{aligned} \tag{3.59}$$

where $h(\varepsilon,\theta) = \varepsilon^3 K(\theta) + \varepsilon^4 L(\theta)$, and

$$K(\theta) = 2a^2C(S-C)|a^2C+S| + \frac{2b}{a-1}(S-C)(a^2C+S)|a^2C+S|, \tag{3.60}$$

$$L(\theta) = \left(1+\frac{1}{a^2}\right)\left[a^4C^2(a^2C+S)^2 + \left(\frac{b}{a-1}\right)^2(a^2C+S)^4\right], \tag{3.61}$$

with $C = \cos\theta$, $S = \sin\theta$. When investigating whether $R(H(u,v))$ is greater then ε^2, the sum of the second and the third term of (3.59) plays an important role. Note that the second term of (3.59) is nonegative for every (ε,θ). In addition, it is positive for $\cos^2\theta \neq 0$. In the case when $\cos^2\theta = 0$, the term $h(\varepsilon,\theta)$ and particularly $K(\theta)$ plays a crucial role. The following equation and inequalities hold true:

$$\lim_{\varepsilon\to 0}\frac{h(\varepsilon,\theta)}{(a^2-1)\varepsilon^2\cos^2\theta} = 0 \qquad \text{for } \theta \neq \pm\tfrac{\pi}{2}, \tag{3.62}$$

$$K\left(+\frac{\pi}{2}\right) = K\left(-\frac{\pi}{2}\right) > 0, \tag{3.63}$$

$$L\left(+\frac{\pi}{2}\right) = L\left(-\frac{\pi}{2}\right) > 0. \tag{3.64}$$

The validity of (3.62) is implied by the fact that the second term of (3.59) has lower order of ε than $h(\varepsilon,\theta)$ and K and L are bounded. To prove it, observe that

$$\begin{aligned} |K| &= \left|2a^2C(S-C)|a^2C+S| + \frac{2b}{a-1}(S-C)(a^2C+S)|a^2C+S|\right| \\ &\leq 2a^2|C(S-C)||a^2C+S| + \frac{2b}{a-1}|S-C|(a^2C+S)^2 \\ &\leq 4a^2(a^2+1) + \frac{4b}{a-1}(a^2+1)^2 = M_K, \end{aligned}$$

with $M_k > 0$ and similarly

$$\begin{aligned} |L| &= \left(1+\frac{1}{a^2}\right)\left|a^4C^2(a^2C+S)^2 + \left(\frac{b}{a-1}\right)^2(a^2C+S)^4\right|, \\ &\leq a^2(a^2+1)^3 + \frac{1}{a^2}\left(\frac{b}{a-1}\right)^2(a^2+1)^5 = M_L, \end{aligned}$$

where $M_L > 0$. If $\theta \neq \pm\frac{\pi}{2}$ then the absolute value of the limit in (3.62) is

$$\begin{aligned}\left|\lim_{\varepsilon\to 0}\frac{h(\varepsilon,\theta)}{(a^2-1)\varepsilon^2\cos^2\theta}\right| &= \left|\lim_{\varepsilon\to 0}\frac{\varepsilon^3K(\theta)+\varepsilon^4L(\theta)}{(a^2-1)\varepsilon^2\cos^2\theta}\right| \\ &\leq \lim_{\varepsilon\to 0}\frac{\varepsilon|K(\theta)|}{(a^2-1)\cos^2\theta}+\lim_{\varepsilon\to 0}\frac{\varepsilon^2|L(\theta)|}{(a^2-1)\cos^2\theta} \\ &\leq \frac{M_K}{(a^2-1)\cos^2\theta}\lim_{\varepsilon\to 0}\varepsilon+\frac{M_L}{(a^2-1)\cos^2\theta}\lim_{\varepsilon\to 0}\varepsilon^2 \\ &= 0.\end{aligned}$$

Therefore the limit itself is equal to 0. The validity of inequalities (3.63) and (3.64) may be verified by computing them directly from the system (3.58), simplified for $\theta = \pm\frac{\pi}{2}$, i.e.

$$u = \mp\frac{b}{a-1}\varepsilon^2, \qquad v = \pm\varepsilon \pm \frac{b}{a-1}\varepsilon^2. \tag{3.65}$$

It is easy to see that

$$K\left(+\frac{\pi}{2}\right) = K\left(-\frac{\pi}{2}\right) = \frac{2b}{a-1} > 0, \tag{3.66}$$

$$L\left(+\frac{\pi}{2}\right) = L\left(-\frac{\pi}{2}\right) = \left(1+\frac{1}{a^2}\right)\left(\frac{b}{a-1}\right)^2 > 0. \tag{3.67}$$

Therefore there has to exist some $\delta^2 > 0$ such that for θ from the open neighborhoods $I_\delta = \{\theta \in [0,2\pi] : \cos^2\theta < \delta^2\}$ of $\pm\frac{\pi}{2}$ the inequalities $K(\theta) > \frac{b}{a-1} > 0$ and $L(\theta) > \frac{1}{2}(1+\frac{1}{a^2})(\frac{b}{a-1})^2 > 0$ hold true. For $\theta \in I_\delta$ and $\varepsilon > 0$ one has

$$\begin{aligned}R(H(u,v)) &= \varepsilon^2+(a^2-1)\varepsilon^2\cos\theta^2+\varepsilon^3K(\theta)+\varepsilon^4L(\theta) \\ &\geq \varepsilon^2+\varepsilon^3K(\theta)+\varepsilon^4L(\theta) \\ &> \varepsilon^2+\varepsilon^3K(\theta) > \varepsilon^2.\end{aligned} \tag{3.68}$$

Now we inspect the remaining interval for θ, which is $[0,2\pi]\setminus I_\delta$ or equivalently $\{\theta \in [0,2\pi] : \cos^2\theta \geq \delta^2\}$. From (3.62) we know, that there exists sufficiently small $\varepsilon_0 > 0$, such that

$$\left|\frac{h(\varepsilon_0,\theta)}{(a^2-1)\varepsilon_0^2\cos^2\theta}\right| < \frac{1}{2}, \tag{3.69}$$

for every $\varepsilon < \varepsilon_0$. This condition may be also written as follows

$$|h(\varepsilon,\theta)| < \frac{1}{2}(a^2-1)\varepsilon^2\cos^2\theta. \tag{3.70}$$

If $h(\varepsilon, \theta)$ is nonegative, then

$$R(H(u,v)) = \varepsilon^2 + (a^2 - 1)\varepsilon^2 \cos^2\theta + h(\varepsilon,\theta) > \varepsilon^2. \tag{3.71}$$

In the other case $h(\varepsilon, \theta)$ satisfies

$$h(\varepsilon,\theta) > -\frac{1}{2}(a^2 - 1)\varepsilon^2 \cos^2\theta, \tag{3.72}$$

and consequently

$$\begin{aligned} R(H(u,v)) &= \varepsilon^2 + (a^2 - 1)\varepsilon^2 \cos^2\theta + h(\varepsilon,\theta) \\ &> \varepsilon^2 + (a^2 - 1)\varepsilon^2 \cos^2\theta - \frac{1}{2}(a^2 - 1)\varepsilon^2 \cos^2\theta \\ &= \varepsilon^2 + \frac{1}{2}(a^2 - 1)\varepsilon^2 \cos^2\theta > \varepsilon^2. \end{aligned} \tag{3.73}$$

The inequalities (3.68), (3.71) and (3.73) complete the first part of the proof.

In the second part we prove that $N(\varepsilon) \subset H(N(\varepsilon))$. Applying the first part of the theorem we obtain a set $N(\varepsilon)$ (we can take $\varepsilon < a$) such that $\partial H(N(\varepsilon)) \cap N(\varepsilon) = \emptyset$. Since $M(\varepsilon)$ is an ellipse, it is homeomorphic to a circle. Therefore, from the Jordan Theorem, it partitions $\mathbb{R}^2 \setminus M(\varepsilon)$ into two different components with their common border $M(\varepsilon)$. $N(\varepsilon)$ is the component, which contains the origin and hence, it is bounded. Similarly, $\partial H(N(\varepsilon))$ contains the origin and is bounded. Since H is a homeomorphism on $N(\varepsilon)$ (because $N(\varepsilon) \subset \{(x,y) \in \mathbb{R}^2 : |x| < 1\}$) and $N(\varepsilon)$ is compact, bounded and connected, also $H(N(\varepsilon))$ is compact, bounded and connected. The condition $\partial H(N(\varepsilon)) \cap N(\varepsilon) = \emptyset$ gives us the result that $N(\varepsilon) \subset H(N(\varepsilon))$. □

Theorem 8 (Limit Sequence). *For G defined by (3.1)–(3.2) there exists a sequence $K_i \subset \mathbb{R}^2$, where $K_{i+1} = G(K_i)$, such that for every $i \in \mathbb{N}$, the set K_i is connected and compact and satisfies*

(1) K_i is homeomorphic to a circle,

(2) $\partial K_i \cap \partial K_{i+1} = \emptyset$,

(3) $K_i \subset K_{i+1}$,

as long as $K_i \subset \{(x,y) \in \mathbb{R}^2 : |x| < 1\}$

Proof. We prove this theorem using the induction argument. Since we assume $K_i \subset \{(x,y) \in \mathbb{R}^2 : |x| < 1\}$, we consider mapping f instead of G (on this set they are equivalent). In the first step of the induction we find the initial set ($= K_0$) with a border homeomorphic to a circle, which is both compact and connected and satisfies conditions $K_0 \subset f(K_0)$ and $\partial K_0 \cap \partial f(K_0) = \emptyset$. We apply the result of Theorem 7 about the mapping H, obtained from f by the linear change of coordinates (3.56). Let us take the set $N'(\varepsilon)$ defined as

$$N'(\varepsilon) = \{(au + v, u + v) \in \mathbf{R^2} : (u, v) \in N(\varepsilon)\},$$

where $N(\varepsilon)$ is the set from Theorem 7 and $\varepsilon > 0$. The Theorem 7 states that there exists some $\varepsilon' > 0$ such that for every $(u, v) \in \partial N(\varepsilon')$

$$\frac{H_1(u, v)^2}{a^2} + H_2(u, v)^2 > (\varepsilon')^2, \tag{3.74}$$

what implies the fact that $\partial N(\varepsilon') \cap \partial H(N(\varepsilon')) = \emptyset$ and therefore also $\partial N'(\varepsilon') \cap \partial H(N'(\varepsilon')) = \emptyset$. The Theorem 7 also states that $N(\varepsilon') \subset H(N(\varepsilon'))$, what implies $N'(\varepsilon') \subset f(N'(\varepsilon'))$. If K_0 is set as $K_0 = N'(\varepsilon')$, it fulfills all *(1), (2) and (3)*.

Now if $K_n \subset \{(x, y) \in \mathbb{R}^2 : |x| < 1\}$ (f is a homeomorphism on this set) and K_{n-1} satisfies (1), (2), (3) of Theorem 8 (induction assumption) we prove that then also K_n satisfies them. Using properties of homeomorphism (2) and (3) of Theorem 4 we obtain that the image of K_n is connected and compact. In addition, properties (4) and (5) of Theorem 4 imply that F (or F^{-1}) maps the interior of K_n (K_{n+1}) into the interior of K_{n+1} (K_n) and the boundary of K_n (K_{n+1}) into the boundary of K_{n+1}, (K_n). Since F is a homeomorphism on K_n, ∂K_n is homeomorphic to a circle and item (*1*) is proven. We have to note here that the origin is contained in every K_n because it is a fixed point of f.

Now we prove (*2*) in the following form: If $K_n \subset \{(x, y) \in \mathbb{R}^2 : |x| < 1\}$ (what means that f is a homeomorphism) and the induction assumption holds true then

$$\partial K_n \cap \partial K_{n+1} = \emptyset. \tag{3.75}$$

To prove this let us suppose the contrary, i.e., there exists some $X \in \partial K_n \cap \partial K_{n+1}$. Since F is a homeomorphism, there is exactly one $Y \in K_{n-1}$ such that $f(Y) = X$ ($X \in K_n$). Also there exists exactly one $Z \in K_{n-1}$ satisfying $f^2(Z) = X$ (because X was in K_{n+1}). The following holds true

$$X = f^2(Z) = f(Y) \qquad \Leftrightarrow \qquad f(f(Z)) = f(Y). \tag{3.76}$$

We also know that f is one to one, therefore

$$f(Z) = Y, \tag{3.77}$$

what means that $Y \in K_n$. On the other hand K_{n-1} satisfies the induction assumption $\partial K_{n-1} \cap \partial K_n = \emptyset$. But as we proved both $Y \in \partial K_{n-1}$ and $Y \in \partial K_n$ hold true, therefore also $Y \in \partial K_{n-1} \cap \partial K_n \neq \emptyset$. Thus we arrive at a contradiction and (3.75) holds true.

Now we know that $\partial K_n \cap \partial K_{n+1} = \emptyset$. We also know that the set ∂K_{n-1} is homeomorphic to a circle. We prove that (3) is also true. By applying the Jordan theorem 3 to the set $\partial K_n \subset \mathbb{R}^2$ we obtain that ∂K_n partitions the space $\mathbb{R}^2 \setminus \partial K_n$ into two parts, one of which is bounded and the other one is unbounded (the set K_n is the bounded one). From (3.75) it follows that there are only two possibilities of a relation between ∂K_n and ∂K_{n+1}. These possibilities are

$$\partial K_{n+1} \subset K_{n+1} \subset K_n \setminus \partial K_n, \tag{3.78}$$

or

$$\partial K_n \subset K_n \subset K_{n+1} \setminus \partial K_{n+1}. \tag{3.79}$$

As we will show only (3.79) is possible because (3.78) leads to a contradiction. To prove this let us suppose that $K_{n+1} \subset K_n \setminus \partial K_n$ and the induction assumption $K_{n-1} \subset K_n \setminus \partial K_n$ holds true. To every $y \in K_{n+1}$ there exists exactly one $x \in K_{n-1} \setminus \partial K_{n-1}$ such that $f(x) = y$. This is because f is one to one and y lies in the image of K_{n-1}, which is K_n, without its border (if there was also some point in the border of K_{n-1}, it would contradict the assumption $\partial K_{n+1} \cap \partial K_n = \emptyset$). We obtain the following

$$K_n \subset K_{n-1} \setminus \partial K_{n-1} \subsetneq K_{n-1} \subset K_n \setminus \partial K_n \subsetneq K_n, \tag{3.80}$$

what is a contradiction and completes the proof. □

Taking together the boundeness of trajectories with Theorem 8, we obtain that for $a = 1.1$ and $b = 1.3$ trajectories of points, lying in the area P (see Lemma 1) converge to some bounded set in $\mathbb{R}^2$. Moreover, this set is a limit (understood as boundary) of closed curves, homeomorphic to a circle.

3.6 Properties of an invariant set

Consider a planar map $G : \mathbb{R}^2 \to \mathbb{R}^2$, satisfying (3.1)–(3.2). We now define the *positive orbit* γ^+ of a point $x_0 \in \mathbb{R}^2$ as a sequence of images of x_0 under the succesive compositions of G:

$$\gamma^+(x_0) = \{x_0, G(x_0), \ldots, G^n(x_0), \ldots\}. \tag{3.81}$$

If G is invertible, we use the notation G^{-n} to denote the n-fold composition of G^{-1} with itself. Define the *negative orbit* γ^- of x_0 to be

$$\gamma^-(x_0) = \{x_0, G^{-1}(x_0), \ldots, G^{-n}(x_0), \ldots\}. \tag{3.82}$$

Definition 3. *A point y is called an ω-limit point of the positive orbit $\gamma^+(x_0)$ of x_0 if there is a sequence of positive integers n_i such that $n_i \to +\infty$ and $G^{n_i}(x_o) \to y$ as $i \to +\infty$. The ω-limit set $\omega(x_0)$ of $\gamma^+(x_0)$ is the set of all ω-limit points. In the case G is invertible, the α-limit set of $\gamma^-(x_0)$ is defined similarly by taking n_i to be negative integers.*

Note that the set

$$\partial K = \partial \left(\bigcup_{n=0}^{\infty} K_n \right), \tag{3.83}$$

which we call invariant set is in fact the ω-limit set of some small area around the origin (we can take K_0 as the area mentioned), with regards to the mapping G. The important question which arises in this context is how ∂K looks like. We already call the set as the invariant set, although we have no theoretical evidence that ∂K is invariant. Therefore we present a theorem, stated in *Robinson* (1995) [14], pp. 23.

Theorem 9. *Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map on $\mathbb{R}^2$. For any x, $\omega(x)$ is closed and invariant, if (i) $\gamma^+(x)$ is contained in some compact subset of X or (ii) G is one to one.*

Since we proved in Section 3.4 that $\partial K \subset P$, where P is a compact subset of $\mathbb{R}^2$, we can conclude that ∂K is invariant under G (all assumptions of the theorem are satisfied since G is continuous).

In Section 3.1 we observed similar properties between the mapping G, restricted to the invariant set, and homeomorphisms on circle. Therefore we can also inspect whether G is an orientation preserving homeomorphism but this issue is by no means obvious and we only can guess the answer. On the other side, numerical simulations, presented in Section 3.1 led us to a conjecture that G is an orientation preserving map.

3.7 Summary

The following theorem summarize the whole section on an invariant set.

Theorem 10 (Invariant Set). *Let $a = 1.1$, $b = 1.3$. There exists $c > 0$ such that the following holds true. If x_0, y_0 are the initial values for a difference system given by (2.1) and (2.2), satisfying $|x_0| + |y_0| < c$ then the trajectory of (2.1) and (2.2), satisfying the initial condition is bounded and converges to the invariant set K, which is an object in $\mathbb{R}^2$ with property $G(K) = K$.*

4 CONCLUSIONS

In this thesis we study the second order recurrent equation in the form

$$x_{n+1} = x_n + a(x_n - x_{n-1})(1 - |x_n|)^+ - bx_n|x_n|, \tag{4.1}$$

and its equivalent forms as a two-dimensional first order difference recurrent equation. The model, despite its simplicity shows complicated dynamics patterns.

We explore asymptotic stability of the only fixed point – the origin using the center manifold reduction. For $a > 1$ the origin is unstable whereas for $a < 1$ the origin is asymptotically stable. The case $a = 1$ remains an open question and provides the area for the further investigation.

Particularly interesting are explorations in the case when the origin is unstable. The crucial role is played by numerical simulations, uncovering the boundeness of the ω-limit set. For special values of parameters we found a compact area mapped into itself proved its existence by the rigorous computer assisted proof. On the other hand, the existence of a compact set lying in its own image, with a boundary homeomorphic to a circle, was proven using local properties of mapping in a neighborhood of the origin. These two results imply the existence of a monotone sequence of closed curves, homeomorphic to a circle, "converging" to the ω-limit set. This set is proved to be invariant due to the boundeness of trajectories.

The simulations of the invariant ω-limit set reveal rather surprising phenomena. The invariant set, given by (3.83), splits into only a several points (6, 8, 12 or even more) for special values of parameter a. Occurrence of these points, representing the periodic orbits of different periods, suggests similarities between the restriction of the mapping to the invariant set and homeomorphisms of circles. With respect to the theory concerning rotation number, the graph of the rotation number, depended on the parameter a is evaluated. This graph resembles the so called Devil's Staircase, a property of diffeomorphisms of circle. So far there are only suggestions and empirical evidence of the ω-limit set being homeomorphic to the circle, but perhaps in the future this puzzle will be solved.

From the economical point of view, the proposed model is too simple to explain truly the movements of the real exchange rate, therefore the analysis of the real data is not provided in the paper. The model shows that the convergence of the real exchange rate transforms into the presence of persistent fluctuations when the elasticity of agents on the market a crosses the threshold value 1 and according to the simulations, it diverges to the infinity when this elasticity is too high.

APPENDIX

The rotation number and the Devil's Staircase

In this section we outline the theory of orientation preserving homeomorphisms of a circle. Their dynamics can be described in terms of a single number. This number, called the rotation number measures the asymptotic average angle a point rotates per iterate of the homeomorphism. The theory about the rotation number and the devil's staircase is presented in *Robinson* (1995) [14], pp. 49–57 and *Hale, J. and Koçak, H.* (1991) [9], pp. 155–165.

We denote by S^1 the unit circle. We can represent S^1 as a function of a parameter t such that $\phi(t) = e^{i2\pi t}$. Assume that $f : S^1 \to S^1$ is an orientation preserving homeomorphism. Then there is a (nonunique) map $F : \mathbb{R} \to \mathbb{R}$ which is called a *lift* of f such that $\phi \circ F = f \circ \phi$. A lift satisfies

- (i) F is monotonically increasing and
- (ii) $F(t+1) = F(t) + 1$ for all t.

Definition 4. *Let f and F be defined as above. Denote*

$$\rho_0(F,t) = \lim_{n\to\infty} \frac{F^n(t) - t}{n} \,. \tag{4.2}$$

The number

$$\rho(f) = \rho_0(F,t) \mod 1 \,, \tag{4.3}$$

is called the rotation number of a mapping f.

As a justification of the previous definition serves the next theorem.

Theorem 11. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with lift F. Then*

(1) for $t \in \mathbb{R}$ the limit defining $\rho_0(F,t)$ exists and is independent of t,

(2) if $\rho(f) = \rho_0(F,t) \mod 1$, then it is independent of the lift F, and

(3) $\rho(f)$ depends continuously on f.

The rotation number is an invariant of a mapping, which characterizes much of the qualitative features of orientation preserving homeomorphism of the circle. The crucial fact is that the rotation number can be used to determine whether the mapping has any periodic points or not.

Theorem 12. *The rotation number $\rho(f)$ is rational if and only if f has a periodic point. In fact, $\rho(f) = p/q$ if and only if f has a point of period q. (Here p/q is assumed to be in a reduced form with p and q integers and q positive.)*

In addition, if f is a continuous orientation preserving homeomorphism and $\rho(f)$ is irrational, the following properties of the omega limit set of f can be proven:

(1) $\omega(x)$ is independent of x,

(2) $\omega(x)$ is a minimal set, and

(3) $\omega(x)$ is either *(i)* all of S^1 of *(ii)* a Cantor subset of S^1

Let now f be a function of two variables x, a and for any fixed a is an orientation preserving homeomorphism of the circle. Denote $\rho(a) = \rho(f(\cdot, a))$. If f is C^2, then $\rho(\cdot)$ has typically the following properties:

- ρ is continuous with finite variation but not absolutely continuous,
- for each rational number p/q there is an interval $I_{p/q}$ with nonempty interior such that for $a \in I_{p/q}$ we have $\rho(a, \bar{b}) = p/q$, and
- if $\rho(\alpha)$ is irrational then ρ is monotonic but not constant in the neighborhood of α.

A function with these properties is an example of a Cantor function and is commonly called *"Devil's Staircase"*.

References

[1] Azariadis, C. (1993), *Intertemporal Macroeconomics*, Oxford: Blackwell Publishers, pp. 49–51

[2] Baxter, M., Stockmann, A. C. (1989), *Business Cycles and the Exchange Rate System*, Journal of Monetary Economics, **23** pp. 377–400.

[3] Brunovský, P., Erdélyi, A., Walther, H. O. (submitted 2004), *On a Model of a Currency Exchange Rate – Local Stability and Periodic Solutions.*

[4] Brunovský, P., *Dynamické systémy a Diferenciálne rovnice*, Študjný materiál, FMFI UK, Bratislava, http://pc2.iam.fmph.uniba.sk/skripta

[5] Carr, J. (1981), *Applications of Center Manifold Theory*, Springer - Verlag, New York, pp. 33–36.

[6] Černý, I. (1983), *Analýza v komplexním oboru*, Československá Akademie Věd, Praha.

[7] Erdélyi, A. (2003), *A Delay Differential Equation Model of Oscillations of Exchange Rates*, Diploma Thesis FMFI UK, advisor Brunovský, P., Bratislava

[8] De Grauwe, P., Grimaldi, M. (2002) *The Exchange Rate and its Fundamentals. A Chaotic Perspective.* , CESIfo Discussion

[9] Hale, J., and Koçak, H. (1991) *Dynamics and Bifurcations (TAM 3)*, Springer - Verlag, New York, Heidelberg, Berlin, pp. 155–165, 444–445.

[10] Iooss, G. (1979) *Bifurcation of Maps and Applications,* North Holland Publishing Company, Amsterdam, New York, Oxford, Ch. 5.2.

[11] Jeanne, O., Rose, A.K. (2002), *Noise Trading and Exchange Rate Regimes*, Quarterly Journal of Economics 117, May, pp.537-69

[12] Krugman, P., Obstfeld, M. (2000) *International Economics: Theory and Policy,* 5th Edition, Addison-Wesley Publishing Company, Part 3

[13] Naylor, A. and Sell, G. (1971) *Linear Operator Theory in Engineering and Science*, Holt, Reinhart and Winston, Inc., Ch. 3

[14] Robinson, C. (1995) *Dynamical Systems - Stability, Symbolic Dynamics, and Chaos*, CRC Press, Inc., pp. 49–57.