# Faculty of Mathematics, Physics and Informatics COMENIUS UnIVERSITY in Bratislava 



## DYNAMICS OF LEARNING the Rational Expectation Equilibrium Orbit

# Fakulta Matematiky, Fyziky a Informatiky Univerzity Komenského v BRATISLAVE 

Ekonomická a finančná matematika

# Dynamika UČEnia Rovnovážnej Trajektórie Racionálnych Očakávaní 

Diplomová práca

Prehlasujem, že som diplomovú prácu napísala samostatne a použila som iba literatúru uvedenú v zozname.

Ďakujem svojmu školiteľovi, prof. RNDr. Pavlovi Brunovskému, DrSc.
za cenné rady a námety pri písaní
tejto práce a za poskytnutie potrebnej literatúry.

## Contents

1 Preface ..... 1
2 Short introduction of the model ..... 2
3 General theory ..... 3
3.1 Difference equations ..... 3
3.2 Logistic equation ..... 8
3.3 Rational expectations ..... 11
4 Model analysis ..... 15
4.1 Description of the model ..... 15
4.2 Introduction of the learning system ..... 21
4.3 Analysis of the dynamics of the first learning system ..... 23
4.4 Analysis of the dynamics of the second learning system ..... 30
5 Conclusion ..... 34
References ..... 35

## 1 Preface

In [5] a difference model with rational expectation for a current state of the economy is introduced. Afterwards, a formula with unknown parameter is given, which individuals use for forming their rational expectations at a given time. The individuals guess this parameter and revise it each time the new state of the economy is known. For this correction they use a general learning system.

We pursue several aims. One of them is to fill all the details which were not straightforward and their explanations were omitted in [5]. These details are given in Subsection 4.1. The second one was to study the dynamics of the simplified versions of the learning system proposed in [5], which describes how the individuals try to learn the parameter. Subsections 4.3 and 4.4 deal with this most important goal, which is the core of this thesis. The last goal is to show that the individuals can find also other values of the parameter and not the one derived in [5] as locally stable. Numerical experiments are given too.

For these purposes, we shortly introduce the model in Section 2 and summarize the necessary theory in Section 3. This basic theory consists of the theory of difference equations in $R^{2}$ in Subsection 3.1, a special case of difference equation in $R$ in Subsection 3.2 and the theory of rational expectations in Subsection 3.3. Afterwards the mentioned analysis of the model is carried out.

The first reason why we have focused on this learning system is that in special cases it performs a strange and interesting behavior. This behavior can be described as follows: individuals state such rational expectations that before the final decrease of the deviation from the stationary equilibrium, this deviation may increase for some time. Ultimately the individuals state such rational expectations that the economy converges to its stationary equilibrium. The analysis of the dynamics of this learning process is not carried out in detail in [5] and the aim of Section 4 is to analyze this dynamics. The second reason is, that in the models with rational expectation linear models are usually studied, because then Certainty Equivalence holds and the perfect foresight orbit is the orbit under the rational expectation up to small random fluctuation. This learning process is not linear and therefore the dynamics is even more interesting.

## 2 Short introduction of the model

Assume that the economy is characterized by a scalar variable $x_{t}$. The current state is a real number $x_{t}$ linked with the forecast of a next state $x_{t+1}^{e}$ and with a previous state $x_{t-1}$ through the following relation,

$$
\begin{equation*}
\gamma x_{t+1}^{e}+x_{t}+\delta x_{t-1}=0 \tag{2.1}
\end{equation*}
$$

This equation stands for a first order approximation of a equilibrium dynamics in a small neighborhood of a locally unique stationary state $(\bar{x}=0)$ [5], $x_{t}$ representing the deviation from this stationary equilibrium $\bar{x}=0$. We see that in this equation the current state of the real variable $x_{t}$ is a weighted average of the previous state $x_{t-1}$ and the forecast of the next state $x_{t+1}^{e}$, where $\gamma$ and $\delta$ represents the relative weights of future and past respectively. We suppose relative weights of the future to be different from zero $\gamma \neq 0$ to be meaningful to talk about rational expectations.

Individuals suppose that the economy is developing according to the rule $x_{t}=\beta_{t} x_{t-1}$ and they try to find the parameter $\beta$. When the expectation fullfils, i.e. $x_{t+1}^{e}=x_{t+1}$, (2.1) becomes a difference equation and we can compute the roots of its characteristic polynomial $\lambda_{1}$ and $\lambda_{2}$. If $\left|\lambda_{1}\right|<1<\left|\lambda_{2}\right|$, then individuals try to find $\lambda_{1}$, because the only converging equilibrium path lies on the line defined by $x_{t}=\lambda_{1} x_{t-1}$. In [5] the local stability of $\lambda_{1}$ is proved. We can ask a question: when the individuals will find this parameter? We deal with the answer in Section 4.

## 3 General theory

### 3.1 Difference equations

In this section we present a short summary of the basic theory of difference equations in $R^{2}$. These results can be found in many books, from which we have used mostly [8]. At the very beginning the basic concepts of the theory are defined and then the important cases depending on the moduli of eigenvalues of the linear systems are discussed. At last we note the differences which arise, when the difference equation in $R^{2}$ is nonlinear.

For a given function $f: R^{2} \rightarrow R^{2}$ a difference equation is given by

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right), \tag{3.1}
\end{equation*}
$$

i.e. as a recurrence relation generated by the map $f$. In its economic interpretation the variable $x_{n}$ represents the state of the economy which is observed in equidistant times $\{0,1,2, \ldots\}$.

The positive orbit $\gamma^{+}$of a point $x_{0}$ in $R^{2}$ is a sequence of images of $x_{0}$ under the successive iterates of the map $f$ :

$$
\gamma^{+}\left(x_{0}\right)=\left\{x_{0}, f\left(x_{0}\right), \ldots, f^{n}\left(x_{0}\right), \ldots\right\}
$$

If the map $f$ is invertible, then the negative orbit $\gamma^{-}$of the point $x_{0}$ is:

$$
\gamma^{-}\left(x_{0}\right)=\left\{x_{0}, f^{-1}\left(x_{0}\right), \ldots, f^{-n}\left(x_{0}\right), \ldots\right\}
$$

where $f^{-n}$ denotes the n -th iterate of $f^{-1}$. When the positive and negative orbits exist, the orbit $\gamma$ of $x_{0}$ is defined by: $\gamma\left(x_{0}\right)=\gamma^{+}\left(x_{0}\right) \cup \gamma^{-}\left(x_{0}\right)$. The negative orbit is the set of past states and the positive orbit is the set of future states including the current one.

Now we provide the summary of the stability theory of fixed points, which plays an important role in practice.

Definition 1 ([8],p.444) A point $\bar{x} \in R^{2}$ is called a fixed point of $f$ if $f(\bar{x})=\bar{x}$. A fixed point $\bar{x}$ of $f$ is said to be stable if, for any $\epsilon>0$, there is a $\delta>0$ such that, for every $x_{0}$ for which $\left\|x_{0}-\bar{x}\right\|<\delta$, the iterates of $x_{0}$ satisfy $\left\|f^{n}\left(x_{0}\right)-\bar{x}\right\|<\epsilon$ for all $n \geq 0$. A fixed point $\bar{x}$ is said to be unstable if it is not stable. A fixed point is said to be asymptotically stable if it is stable and, in addition, there is an $r>0$ such that $f^{n}\left(x_{0}\right) \rightarrow \bar{x}$ as $n \rightarrow+\infty$ for all $x_{0}$ satisfying $\left\|x_{0}-\bar{x}\right\|<r$.

Definition 2 ([8],p.445) A point $\bar{x} \in R^{2}$ is called periodic of minimal period $n$ if $f^{n}(\bar{x})=\bar{x}$ and $n$ is the least such positive integer. The set of all iterates of a periodic point is called a periodic orbit.

Definition 3 ([8],p.445) A set $M$ in $R^{2}$ is said to be invariant under the map $f$ if $f(M)=$ $M$, that is, for any $x \in M$ we have $f(x) \in M$ and there is a point $y \in M$ such that $f(y)=x$.

A special case of difference equations are linear equations which are given by a recurrence equation

$$
\begin{equation*}
x_{n+1}=A x_{n} \tag{3.2}
\end{equation*}
$$

for some $2 \times 2$ matrix $A$. If $I-A$ is regular then the only fixed point of this system is $\bar{x}=\binom{0}{0}$. Computation of orbits of linear equation can be greatly simplified using the normal form of the matrix $A$. If $\Lambda$ denotes the Jordan Normal Form of the matrix $A$, i.e. $A=P \Lambda P^{-1}$, then $A^{n}=\left(P \Lambda P^{-1}\right)^{n}=P \Lambda^{n} P^{-1}$. Using the transformation $x=P y$ (e.g. [4]) we can rewrite the equation (3.2):

$$
P y_{n+1}=P \Lambda y_{n}
$$

and after the multiplication by regular $P^{-1}$ from the left

$$
\begin{equation*}
y_{n+1}=\Lambda y_{n} \tag{3.3}
\end{equation*}
$$

For two distinct eigenvalues $\lambda_{1}, \lambda_{2}$ of the matrix $A$ the Jordan Normal Form of the matrix $A$ is $\Lambda=\left(\begin{array}{cc}\lambda_{1} & 0 \\ 0 & \lambda_{2}\end{array}\right)$ ([3], p.284).

Theorem 1 ([8], p.451) The fixed point $\bar{x}=\binom{0}{0}$ of the linear equation (3.2) is asymptotically stable if and only if the eigenvalues of A have moduli less than one. If at least one eigenvalue of $A$ has modulus greater than one then the fixed point is unstable.

Definition 4 ([8], p.451) A linear planar map is called hyperbolic if the eigenvalues of $A$ have moduli different than one.

Below, we characterize hyperbolic planar maps. For two distinct eigenvalues $\lambda_{1} \neq \lambda_{2}$ (3.3) is equivalent to:

$$
\begin{aligned}
y_{n+1}^{1} & =\lambda_{1} y_{n}^{1} \\
y_{n+1}^{2} & =\lambda_{2} y_{n}^{2}
\end{aligned}
$$

From this system we see that for $y_{n}$

$$
\begin{align*}
& y_{n}^{1}=\lambda_{1}^{n} y_{0}^{1}  \tag{3.4}\\
& y_{n}^{2}=\lambda_{2}^{n} y_{0}^{2},
\end{align*}
$$

holds.
A Hyperbolic sink:
If both $\left|\lambda_{1}\right|<1$ and $\left|\lambda_{2}\right|<1$, we can see, that both components of the vector $y$ approach zero as $n \rightarrow \infty$. Asymptotical stability can be proved as follows.


Figure 1: A single positive orbit of a hyperbolic sink (both eigenvalues are positive).


Figure 2: A single positive orbit of a hyper-
Figure 3: Phase portrait of a hyperbolic sink bolic sink when one eigenvalue is negative.

Proof: If we denote $\|$.$\| as the norm in R^{2}$ and $\bar{\lambda}$ is the maximum from $\left|\lambda_{1}\right|$ and $\left|\lambda_{2}\right|$, for the transformed system $y_{n}=\Lambda^{n} y_{0}$ one has that $\left\|y_{n}\right\| \leq \bar{\lambda}^{n}\left\|y_{0}\right\|$. Furthermore from the properties of the norm we know that $\left\|x_{n}\right\| \leq\|P\|\left\|y_{n}\right\|$ and $\left\|y_{0}\right\| \leq\left\|P^{-1}\right\|\left\|x_{0}\right\|$. Altogether this allows us to write $\left\|x_{n}\right\| \leq\|P\|\left\|y_{n}\right\| \leq \bar{\lambda}^{n}\|P\|\left\|y_{0}\right\| \leq \bar{\lambda}^{n}\|P\|\left\|P^{-1}\right\|\left\|x_{0}\right\|$. Because $|\bar{\lambda}|<1$, the origin is asymptotically stable (figure 1 ).
In case that at least one of the eigenvalues is negative, the single orbit jumps back and forth across the axes, see figure 2. It is important to be aware that the orbit is a sequence of discrete points and not a connected curve. The phase portrait of a sink can be seen in the figure 3 .

## A Hyperbolic source:

On the contrary, from (3.4) we see that when both $\left|\lambda_{1}\right|>1$ and $\left|\lambda_{2}\right|>1$, both $y^{1}$ and $y^{2}$ tend to infinity as $n \rightarrow \infty$ and the origin is an unstable fixed point. Phase portrait is in the figure 4.


Figure 4: Phase portrait of a hyperbolic source.


Figure 5: Phase portrait of a hyperbolic saddle.

Figure 7: A single positive orbit of a linear map with complex eigenvalues for which $\sqrt{\alpha^{2}+\beta^{2}}>1$ holds.

## A Hyperbolic saddle:

This is the case, when $\left|\lambda_{2}\right|>1>\left|\lambda_{1}\right|$. The component $y^{1}$ tends to zero and the second component $y^{2}$ tends to infinity (figure 5). The origin is unstable. Only orbits of initial points with $y_{0}^{2}=0$ approach zero and these orbits interest us in this paper.

Linear maps with complex eigenvalues:
Let $\alpha \pm i \beta$ be the eigenvalues of matrix $A$. In their polar representation they are rewritten as $\alpha=\sqrt{\alpha^{2}+\beta^{2}} \cos \omega$ and $\beta=\sqrt{\alpha^{2}+\beta^{2}} \sin \omega$ for some $\omega \in(-\pi, \pi\rangle$. The matrix $A$ then becomes

$$
A=\sqrt{\alpha^{2}+\beta^{2}}\left(\begin{array}{cc}
\cos \omega & \sin \omega \\
-\sin \omega & \cos \omega
\end{array}\right) .
$$

Multiplying of a vector $x$ by matrix $A$ rotates $x$ by the angle $\omega$ and multiplies its modulus by $\sqrt{\alpha^{2}+\beta^{2}}$. Depending on the value $\sqrt{\alpha^{2}+\beta^{2}}$, the components of $x$ will decrease and converge to 0 or diverge. For $\sqrt{\alpha^{2}+\beta^{2}}<1$ is the origin asymptotically stable (see figure 6 ) and for $\sqrt{\alpha^{2}+\beta^{2}}>1$ it is unstable (see figure 7) ([8], p. 451). If $\sqrt{\alpha^{2}+\beta^{2}}=1$

Figure 8: A single positive orbit of a linear map with complex eigenvalues for which $\sqrt{\alpha^{2}+\beta^{2}}=1$ holds.


Figure 9: A phase portrait of a nonhyperbolic map with one eigenvalue 1.


Figure 10: A single positive orbit of a nonhyperbolic map with one eigenvalue -1 .
this becomes a nonhyperbolic map and the orbit of $x_{0}$ lies on the circle with radius $\left\|x_{0}\right\|$ centered at the origin (figure 8). If $\frac{\omega}{2 \pi}$ is rational then this orbit is periodic, otherwise it is dense ([8], p. 451).

Although we are not interested in nonhyperbolic planar maps in this paper, we discussed them in short, because we want to provide the survey of all the possibilities.

A Nonhyperbolic linear maps:
Nonhyperbolic linear maps are maps with an eigenvalue 1 or -1 . At first we focus at the eigenvalue 1. If $\lambda_{1}=1$, from (3.4) we see that $y_{n}^{1}$ stays the same and $y_{n}^{2}$ converges to the origin or diverges depending upon the modulus of the second eigenvalue. We can observe that in addition to the origin also each point at the $y^{1}$ axis is a fixed point (see figure 9).

The eigenvalue $\lambda_{1}=-1$ causes the reflection of $y^{1}$, therefore only the origin is the fixed point, and every point at $y^{1}$ axis is a periodic orbit of period 2 (figure 10). Again the stability of the origin depends on the modulus of the second eigenvalue.

We shall continue with the nonlinear system (3.1). Stability is determined on the basis of linearization defined in the following definition.

Definition 5 If $\bar{x}$ is a fixed point of a $C^{1}$ map (3.1) then the linear map

$$
\begin{equation*}
x_{n+1}=D f(\bar{x}) x_{n} \tag{3.5}
\end{equation*}
$$

where $D f(\bar{x})$ is the Jacobian matrix

$$
\left(\begin{array}{ll}
\frac{\partial f_{1}}{\partial x_{1}}(\bar{x}) & \frac{\partial f_{1}}{\partial x_{2}}(\bar{x}) \\
\frac{\partial \partial_{2}}{\partial x_{1}}(\bar{x}) & \frac{\partial f_{2}}{\partial x_{2}}(\bar{x})
\end{array}\right),
$$

is called the linearization of the map $f$ at the fixed point $\bar{x}$.
Definition 6 A fixed point $\bar{x}$ of (3.1) is said to be hyperbolic if the linear map (3.5) is hyperbolic, that is, if the Jacobian matrix $D f(\bar{x})$ at $\bar{x}$ has no eigenvalues with modulus one.

Because (3.5) is only an approximation derived from the Taylor series of $f$ around $\bar{x}$, for the stability of $\bar{x}$ a stronger condition is necessary than for a linear map (e.g. eigenvalues $\lambda_{1}=1$ and $\left|\lambda_{2}\right|<1$ does not suffice for stability of $\bar{x}$ for nonlinear map, but suffice for a linear map). The stability is stated in the following theorem.
Theorem 2 Let $f$ be a $C^{1}$ function with a fixed point $\bar{x}$.
(i) If all the eigenvalues of the Jacobian matrix $D f(\bar{x})$ have moduli less than one, then the fixed point $\bar{x}$ is asymptotically stable.
(ii) If at least one of the eigenvalues of $D f(\bar{x})$ has modulus greater than one, then $\bar{x}$ is unstable.

### 3.2 Logistic equation

Thus we have stated the basics of difference equation in $R^{2}$ and now we introduce an important special case of difference equation in $R$, which is the logistic equation. As we will see later, one of our learning system will turn out to be the logistic equation and therefore we introduce the known analysis of this equation (these results are adopted from [4]). The logistic map is defined by:

$$
\begin{equation*}
x_{n+1}=f_{\mu}\left(x_{n}\right), \tag{3.6}
\end{equation*}
$$

where $f_{\mu}(x)=\mu x(1-x)$. If $0 \leq \mu \leq 4$, then for $x_{0} \in\langle 0,1\rangle f_{\mu}$ maps $\langle 0,1\rangle$ into itself. This follows from the fact, that for such $\mu$ the maximum of $f_{\mu}(x)$ is from $\langle 0,1\rangle$.

$$
\max f=f(1 / 2)=\mu / 4 \leq 1
$$

Now we continue by locating fixed points and studying their stability for various $\mu$. Solutions of the following equation for fixed points

$$
\bar{x}=\mu \bar{x}(1-\bar{x})
$$

are $\bar{x}_{1}=0$ and for $\mu>1$ also $\bar{x}_{2}=1-1 / \mu$. Thus we can prove the following property of the logistic equation.

Lemma 1 ([8], p.93)
Suppose that $\mu>1$. If $x_{0}<0$ or $x_{0}>1$, then $f_{\mu}^{n}\left(x_{0}\right) \rightarrow-\infty$ as $n \rightarrow \infty$.
Proof: If $x_{0}<0$ or $x_{0}>1$, then $f_{\mu}\left(x_{0}\right)<x_{0}$. Thus $f_{\mu}^{n}\left(x_{0}\right)$ is a decreasing sequence. This sequence cannot converge because $f$ has no negative fixed points.

To determine the stability of the two fixed points we find the derivative of $f_{\mu}$ :

$$
f_{\mu}^{\prime}(x)=\mu-2 \mu x
$$

Since $f_{\mu}^{\prime}(0)=\mu, \bar{x}_{1}$ is asymptotically stable for $\mu \in\langle 0,1)$ and unstable for $\mu>1$. Further for $\bar{x}_{2}$ the derivative is $f_{\mu}^{\prime}(1-1 / \mu)=2-\mu$ and therefore $\bar{x}_{2}$ is asymptotically stable for $1<\mu<3$ and for $\mu>3$ it is unstable. This model starts to be interesting when studying what happens for $\mu=3\left(\bar{x}_{2}=\frac{2}{3}\right.$ then $)$. We will find that $\bar{x}_{2}$ becomes unstable and two periodic points of period 2 appear. These periodic points are fixed points of $f_{\mu}^{2}$ and are not fixed points of $f_{\mu}$.

$$
f_{\mu}^{2}(x)=\mu[\mu x(1-x)][1-\mu x(1-x)]=-\mu^{3} x^{4}+2 \mu^{3} x^{3}-\left(\mu^{2}+\mu^{3}\right) x^{2}+\mu^{2} x
$$

For the fixed points of $f_{\mu}^{2}$ the following equation holds:

$$
-\mu^{3} x^{4}+2 \mu^{3} x^{3}-\left(\mu^{2}+\mu^{3}\right) x^{2}+\mu^{2} x-x=0
$$

We know that the roots are also 0 and $1-1 / \mu$, so we divide this equation by $\mu x(x-1+1 / \mu)$ and we obtain:

$$
-\mu^{2} x^{2}+\left(\mu^{2}+\mu\right) x-\mu-1=0
$$

We can consider $x$ as given implicitly by $g(x, \mu)=0$, where $g(x, \mu)=\mu^{2} x^{2}-\left(\mu^{2}+\mu\right) x+\mu+1$. Because

$$
\begin{equation*}
\left.\frac{\partial g}{\partial x}\right|_{\left(\frac{2}{3}, 3\right)}=2 \mu^{2} x-\mu^{2}-\left.\mu\right|_{\left(\frac{2}{3}, 3\right)}=0 \tag{3.7}
\end{equation*}
$$

we have to find another way how to find the solutions and analyze them. $g(x, \mu)$ and $\frac{\partial g}{\partial \mu}$ are continuous and $\left.\frac{\partial g}{\partial \mu}\right|_{\left(\frac{2}{3}, 3\right)} \neq 0$ holds. According to the implicit function theorem we see that in the neighborhood of $2 / 3$ there exists a unique function $\mu=\phi(x)$ such that in this neighborhood it is continuous and $g(x, \phi(x))=0$. This function $\phi(x)$ can be found approximately utilizing the Taylor Series. Partial derivatives of $g(x, \mu)$ are continuous and $\frac{\partial g}{\partial \mu} \neq 0$ in this neighborhood of the point $2 / 3$. Thus the first derivative in $2 / 3$ exists and equals

$$
\Phi^{\prime}\left(\frac{2}{3}\right)=-\frac{\left.\frac{\partial g}{\partial x}\right|_{\left(\frac{2}{3}, 3\right)}}{\left.\frac{\partial g}{\partial \mu}\right|_{\left(\frac{2}{3}, 3\right)}}=0
$$

because of (3.7). The second derivative is

$$
\left.\frac{\partial^{2} \Phi}{\partial x^{2}}\right|_{\left(\frac{2}{3}\right)}=-\left.\frac{\frac{\partial^{2} g}{\partial x^{2}} \cdot \frac{\partial g}{\partial m u}-\frac{\partial g}{\partial x} \cdot \frac{\partial^{2} g}{\partial m u^{2}}}{\left(\frac{\partial g}{\partial m u}\right)^{2}}\right|_{\left(\frac{2}{3}, 3\right)}=-\left.\frac{\frac{\partial^{2} g}{\partial x^{2}}}{\frac{\partial g}{\partial m u}}\right|_{\left(\frac{2}{3}, 3\right)}=\left.\frac{-2 \mu^{2}}{2 \mu x^{2}-(2 \mu+1) x+1}\right|_{\left(\frac{2}{3}, 3\right)}=18
$$

Locally in a neighborhood of $\left(\frac{2}{3}, 3\right)$ then

$$
\begin{equation*}
\mu=\phi(x)=3+9\left(x-\frac{2}{3}\right)^{2}+o\left(x-\frac{2}{3}\right)^{2} \tag{3.8}
\end{equation*}
$$

holds. It is a parabolic shaped curve open toward the direction of $\mu$. It can be shown that this periodic orbit of period two is asymptotically stable in the following way.
Proof: We denote $h(x)=\left.\left(f_{\mu}^{2}\right)^{\prime}(x)\right|_{\mu=\phi(x)}$ and prove that $-1<h(x)<1$ for $x \neq \frac{2}{3}$ in some neighborhood of $\frac{2}{3}$. Therefore we compute following derivatives and their values in $x=\frac{2}{3}\left(\right.$ from $\left.(3.8) \mu=\phi\left(\frac{2}{3}\right)=3\right)$ :

$$
\begin{aligned}
& \left.h\left(\frac{2}{3}\right)\right|_{\mu=3}=-4 \mu^{3} x^{3}+6 \mu^{3} x^{2}-2\left(\mu^{2}+\mu^{3}\right) x+\left.\mu^{2}\right|_{x=\frac{2}{3}, \mu=3}=1, \\
& \left.h^{\prime}\left(\frac{2}{3}\right)\right|_{\mu=3}=-12 \mu^{3} x^{2}+12 \mu^{3} x-\left.2\left(\mu^{2}+\mu^{3}\right)\right|_{x=\frac{2}{3}, \mu=3}=0, \\
& \left.h^{\prime \prime}\left(\frac{2}{3}\right)\right|_{\mu=3}=-24 \mu^{3} x+\left.12 \mu^{3}\right|_{x=\frac{2}{3}, \mu=3}=-108 .
\end{aligned}
$$

Thus we see that in $\frac{2}{3}$ is a local maximum of the function $h(x)$, i.e. $h(x)<1$ for $x \neq \frac{2}{3}$ in some neighborhood of $\frac{2}{3}$. In some neighborhood of $\frac{2}{3}$ also $-1<h(x)$ holds. That means $\left|\left(f_{\mu}^{2}\right)^{\prime}(x)\right|_{\mu=\phi(x)} \mid<1$ for $x \neq \frac{2}{3}$ near $\frac{2}{3}$, or in other words, the cycle of period two is asymptotically stable for $x$ close to $\frac{2}{3}$.

The value of the parameter $\mu$ for which asymptotically stable fixed point loses its stability, which is transferred to an another fixed point (or periodic orbit) is called a bifurcation value. The value $\mu=3$ is an example of such bifurcation value.
[8] provides the following survey of this bifurcation analysis of the logistic map. For $3<\mu<1+\sqrt{6}$ there exists the asymptotically stable periodic orbit of minimal period two. For $3.449<\mu<3.544$ this periodic orbit loses its stability and an asymptotically stable orbit of minimal period four occurs. For $3.544<\mu<3.564$ another asymptotically stable orbit of minimal period eight appears. For $3.570<\mu$ there is no asymptotically stable orbit, but for $\mu$ great enough which is suficiently close to 4 there exists an orbit of minimal period three.

To continue we need to introduce Sharkovskii ordering of positive integers defined as follows:

$$
3<5<7<\ldots<2 \cdot 3<2 \cdot 5<\ldots<2^{k} \cdot 3<2^{k} \cdot 5<\ldots<2^{3}<2^{2}<2<1
$$

Theorem 3 (Sharkovskii,[8] p. 99) Let $f: R \rightarrow R$ be a continuous map. Suppose that $f$ has a periodic point of minimal period $m$. If $m<n$ in the Sharkovskii ordering, then $f$ also has a periodic point of minimal period $n$.


Figure 11: Bifurcation diagram of logistic equation. At the horizontal axis are the values of parameter $\mu$ and at the vertical axis are the values of fixed points.

If there exists a periodic orbit of period three, then there exists a periodic orbit of arbitrary period and moreover the system demonstrates a chaotic behavior. For our need we only mention one property of such behavior. There exists $\epsilon>0$, that if $x_{0}$ and $y_{0}$ are two initial points such that $0<\left|x_{0}-y_{0}\right|<\epsilon$, then there exists $K$ for which $\left|x_{K}-y_{K}\right|>\epsilon$. This means that the system is highly sensitive to initial conditions.

### 3.3 Rational expectations

In this part we present the basics of rational expectation theory. We start with the rational expectations, where we will presume perfect certainty at first, and then we admit uncertainty. This theory is adopted from [1], where it is perfectly explained. We also present this theory in the special case of our model (2.1).

The most important assumption of rational expectation is that individuals should not make systematic errors in expectation formation, otherwise there exists an incentive to diagnose the source of mistakes and amend the forecasting rule ([1], p. 12). Individuals do not have to form accurate forecasts, but their guesses should be correct in average. The hypothesis of rational expectations asserts that the unobservable subjective expectations of individuals are exactly the true mathematical conditional expectations implied by the model itself. Individuals act as if they know the model and form expectations accordingly ([1], p. 30).

## Perfect foresight equilibrium orbit

When there is not any uncertainty and information is complete, then the rational expectations reduce to the special case of perfect foresight. In this case the forecasts of individuals are accurate and the same. The forecast fulfils. This approach to the study of rational expectations is useful mainly when systematic factors are much more important than random factors, because then we have a deterministic model and its study is more simple and the standard questions of existence, uniqueness and stability may be analyzed most easily.

Forward looking individuals use past values of variables and solve the model over all future time. Thus the convergence is not only the property of the model but a part of the process of expectation formation itself.

Consider now the system (2.1). Given a point $x_{0}$, by its equilibrium orbit we understand the orbit of the point $\left(x_{-1}, x_{0}\right)$ which converges to the stationary equilibrium $\bar{x}=0$. Suppose further that $\bar{x}=0$ is stable. The economy will converge to this stationary equilibrium no matter what the expectations are. But in this case, the expectation formation is very difficult ([1], p. 36), because an infinity of perfect foresight equilibrium orbits converge to this stationary equilibrium. When the individuals choose the starting point (in our model if the individuals know the value $x_{t-1}$, they have to state their expectations $x_{t}$ ) and the economy has proceeded along this chosen orbit for some time, afterwards the individuals let bygones be bygones and consider the new situation as before, choosing the new initial point (there is a new $x_{t^{\prime}-1}$ and they determine a new $x_{t^{\prime}+1}^{e}=x_{t^{\prime}+1}$ ). Thus the economy can jump to another equilibrium orbit. So when the stationary equilibrium is globally stable there does not exist a unique equilibrium orbit when expectations are forward looking. If there exist infinitely many perfect foresight equilibrium orbits all converging asymptotically, we say that the stationary equilibrium is indeterminate.

If the stationary equilibrium of the model is globally unstable, there is no equilibrium orbit to this stationary equilibrium, unless the economy occurs by chance in the stationary equilibrium. We say that a perfect foresight equilibrium orbit does not exist for any initial point. After some time the individuals would find out that the orbit is explosive and at some stage the structure of the model would change, for example by an intervention of the government ([1], p.39).

The saddle point is often found in economy models and its important feature is that for a given value of predetermined variable (in our model the value $x_{t-1}$ ) there exists a unique equilibrium orbit to the steady state. In the world without uncertainty, the economy would be on this equilibrium orbit only by chance. However the individuals have the chance to locate the economy on this equilibrium orbit choosing the appropriate level of free variable (in our model it is $x_{t}$ ). If for example the economy starts at the explosive orbit, at some
time the individuals will consider the situation as intolerable and they will adjust the free variable to move the economy to the equilibrium orbit.

Now we can generalize this to a system of $n$ variables, $s$ of which are predetermined. Suppose that convergent subspace has dimension $s$ and is in a general position, that is it projects to the space of determined variables surjectively along the space of free variables. Then there is a unique equilibrium orbit. ${ }^{1}$ If the dimension is greater then $s$, the perfect foresight equilibrium orbit is not unique. If the dimension is less then $s$, then a perfect foresight equilibrium orbit does not exist in general.

As long as the information remains the same, the economy follows the perfect foresight equilibrium orbit. When a new information occurs, individuals revise their expectations and the immediate response is a jump to a new equilibrium orbit. The economy stays on this orbit until the time that some further information becomes available ([1], p. 49).

## Rational expectation equilibrium orbit

Under the rational expectation hypothesis the individuals' expectations are precisely the mathematical expectations formed by using the model and conditional on available information at the date expectations are formed. Behavior in one period depends in part on the behavior expected in subsequent periods. Expectation can be obtain by solving the model over all future time to derive consistent or self-fulfilling set of expectations. When the solution of a stochastic model differs from the solution of a deterministic or non-random model only in the trivial respect that actual values of future variables are replaced by the current expectations of these future variables, we say that the random model exhibits Certainty Equivalence([1], p.52). The functional form and parameter values of the solutions to the two models are the same. Analysis of the model as if under perfect foresight will then display the essential structure of the solution to the analogous stochastic model under rational expectations. Admitting uncertainty leads merely to random fluctuations around the perfect foresight path. The condition under which certainty equivalence will be obtained is that the equations are linear and contain additive random disturbance with a mean zero ([1], p. 51,52). The expectation of non-linear function is becoming considerably more complicated.

Let $I_{t}$ denote the information set available at time $t$. This set has three parts: knowledge of the structure of the model, knowledge of government policies in operation and knowledge of past values of economic variables. The rational expectation at time $t$ of the variable $x$ at time $t+k$ can be denoted as ${ }_{t} x_{t+k}^{e}$ or as $E\left(x_{t+k} \mid I_{t}\right)$. Now we will rather use the latter to emphasize the condition on the information set. But often, and except of this theory section we use it in this paper, for $k=1$ this can be shorten to $x_{t+1}^{e}$ which means the

[^0]rational expectation of the variable $x$ at time $t+1$ formed one period sooner, i.e. at time $t$. There are four basic properties ([1], p.72,73) of the rational expectations models:

Property $1 E\left\{\left[E\left(x_{t+i+j} \mid I_{t+i}\right)\right] \mid I_{t}\right\}=E\left(x_{t+i+j} \mid I_{t}\right)$.
This property asserts that individuals can not predict how they will change their expectations therefore their best guess about the value of $x_{t+i+j}$ at time $t+i$, if they are at time $t$, is the guess they are forming at time $t$ about the value at time $t+i+j$.

Property $2 E\left\{\left[x_{t+i}-E\left(x_{t+i} \mid I_{t}\right)\right] \mid S_{t}\right\}=0$,
where $S_{t}$ is some subset at time $t$ of the full information set $I_{t}$ used by individuals at time $t$. This property states, that forecasting error is uncorrelated with each and every component $S_{t}$ of the information set $I_{t}$. It asserts that there is no information that may be used systematically to improve forecasting errors if expectations are rational.

Property $3\left\{x_{t+1}-E\left(x_{t+1} \mid I_{t}\right)\right\}$ is serially uncorrelated with mean zero.
This is a special case of $(2)$, where $S_{t}$ contains data on previous forecasting errors.
Property 4 In linear models Chain Rule of Forecasting holds.
This can be explained on an example of the expectation formation used in our model. Suppose that it is known that

$$
x_{t}=\beta x_{t-1}+u_{t}
$$

where the constant parameter $\beta$ is a positive fraction and $u_{t}$ is a random disturbance which is serially uncorrelated with mean zero. $I_{t}$ comprises past values of $u_{t}$ and of $x_{t}$, but they are of no use for predicting of the current value $u_{t}$. At the beginning of time $t$, before $u_{t}$ is known, the Rational Expectation of $x_{t}$ is given by

$$
E\left(x_{t} \mid I_{t}\right)=E\left(\beta x_{t-1}+u_{t} \mid I_{t}\right)=\beta x_{t-1}
$$

At the same date, the Rational Expectation of $x_{t+1}$ may be formed as:

$$
E\left(x_{t+1} \mid I_{t}\right)=E\left(\beta x_{t}+u_{t+1} \mid I_{t}\right)=\beta E\left(x_{t} \mid I_{t}\right)=\beta^{2} x_{t-1}
$$

By analogy we can compute the expectation of $x_{t+2}^{e}$, and we would obtain $\beta^{3} x_{t-1}$ as the solution. This iterating of computation of the expectation of future values of $x_{t^{\prime}}^{e}$, $t^{\prime}=\{t+1, t+2 \ldots\}$ is called Chain Rule of Forecasting.

## 4 Model analysis

### 4.1 Description of the model

We remind the form of the model. The current state of the economy is determined from the equation (2.1), where $x_{t}$ characterizes the economy at time $t$ and depends on the previous state and on rational expectation of the future state. Our first aim is to describe the local perfect foresight dynamics around $\bar{x}=0$ in cases of distinct real roots of the characteristic polynomial. Several properties of these orbits will be discussed. Then we will decompose the plane of $\gamma$ and $\delta$ into regions, where $\bar{x}=0$ is stable or unstable and (2.1) is the difference equation with complex eigenvalues and regions where $\bar{x}=0$ sink, source or saddle and (2.1) is the difference equation with real eigenvalues.

The perfect foresight dynamics is the dynamics when each foresight fulfills, i.e. $x_{t+1}^{e}=$ $x_{t+1}$. Then (2.1) becomes a difference equation:

$$
\begin{equation*}
\gamma x_{t+1}+x_{t}+\delta x_{t-1}=0 \tag{4.1}
\end{equation*}
$$

and its characteristic polynomial is

$$
\begin{equation*}
\gamma \lambda^{2}+\lambda+\delta=0 \tag{4.2}
\end{equation*}
$$

We are interested in the case when (4.2) has two distinct real roots $\lambda_{1}=\frac{-1+\sqrt{1-4 \gamma \delta}}{2 \gamma}$ and $\lambda_{2}=\frac{-1-\sqrt{1-4 \gamma \delta}}{2 \gamma}$. For them $\left|\lambda_{1}\right|<\left|\lambda_{2}\right|$ holds. This follows from the fact that $\mid-1+$ $\sqrt{1-4 \gamma \delta}|<|-1-\sqrt{1-4 \gamma \delta}|$. Eigenvectors can be computed from the matrix form of (4.1) which is:

$$
\binom{x_{t}}{x_{t+1}}=\left(\begin{array}{cc}
0 & 1  \tag{4.3}\\
\frac{-\delta}{\gamma} & \frac{-1}{\gamma}
\end{array}\right) \cdot\binom{x_{t-1}}{x_{t}}
$$

From the equation for the eigenvector of the matrix in (4.3) $v^{2}=\lambda v^{1}$ we can compute that eigenvectors are $\left(1, \lambda_{1}\right)$ and $\left(1, \lambda_{2}\right)$ generating the lines

$$
\begin{equation*}
x_{t+1}=\lambda_{1} x_{t} \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{t+1}=\lambda_{2} x_{t} . \tag{4.5}
\end{equation*}
$$

The stationary equilibrium $\bar{x}=0$ is a source if $\left|\lambda_{1}\right|>1$ (i.e. $\left|\lambda_{2}\right|>\left|\lambda_{1}\right|>1$ ). $\bar{x}$ is then unstable, i.e. all the orbits diverges unless the economy was in the stationary equilibrium at the beginning.

If $\left|\lambda_{1}\right|<1<\left|\lambda_{2}\right|, \bar{x}=0$ is a saddle and from the theory of rational expectations we know that if one basic condition holds, then there exists a unique convergent orbit.


Figure 12: Determinacy in the case of a saddle.

Precisely this condition is that the convergent line defined by (4.4) projects to the axis of given variable $x_{t-1}$ surjectively along the axis of the free variable $x_{t}$. The existence of the unique convergent orbit follows from the fact (see figure 12) that line determined by (4.4) and axis $x_{t-1}$ are not perpendicular (scalar product of $\left(1, \lambda_{1}\right)$ and $(1,0)$ is 1 ). Therefore if we know $x_{-1}$ there is a unique $x_{0}=\lambda_{1} x_{-1}$, which gives a unique sequence converging to $\bar{x}=0$ satisfying (4.1). It is interesting that the second eigenvector is not perpendicular to $(1,0)$ as well, and also both eigenvectors are not perpendicular to the second axis $(0,1)$ apart from the specific case when $\delta=0$ (i.e. the current state does not depend on the past states), when scalar products equal $\frac{-1 \pm 1}{2 \gamma}$.

At last, if $\left|\lambda_{1}\right|<\left|\lambda_{2}\right|<1$ then $\bar{x}=0$ is a sink and is indeterminate, so for any initial condition $x_{-1}$ near $\bar{x}$, there exist infinitely many perfect foresight equilibrium orbits staying in $V(\bar{x})$ for all $t>0$. Further we can show that if we look for the orbit in the form $x_{t}=c_{1} \lambda_{1}^{t}+c_{2} \lambda_{2}^{t}$, then either $c_{2} \neq 0$ and $x_{t+1}=\left(\lambda_{2}+\omega\left(t, x_{t}\right)\right) x_{t}$, where $\omega\left(t, \lambda_{1}, \lambda_{2}\right) \xrightarrow{t \rightarrow \infty} 0$ or $c_{2}=0$ and $x_{t+1}=\lambda_{1} x_{t}$ (i.e. the equilibrium orbit lays on the line determined by (4.4)).

Proof: The case $c_{2} \neq 0$ can be proved in this way:

$$
\begin{aligned}
\frac{x_{t+1}}{x_{t}} & =\frac{c_{1} \lambda_{1}^{t+1}+c_{2} \lambda_{2}^{t+1}}{c_{1} \lambda_{1}^{t}+c_{2} \lambda_{2}^{t}}=\lambda_{2}+\frac{c_{1} \lambda_{1}^{t+1}+c_{2} \lambda_{2}^{t+1}-c_{1} \lambda_{1}^{t} \lambda_{2}-c_{2} \lambda_{2}^{t+1}}{c_{1} \lambda_{1}^{t}+c_{2} \lambda_{2}^{t}}= \\
& =\lambda_{2}+c_{1}\left(\lambda_{1}-\lambda_{2}\right) \frac{\lambda_{1}^{t}}{c_{1} \lambda_{1}^{t}+c_{2} \lambda_{2}^{t}}=\lambda_{2}+c_{1}\left(\lambda_{1}-\lambda_{2}\right) \frac{1}{c_{1}+c_{2}\left(\frac{\lambda_{2}}{\lambda_{1}}\right)^{t}}=\lambda_{2}+\omega\left(t, \lambda_{1}, \lambda_{2}\right),
\end{aligned}
$$

where $\omega\left(t, \lambda_{1}, \lambda_{2}\right) \xrightarrow{t \rightarrow \infty} 0$. The case $c_{2}=0$ is obvious.
Interpretation of this property is that either the orbit lies whole on the eigenvector determined by $\lambda_{1}$ (i.e. the initial point lies on this line) or for other initial points the orbit becomes "attached" to the second eigenvector corresponding to $\lambda_{2}$.

We are interested in such combinations of $\gamma$ and $\delta$, that $\bar{x}=0$ will be a saddle, because


Figure 13: Saddle (the darkest region), source (the middle dark region) and sink (the lightest region) depending on the combinations of $\gamma$ and $\delta$. The region with vertical (horizontal) lines denotes such $\gamma$ and $\delta$ that the linear map (4.1) has two complex eigenvalues and $\bar{x}=0$ is asymptotically stable (unstable).
then a unique equilibrium orbit which converges to this stationary equilibrium exists. In the sink case there are infinitely many converging orbits. The source case is not interesting, because in that case economy explodes and there is no converging orbit. We therefore decompose the plane of parameters $(\gamma, \delta)$ into regions with different types of equilibrium orbits, i.e find conditions on $\gamma$ and $\delta$ under which $\bar{x}=0$ is a sink, a saddle or a source (figure 13).

For real roots of (4.2), discriminant $1-4 \gamma \delta$ must be nonnegative. To have two distinct eigenvalues, even positive. The equation $\delta=\frac{1}{4 \gamma}$ is the boundary, which partitions the plane $R^{2}$ into three areas (see figure 14), two giving complex eigenvalues and one between them with real eigenvalues.

Regions with zero, one or two real eigenvalues of modulus different from 1 are separated by the set of points $(\gamma, \delta)$ for which one eigenvalue is 1 or -1 . Denote $M$ the set where at least one eigenvalue has modulus 1. For the two roots of (4.2) $\lambda_{1}$ and $\lambda_{2}$ following Vièta's


Figure 14: Three areas of $R^{2}$, two of complex eigenvalues and one of real eigenvalues.
formulas:

$$
\begin{align*}
\lambda_{1}+\lambda_{2} & =\frac{-1}{\gamma}  \tag{4.6}\\
\lambda_{1} \lambda_{2} & =\frac{\delta}{\gamma} \tag{4.7}
\end{align*}
$$

hold. If $\lambda_{1}=1$ then Vièta's formulas give:

$$
\begin{align*}
1+\lambda_{2} & =\frac{-1}{\gamma}  \tag{4.8}\\
\lambda_{2} & =\frac{\delta}{\gamma} \tag{4.9}
\end{align*}
$$

and after solving of (4.8), (4.9) we obtain the set of $\gamma$ and $\delta$ such that $1+\gamma+\delta=0$ holds, the eigenvalues being $-1-\frac{1}{\gamma}$ and 1 . This case is interesting also from another point of view. The system (2.1) with condition $1+\gamma+\delta=0$ has arbitrary $\bar{x}$ as a stationary equilibrium. Moreover the relative weights of the future and past give together the weight 1. By analogy we can rewrite the formulas for the eigenvalue -1 :

$$
\begin{align*}
-1+\lambda_{2} & =\frac{-1}{\gamma}  \tag{4.10}\\
-\lambda_{2} & =\frac{\delta}{\gamma} \tag{4.11}
\end{align*}
$$

and obtain the equation $-1+\gamma+\delta=0$. This describes the case when the sum of the relative weights of future and past equals -1 . The eigenvalues are -1 and $\frac{-1}{\gamma}+1$. Therefore $M=\{(\gamma, \delta) ; 1+\gamma+\delta=0 \vee-1+\gamma+\delta=0\}$.

Thus $M$ contains all $(\gamma, \delta)$ for which $\bar{x}=0$ is nonhyperbolic and the eigenvalues are real. Conditions on $\gamma$ and $\delta$ for which $\lambda_{1}, \lambda_{2}$ are real and $\bar{x}=0$ is hyperbolic are stated
in the following lemma in points (i), (ii), (iii). And conditions that the eigenvalues $\lambda_{1}, \lambda_{2}$ are complex numbers and $\bar{x}=0$ is again hyperbolic are in the points (iv), (v).

Lemma 2 The stationary equilibrium $\bar{x}=0$ of the difference equation (4.1) is:
(i) saddle, if $\delta \in(-1-\gamma, 1-\gamma)$ and $\gamma \in R$,
(ii) sink, if $\delta \in\left(\frac{1}{4 \gamma},-1-\gamma\right)$ and $\gamma \in\left(-\infty, \frac{-1}{2}\right)$ or if $\delta \in\left(1-\gamma, \frac{1}{4 \gamma}\right)$ and $\gamma \in\left(\frac{1}{2}, \infty\right)$,
(iii) source, if $\delta \in(1-\gamma, \infty)$ and $\gamma \in\left(-\infty, \frac{1}{2}\right)$ or if $\delta \in(-\infty,-1-\gamma)$ and $\gamma \in\left(\frac{-1}{2}, \infty\right)$.
(iv) asymptotically stable and eigenvalues are complex, if $\gamma<\frac{-1}{2}$ and $\delta \in\left(\gamma, \frac{1}{4 \gamma}\right)$, or $\gamma>\frac{1}{2}$ and $\delta \in\left(\frac{1}{4 \gamma}, \gamma\right)$
(v) unstable and eigenvalues are complex, if $\gamma<0$ and $\delta \in(-\infty, \gamma)$, or if $\gamma>0$ and $\delta \in(\gamma, \infty)$.

Proof: First we concentrate on the real eigenvalues. We will utilize the following interpretation of points (i), (ii), (iii). Equivalent of (i) is to say that the points $(\gamma, \delta)$ lies between the lines $-1+\gamma+\delta=0$ and $1+\gamma+\delta=0$. Equally (ii) can be interpreted as 1. $(\gamma, \delta)$ lies above the line $-1+\gamma+\delta=0$ and $\gamma>\frac{1}{2}$ or 2 . $(\gamma, \delta)$ lies below $1+\gamma+\delta=0$ and $\gamma<\frac{-1}{2}$. And at last (iii) is in other words such $(\gamma, \delta)$ that 1 . this point lies above $-1+\gamma+\delta=0$ and $\gamma<\frac{1}{2}$ or 2 . it lies below $1+\gamma+\delta=0$ and $\gamma>\frac{-1}{2}$.

We first show radial symmetry in absolute values of the eigenvalues with respect to the point $(0,0)$, i.e. the eigenvalues change their sign, when we exchange $\gamma$ for $-\gamma$ and $\delta$ for $-\delta$. If the parameters in (4.6) and (4.7) are $-\gamma$ and $-\delta$ instead of $\gamma$ and $\delta$, the following formulas hold for the roots of the equation (4.2):

$$
\begin{align*}
\bar{\lambda}_{1}+\bar{\lambda}_{2} & =\frac{1}{\gamma}  \tag{4.12}\\
\bar{\lambda}_{1} \bar{\lambda}_{2} & =\frac{\delta}{\gamma} . \tag{4.13}
\end{align*}
$$

Comparing (4.6) with (4.12) and (4.7) with (4.13), we see that $\bar{\lambda}_{1}=-\lambda_{1}$ and $\bar{\lambda}_{2}=-\lambda_{2}$ solve (4.12) and (4.13). The sink, saddle or source are determined upon the absolute values and thus thank to the symmetry in absolute value it is sufficient to prove the lemma for $\gamma>0$. For such $\gamma$ the two eigenvalues $\lambda_{1,2}=\frac{-1 \pm \sqrt{1-4 \gamma \delta}}{2 \gamma}$ are continuous functions of $\gamma$ and $\delta$. The set M partitions the half plane $\gamma>0$ into four regions $1,2,3$ and 4 which are depicted in the figure (15), since moduli do not change their position with respect to $\pm 1$ within these regions. Now it is sufficient to pick one point from each area and compute the roots to find out the moduli of eigenvalues. Or, we can alternatively compute the derivatives of the eigenvalues for the points from $M . \lambda$ is given implicitly by $F(\delta, \lambda)=\gamma \lambda^{2}+\lambda+\delta=0$, so its derivative with respect to $\delta$ is:

$$
\frac{\partial \lambda}{\partial \delta}=-\frac{\frac{\partial F}{\partial \delta}}{\frac{\partial F}{\partial \lambda}}=\frac{-1}{2 \gamma \lambda+1}
$$

When we substitute the definitions of $\lambda_{1,2}$ we see that $\lambda_{1}$ is decreasing and $\lambda_{2}$ is increasing


Figure 15: Regions of sink, saddle and source.
with respect to $\delta$ :

$$
\begin{aligned}
& \frac{\partial \lambda_{1}}{\partial \delta}=\frac{-1}{2 \gamma \frac{-1+\sqrt{1-4 \gamma \delta}}{2 \gamma}+1}=\frac{-1}{\sqrt{1-4 \gamma \delta}}<0, \\
& \frac{\partial \lambda_{2}}{\partial \delta}=\frac{-1}{2 \gamma \frac{-1-\sqrt{1-4 \gamma \delta}}{2 \gamma}+1}=\frac{1}{\sqrt{1-4 \gamma \delta}}>0 .
\end{aligned}
$$

We repeat the known facts:

1. If $\gamma+\delta+1=0$ then $\lambda_{1}=1, \lambda_{2}=-1-\frac{1}{\gamma}$.
2. If $\gamma+\delta-1=0$ and $0<\gamma<\frac{1}{2}$, then $\lambda_{1}=-1$ and $\lambda_{2}=1-\frac{1}{\gamma}$.
3. If $\gamma+\delta-1=0$ and $\gamma>\frac{1}{2}$, then $\lambda_{2}=-1$ and $\lambda_{1}=1-\frac{1}{\gamma}$.

Thus for $0<\gamma<\frac{1}{2}, \lambda_{2}$ is always smaller than -1 , i.e. $\left|\lambda_{2}\right|>1$. For such $\gamma \lambda_{1}$ decreases with increasing $\delta$ and equals 1 on the line $\gamma+\delta+1=0$ and -1 on the line $\gamma+\delta-1=0$. Therefore below $\gamma+\delta+1=0$ and above the line $\gamma+\delta-1=0$ is $\left|\lambda_{1}\right|>1$, between them it is $\left|\lambda_{1}\right|<1$. For $0<\gamma<\frac{1}{2}$ are (i), (ii), (iii) proved.

We continue with $\gamma>\frac{1}{2}$. Below $\gamma+\delta+1=0 \lambda_{1}>1$ holds and above this line $\left|\lambda_{1}\right|<1$, because eigenvalues are continuous and change their moduli only for $(\gamma, \delta) \in M$. Similarly $\lambda_{2}$ is increasing and on $\gamma+\delta-1=0$ equals -1 . Altogether for $\gamma>\frac{1}{2}$ and points below $\gamma+\delta+1=0\left|\lambda_{1}\right|>1$ and $\left|\lambda_{2}\right|>1$ hold, between the lines $\gamma+\delta+1=0$ and $\gamma+\delta-1=0$ $\left|\lambda_{1}\right|<1$ and $\left|\lambda_{2}\right|>1$ hold and finally above $\gamma+\delta-1=0\left|\lambda_{1}\right|<1$ and $\left|\lambda_{2}\right|>1$ hold. The proof is for real eigenvalues finished.

The eigenvalues of (4.1) are complex when $\gamma<0$ and $\delta<\frac{1}{4 \gamma}$ or $\gamma>0$ and $\delta>\frac{1}{4 \gamma}$. The eigenvalues are $\frac{-1 \pm i \sqrt{4 \gamma \delta-1}}{2 \gamma}$ with the real part $\frac{-1}{2 \gamma}$ and the complex part $\frac{ \pm \sqrt{4 \gamma \delta-1}}{2 \gamma}$. Thus we
determine the stability upon the value of:

$$
\begin{equation*}
\sqrt{\left(\frac{-1}{2 \gamma}\right)^{2}+\left(\frac{\sqrt{4 \gamma \delta-1}}{2 \gamma}\right)^{2}}=\sqrt{\frac{\delta}{\gamma}} \tag{4.14}
\end{equation*}
$$

If $|\delta|>|\gamma|$ then $\sqrt{\frac{\delta}{\gamma}}>1$ and $\bar{x}=0$ is unstable. If $|\delta|<|\gamma|$ then $\sqrt{\frac{\delta}{\gamma}}<1$ and $\bar{x}=0$ is asymptotically stable. Hereby the proof is finished also for the complex eigenvalues.

We have analyzed all the points $(\gamma, \delta)$ in the plane, so we know which $\gamma$ and $\delta$ are such that $\bar{x}=0$ will be a saddle stationary equilibrium.

### 4.2 Introduction of the learning system

We now include the expectation formation in the model and introduce the general learning system for finding the parameter as it was done in [5]. From now on, we consider the saddle case. Then the unique convergent orbit under the perfect foresight can be described as follows:

$$
\begin{equation*}
x_{t}=\beta x_{t-1} \tag{4.15}
\end{equation*}
$$

for every $x_{t-1}$ in neighborhood of $\bar{x}=0$ and every $t \geq 0$ and the growth rate $\beta$ equals $\lambda_{1}\left(\left|\lambda_{1}\right|<1\right)$ for the saddle case. For both $\lambda_{1}$ and $\lambda_{2}$ is (4.15) the invariant line in the $\left(x_{t-1}, x_{t}\right)$-space. If the initial point lies on the line $x_{t}=\lambda_{1} x_{t-1}$ then dynamics evolve along the eigenvector and converges to the stationary equilibrium $\bar{x}=0$. Individuals form their expectations by iterating twice (Property 4 in Subsection 3.3) at time $t$, therefore their expectation is $x_{t+1}^{e}=\beta^{2} x_{t-1}$. The economy is governed by (2.1), therefore the current state is

$$
\begin{equation*}
x_{t}=-\left(\gamma \beta^{2}+\delta\right) x_{t-1}, \tag{4.16}
\end{equation*}
$$

where we have substituted their expectation $x_{t+1}^{e}=\beta^{2} x_{t-1}$ into (2.1). When $\beta$ equals $\lambda_{1}$ or $\lambda_{2}$ from (4.2) $-\left(\gamma \beta^{2}+\delta\right)=\lambda_{i}, i=1,2$. Hereby the expectation will fulfill.

To maintain the economy on the equilibrium orbit in the saddle case, the agents have to know the parameter $\lambda_{1}$. When the individuals do not know the model and rules how the economy develops, they try to find the parameter $\lambda_{1}$ by a learning process. They form their expectation about $x_{t+1}$, estimating the parameter $\beta_{t}$ at time $t$ and revise this parameter at the beginning of the period $t+1$ according to the observed error $x_{t}-\beta_{t} x_{t-1}$. In [5] the following class of learning algorithms is considered:

$$
\begin{align*}
x_{t} & =-\left(\gamma \beta_{t}^{2}+\delta\right) x_{t-1}  \tag{4.17}\\
\beta_{t+1} & =\beta_{t}+\alpha_{t} h(t) x_{t-1}\left(x_{t}-\beta_{t} x_{t-1}\right) \tag{4.18}
\end{align*}
$$

where $\alpha_{t}>$ tends toward 0 as $t$ becomes large, and $h(t)$ is a function of past history of the state variable. In [5] it is argued that $h(t)>0$ because if the individuals overestimate the
actual growth rate $x_{t}<\beta_{t} x_{t-1}$ they should set $\beta_{t+1}<\beta_{t}$ in (4.18). Here we note, that the argument is right, but the reason is insufficient. For example, if $x_{t}, x_{t-1}$ are negative and $\beta_{t}$ is positive then $h(t)$ should be positive too ( $\beta$ will increase (4.18)) but the inequality $x_{t}<\beta_{t} x_{t-1}$ should be called underestimation now.

We will use the simplified learning process of the latter (4.17), (4.18) and study its dynamics. We substitute (4.17) into (4.18) and obtain the equation for $\beta_{t+1}$ :

$$
\begin{equation*}
\beta_{t+1}=\beta_{t}+\alpha_{t} h(t) x_{t-1}\left[-\left(\gamma \beta_{t}^{2}+\delta\right) x_{t-1}-\beta_{t} x_{t-1}\right] \tag{4.19}
\end{equation*}
$$

or

$$
\begin{equation*}
\beta_{t+1}=\beta_{t}+\alpha_{t} h(t) x_{t-1}^{2}\left(-\gamma \beta_{t}^{2}-\beta_{t}-\delta\right) . \tag{4.20}
\end{equation*}
$$

In [5] the following particular case of this system is described: $\alpha_{t}=\gamma_{t} / t$ and $h(t)=$ $t /\left(\gamma_{t} x_{t-1}^{2}+\ldots+\gamma_{0} x_{-1}^{2}\right)$ with forgetting factors $\gamma_{s} \geq 0$ for $s=0, \ldots, t$. If we use $h(t)=$ $t /\left|x_{t-1}\right|, \alpha_{t}=\alpha / t$ with a parameter $\alpha \in(0,1)$ and the substitution $b_{t}=\beta_{t+1}$, we will obtain the following system:

$$
\begin{align*}
x_{t} & =-\left(\gamma b_{t-1}^{2}+\delta\right) x_{t-1}  \tag{4.21}\\
b_{t} & =b_{t-1}+\alpha\left|x_{t-1}\right|\left(-\gamma b_{t-1}^{2}-b_{t-1}-\delta\right) \tag{4.22}
\end{align*}
$$

If we use $h(t)=t / x_{t-1}^{2}$ the system becomes:

$$
\begin{align*}
x_{t} & =-\left(\gamma b_{t-1}^{2}+\delta\right) x_{t-1}  \tag{4.23}\\
b_{t} & =b_{t-1}+\alpha\left(-\gamma b_{t-1}^{2}-b_{t-1}-\delta\right) . \tag{4.24}
\end{align*}
$$

Both $h(t)$ are positive.
At first we study the fixed points and their stability of the first learning system (4.21) and (4.22), We compute the stationary equilibria:

$$
\begin{aligned}
& \bar{x}=-\left(\gamma \bar{b}^{2}+\delta\right) \bar{x} \\
& \bar{b}=\bar{b}+\alpha|\bar{x}|\left(-\gamma \bar{b}^{2}-\bar{b}-\delta\right) .
\end{aligned}
$$

They are $(0, \bar{b})$ for arbitrary $\bar{b}$. For the linearization we need to compute the derivative with respect to $x$ of $v(x)=b+\alpha|x|\left(-\gamma b^{2}-b-\delta\right)$ in $x=0$. Because the derivative does not exist, we can avoid this problem by using the derivative from the left anf from the right. Then the linearization in $(x, b)$ in $x<0$ subspace is:

$$
\left(\begin{array}{cc}
-\gamma b^{2}-\delta & -2 \gamma b x \\
\alpha\left(\gamma b^{2}+b+\delta\right) & 1+\alpha x(2 \gamma b+1)
\end{array}\right)
$$

and in the point $(0, \bar{b})$ it is:

$$
\left(\begin{array}{cc}
-\gamma \bar{b}^{2}-\delta & 0 \\
\alpha\left(\gamma \bar{b}^{2}+\bar{b}+\delta\right) & 1
\end{array}\right)
$$

The eigenvalues can be computed from the equation

$$
\left(-\gamma \bar{b}^{2}-\delta-\bar{\lambda}\right)(1-\bar{\lambda})=0
$$

The eigenvalues in the stationary equilibria $(0, \bar{b})$ are 1 and $-\gamma \bar{b}^{2}-\delta$. The eigenvectors are $\binom{0}{1}$ for the eigenvalue 1 and $\binom{1+\gamma \bar{b}^{2}+\delta}{-\alpha\left(\gamma \bar{b}^{2}+\bar{b}+\delta\right)}$ for $-\gamma \bar{b}^{2}-\delta$. By analogy we can compute the linearization in $x>0$ subspace and obrtain the same eigenvalues and aigenvectors $\binom{0}{1}$ for the eigenvalue 1 and $\binom{1+\gamma \bar{b}^{2}+\delta}{\alpha\left(\gamma \bar{b}^{2}+\bar{b}+\delta\right)}$ for $-\gamma \bar{b}^{2}-\delta$. We see that for $\bar{b}$, such that $\left|-\gamma \bar{b}^{2}-\delta\right|>1$, these stationary equilibria $(0, \bar{b})$ are unstable. For other $\bar{b}$ we cannot claim anything about the stability.

Now we return to the second system (4.23) and (4.24), which is a little bit easier. We compute the stationary equilibria by analogy from the equations:

$$
\begin{aligned}
& \bar{x}=-\left(\gamma \bar{b}^{2}+\delta\right) \bar{x} \\
& \bar{b}=\bar{b}+\alpha\left(-\gamma \bar{b}^{2}-\bar{b}-\delta\right)
\end{aligned}
$$

which are two points $\left(0, \lambda_{1}\right)$ and $\left(0, \lambda_{2}\right)$. The linearization

$$
\left(\begin{array}{cc}
-\gamma b^{2}-\delta & -2 \gamma b x \\
0 & 1+\alpha(-2 \gamma b-1)
\end{array}\right)
$$

in these stationary equilibria is:

$$
\left(\begin{array}{cc}
-\gamma \lambda_{i}^{2}-\delta & 0 \\
0 & 1+\alpha\left(-2 \gamma \lambda_{i}-1\right)
\end{array}\right)
$$

for $i=1,2$. Before computing the eigenvalues of this system, we simplify this linearization using the properties of $\lambda_{1}$ and $\lambda_{2}$. First we know that $-\gamma \lambda_{i}^{2}-\delta=\lambda_{i}$ because $\lambda_{1}$ and $\lambda_{2}$ are the roots of (4.2) and further we know that $\lambda_{1,2}=\frac{-1 \pm \sqrt{1-4 \gamma \delta}}{2 \gamma}$ therefore $1+\alpha\left(-2 \gamma \lambda_{i}-1\right)=$ $1+\alpha(1 \mp \sqrt{1-4 \gamma \delta}-1)=1 \mp \alpha \sqrt{1-4 \gamma \delta}$. Now the eigenvalues of the linearization are evident and they are $\lambda_{1}$ and $1-\alpha \sqrt{1-4 \gamma \delta}$ in the point $\left(0, \lambda_{1}\right)$ and similarly $\lambda_{2}$ and $1+\alpha \sqrt{1-4 \gamma \delta}$ in $\left(0, \lambda_{2}\right)$. Thus we can see that $\left(0, \lambda_{2}\right)$ is always unstable stationary equilibrium and $\left(0, \lambda_{1}\right)$ is a sink for a sufficiently small parameter $\alpha<\frac{2}{\sqrt{1-4 \gamma \delta}}$, otherwise it is a saddle. The eigenvectors are in both stationary equilibria the same and they are $(1,0)$ and $(0,1)$.

### 4.3 Analysis of the dynamics of the first learning system

The dynamics of the learning system (4.21) and (4.22) is interesting also because its linearization in fixed points has eigenvalue 1. First we depict the orbits of the numerical


Figure 16: An example of the dynamics with parameters: $\gamma=\frac{-5}{3}, \delta=\frac{91}{60}, \alpha=0.2, b_{0}=$ $1.18, x_{0}=2$.


Figure 17: An example of the dynamics with parameters: $\gamma=\frac{5}{3}, \delta=\frac{-91}{60}, \alpha=0.2, b_{0}=$ $-0.94, x_{0}=2$.
simulations of the dynamics in figures 16 and 17. In the figure 16 the eigenvalues of (4.1) are 1.3 and -0.7 . In this dynamics $x_{t}$ converges to numerical zero and $b_{t}$ to -0.700027 in 300 iterations. In the second figure 17 the eigenvalues has opposite signs, i.e. they are -1.3 and 0.7, and in the last iteration $x_{t}$ was also numerical zero and $b_{t}$ was 0.700003 . We consider the dynamics in these figures as an interesting example on how the economy moves to a stationary equilibrium $(0, \bar{b})\left(\left|x_{t}\right|\right.$ decreases $)$ and it turns out that the economy moves off from this stationary equilibrium ( $\left|x_{t}\right|$ increases) for some time to converge later to another stationary equilibrium $\left(0, b^{\prime}\right)$.

Firstly we are interested in the points where $x_{t}$ is changing its monotonicity in absolute value, from decreasing to increasing and vice versa. For this reason we specify several crucial values of $b_{t}$ and prove their order. So look at such $b_{t}$ that $x_{t}=x_{t-1}$. From (4.21) for $b_{t}$ $-\gamma b_{t}^{2}-\delta=1$, i.e. $b_{t}= \pm \sqrt{\frac{-1-\delta}{\gamma}}$, must hold. We denote $B_{1}=\sqrt{\frac{-1-\delta}{\gamma}}$ and $B_{2}=-\sqrt{\frac{-1-\delta}{\gamma}}$. Assume $\gamma$ is positive. If $B_{2}<b_{t}<B_{1}$, then $-\gamma b_{t}^{2}-\delta>1$, therefore from (4.21) we see that $\left|x_{t}\right|$ is increasing for such $b_{t}$. For $\tilde{B}_{2}<b_{t}<B_{2}<0$ and $\tilde{B}_{1}>b_{t}>B_{2}>0$ $-1<-\gamma b_{t}^{2}-\delta<1$ holds, where $\tilde{B}_{1}=\sqrt{\frac{1-\delta}{\gamma}}$ and $\tilde{B}_{2}=-\sqrt{\frac{1-\delta}{\gamma}}$ and therefore for these $b_{t}$ $\left|x_{t}\right|$ is decreasing. Finally if $b_{t}$ is greater than $\tilde{B}_{1}$ or smaller than $\tilde{B}_{2}$, then $\left|x_{t}\right|$ is increasing for $-\gamma b_{t}^{2}-\delta<-1$. This analysis of monotonicity of $\left|x_{t}\right|$ is depicted in figure 18. As a supplement to the figure 18 , we note that, when $-\gamma b_{t}^{2}-\delta=0$, then $x_{t+1}$ changes its sign, i.e. for $b_{t}= \pm \sqrt{\frac{-\delta}{\gamma}}$. Altogether, now it is interesting to see, in which order are the values of all these points.

For positive $\gamma$ and such $(\gamma, \delta)$ that $B_{1}, B_{2}, \tilde{B}_{1}, \tilde{B}_{2}, \sqrt{\frac{-\delta}{\gamma}}$ and $\sqrt{\frac{-\delta}{\gamma}}$ exist the following order holds:

$$
\begin{equation*}
\tilde{B}_{1}>\sqrt{\frac{-\delta}{\gamma}}>\lambda_{1}>B_{1}>B_{2}>-\sqrt{\frac{-\delta}{\gamma}}>\tilde{B}_{2}>\lambda_{2} \tag{4.25}
\end{equation*}
$$

For this order we can depict the eigenvectors as follows in figure 19.


Figure 18: Monotonicity of $x$ according to the $b_{t}$ for $\gamma$ positive.

Proof: Most of the inequalities in (4.25) are trivial, only these three inequalities are not evident:

$$
\begin{gather*}
\tilde{B}_{2}>\lambda_{2}  \tag{4.26}\\
\sqrt{\frac{-\delta}{\gamma}}>\lambda_{1}  \tag{4.27}\\
\lambda_{1}>B_{1} \tag{4.28}
\end{gather*}
$$

It can be shown in several ways. For example we can utilize the function $g\left(b_{t}\right)=\gamma b_{t}^{2}+\delta$ which is decreasing for $b_{t}$ negative. The values of this function in $\tilde{B}_{2}$ and $\lambda_{2}$ are:

$$
\begin{aligned}
& g\left(\tilde{B}_{2}\right)=\gamma \frac{1-\delta}{\gamma}+\delta=1 \\
& g\left(\lambda_{2}\right)=-\lambda_{2}
\end{aligned}
$$

where for computation of the value in $\lambda_{2}$ we used (4.2). We know that $\left|\lambda_{2}\right|>1>\left|\lambda_{1}\right|$. For positive $\gamma$ is $\lambda_{2}$ negative what arises from its definition $\left(\lambda_{2}=\frac{-1-\sqrt{1-4 \gamma \delta}}{2 \gamma}\right)$. Therefore $g\left(\lambda_{2}\right)>g\left(\tilde{B}_{2}\right)$ and $\tilde{B}_{2}>\lambda_{2}$.

By analogy we can prove the inequalities of (4.27) and (4.28). For positive $b_{t}$ this function is increasing. Values of $g\left(b_{t}\right)$ in $\sqrt{\frac{-\delta}{\gamma}}, \lambda_{1}$ and $B_{1}$ are following:

$$
\begin{aligned}
& g\left(\sqrt{\frac{-\delta}{\gamma}}\right)=0, \\
& g\left(\lambda_{1}\right)=-\lambda_{1} \\
& g\left(B_{1}\right)=-1
\end{aligned}
$$

the value in $\lambda_{1}$ follows from (4.2) as before. For proving the inequality (4.27) we utilize the fact that $\delta$ is negative to make $\sqrt{\frac{-\delta}{\gamma}}$ real. Further holds, that for negative $\delta$ and positive $\gamma$


Figure 19: Eigenvalue $-\gamma b^{2}-\delta$ and its corresponding eigenvector for $\gamma>0$. Eigenvalue 1 has corresponding eigenvector $(0,1)$
is $-1+\sqrt{1-4 \gamma \delta}$ positive, therefore $\lambda_{1}$ is positive. Finally for such $\lambda_{1} g\left(\sqrt{\frac{-\delta}{\gamma}}\right)>g\left(\lambda_{1}\right)$ holds, so $\sqrt{\frac{-\delta}{\gamma}}>\lambda_{1}$. The last inequality (4.28) follows from the property $\left|\lambda_{1}\right|<1<\left|\lambda_{2}\right|$, therefore $\lambda_{1}>B_{1}$. Thus for positive $\gamma$ the order is proved.

This study of monotonicity of $\left|x_{t}\right|$ can be repeated for negative $\gamma$. For $b_{t}$ such that $b_{t}>B_{1}$ or $b_{t}<B_{2}$ the expression $-\gamma b_{t}^{2}-\delta$ is smaller than -1 and therefore $\left|x_{t}\right|$ is increasing. For $B_{1}>b_{t}>\tilde{B}_{1}$ or $B_{2}<b_{t}<\tilde{B}_{2}$ the expression $-\gamma b_{t}^{2}-\delta$ is between -1 and 1 and therefore $\left|x_{t}\right|$ is decreasing. And at last when $\tilde{B}_{1}>b_{t}>\tilde{B}_{2}$ then $\left|x_{t}\right|$ increases, because $-\gamma b_{t}^{2}-\delta$ is greater than 1. The figure 18 represents also this case, but we would have to switch $B_{1}$ with $\tilde{B}_{1}$ and $B_{2}$ with $\tilde{B}_{2}$.

For $\gamma$ negative, there is another ordering, namely

$$
\begin{equation*}
\lambda_{2}>B_{1}>\sqrt{\frac{-\delta}{\gamma}}>\tilde{B}_{1}>0>\tilde{B}_{2}>\lambda_{1}>-\sqrt{\frac{-\delta}{\gamma}}>B_{2} . \tag{4.29}
\end{equation*}
$$

Proof: Again we need to show the following inequalities:

$$
\begin{gather*}
\lambda_{2}>B_{1},  \tag{4.30}\\
\tilde{B}_{2}>\lambda_{1},  \tag{4.31}\\
\lambda_{1}>-\sqrt{\frac{-\delta}{\gamma}} . \tag{4.32}
\end{gather*}
$$

The validity of the inequality (4.30) can be shown through the function $g\left(b_{t}\right)$ too, but we have to note that for negative $\gamma$ the monotonicity changes (for negative values it is increasing, for positive decreasing).

$$
\begin{aligned}
& g\left(\lambda_{2}\right)=-\lambda_{2} \\
& g\left(B_{1}\right)=-1
\end{aligned}
$$

$\lambda_{2}$ is positive for negative $\gamma$ and further we know $\left|\lambda_{2}\right|>1$. Therefore $g\left(\lambda_{2}\right)<g\left(B_{1}\right)$, so we have shown (4.30). We continue with the inequalities (4.31) and (4.32).

$$
\begin{aligned}
& g\left(\tilde{B}_{2}\right)=1 \\
& g\left(\lambda_{1}\right)=-\lambda_{1} \\
& g\left(-\sqrt{\frac{-\delta}{\gamma}}\right)=0
\end{aligned}
$$

We presume positive $\delta$ to have real $-\sqrt{\frac{-\delta}{\gamma}}$, therefore $-1+\sqrt{1-4 \gamma \delta}>0$ and $\lambda_{1}$ is negative.
Moreover $0<\left|\lambda_{1}\right|<1$, so $g\left(\tilde{B}_{2}\right)>g\left(\lambda_{1}\right)>g\left(-\sqrt{\frac{-\delta}{\gamma}}\right)$. Therefore $\tilde{B}_{2}>\lambda_{1}>-\sqrt{\frac{-\delta}{\gamma}}$.

Thereby is the proof of the order for negative $\gamma$ finished.
To learn how the dynamics will move in one iteration we have to supplement a similar study of $b_{t}$. We concentrate on the quadratic function $f\left(b_{t}\right)=\gamma b_{t}^{2}+b_{t}+\delta$. For the roots of $f\left(b_{t}\right)=0(4.22), b_{t}=b_{t-1}$ holds. These roots are the same as of the equation (4.2), i.e. the eigenvalues $\lambda_{1}$ and $\lambda_{2}$ from which the individuals want to find $\lambda_{1}$. If $\gamma$ is positive and $b_{t}$ is between $\lambda_{1}$ and $\lambda_{2}, f\left(b_{t}\right)<0$ and thus $b_{t+1}>b_{t}$. If $b_{t}$ is not in this interval, it will decrease. The contrary holds for $\gamma$ negative.

It is useful now to draw a figure rather then to describe it further in words. In these phase portraits (figures 20 and 21) is shown how the initial point from the area where $b_{0}$ is between $-\sqrt{-\delta / \gamma}$ and $B_{2}$ for positive $\gamma$ may converge to $\left(0, \lambda_{1}\right)$ in a " $S$ "-shaped orbit. The same case is for $\gamma$ negative and initial $b_{0}$ between $\sqrt{-\delta / \gamma}$ and $\tilde{B}_{1}$.

We now study the case of positive $\gamma$ and ask when the orbit will converge to some point where $b_{t}<B_{2}$ and on the contrary when the orbit jump over $B_{2}$ (will have the interesting twice bent orbit). The jump over $B_{2}$ in one step can be easily computed straight from the model (4.22) giving the condition for example on the initial point $b_{0}$ (if others parameters are given). This same holds for the parameter $\alpha$ if we already know $b_{0}$. We simply set $b_{t}=B_{2}$ and compute the necessary $b_{0}$ or $\alpha$ that the orbit will jump over $B_{2}$ in one step. But it is much more interesting whether it will jump over $B_{2}$ and we do not ask about the number of steps. This was done numerically and can be seen in the figure 22 . The initial points of two depicted orbits are $(2,-0.95)$ (this orbit converge to $(0,-0.749)$ ) and $(2,-0.91)$ (this orbit converge to $(0,0.70003)$ ).

The second interesting numerical simulation was computing of such $\left(x_{0}, b_{0}\right)$ that they will converge to some point $\left(x^{\prime}, b^{\prime}\right)$, where $x^{\prime}$ is close to zero and $b^{\prime} \in\left(B_{1}, \lambda_{1}\right)$. This simulation is in the figure 23. It is important now to emphasize that these numerical simulations are sensitive to the parameter $\alpha$ which the individuals has set and which describes the weight of the previous error. The smaller the parameter $\alpha$ is the greater $x_{0}$ appear in the area of initial points, which is depicted in figure 23, and vice versa. For parameters given in figure 17 the simulations show that for arbitrary $\alpha \in(0,1)$, the smallest $b_{0}$ from this area is bounded from below by approximately -0.026667 . Thus for greater $\alpha$, the depicted area grows mostly in the $x_{0}$ direction.

This shows the existence of such areas, that if the initial point is from one of these areas then it converges to a point $(0, b)$ where $b \neq \lambda_{1}$. We know that if for the eigenvalue $\left|-\gamma b^{2}-\delta\right|>1$ holds, then $(0, b)$ is unstable fixed point. The individuals will never learn $b$ from the following intervals $\left(B_{1}, B_{2}\right),\left(\tilde{B}_{1}, \infty\right)$ and $\left(-\infty, \tilde{B}_{2}\right)$. Therefore if the economy converges, the individuals will learn the parameter $b \in\left(\tilde{B}_{1}, B_{1}\right)$ or $b \in\left(B_{2}, \tilde{B}_{2}\right)$. We have shown in the numerical simulation depicted in figure 22 that there exist a curve q dividing the area $(x, b)$ for $b \in\left(B_{2}, \tilde{B}_{2}\right)$. Orbits of all the initial points above the line q will converge


Figure 20: Phase portrait for $\gamma$ positive (where the sign of $x_{t}$ is changing, arrows were replaced by the lines with a square at the end).


Figure 21: Phase portrait for $\gamma$ negative (where the sign of $x_{t}$ is changing, arrows were replaced by the lines with a square at the end).


Figure 22: Initial points below the curve q converge to $(0, b), b<B_{2}<0$ ( $y_{0}$ orbit), and those above this line has " S "-shaped orbit converging to $\left(0, b^{\prime}\right)$, where $B_{1}<b^{\prime}<\sqrt{\frac{-\delta}{\gamma}}$ ( $x_{0}$ orbit). Parameters of this simulation are the same as in the figure 17 .


Figure 23: Initial points from the shaded area
converge to $(0, b), B_{1}<b<\lambda_{1}$. Parameters
of this simulation are the same as in the figure
Figure 23: Initial points from the shaded area
converge to $(0, b), B_{1}<b<\lambda_{1}$. Parameters
of this simulation are the same as in the figure
Figure 23: Initial points from the shaded area
converge to $(0, b), B_{1}<b<\lambda_{1}$. Parameters
of this simulation are the same as in the figure 17. 7.
to $\left(0, b^{\prime}\right)$, where $B_{1}<b^{\prime}<\tilde{B}_{1}$ (i.e. possibly to $\lambda_{1}$ ). If ( $x_{0}, b_{0}$ ) will be below q , the orbits will converge to $x_{t}=0$ sooner than $b_{t}$ will be over $B_{2}$ and thus the individuals will never learn the parameter $\lambda_{1}$. But no matter what the parameter is, once $x_{0}$ is zero, the economy is in its stationary equilibrium. Secondly if the initial point is from the area depicted in figure 23, individuals will not learn the parameter $\lambda_{1}$, but $b \in\left(B_{1}, \lambda_{1}\right)$ instead. But also in this case, although the individuals has not set the economy on the equilibrium orbit, the economy is at last in its stationary equilibrium. We have shown that the individuals do not have to find the parameter $\lambda_{1}$.

### 4.4 Analysis of the dynamics of the second learning system

We return to the learning system (4.23) and (4.24). We remind that stationary equilibrium $\left(0, \lambda_{2}\right)$ is always a source and $\left(0, \lambda_{1}\right)$ is a sink for $\alpha<\frac{2}{\sqrt{1-4 \gamma \delta}}$ and otherwise it is a saddle.

At the beginning we have done several simulations. In figure 24 the eigenvalues of (4.1) are 1.3 and -0.7 . In this dynamics $x_{t}$ converges to numerical zero and $b_{t}$ to -0.7 in 300 iterations. In the second figure 25 the eigenvalues have opposite signs, i.e. they are -1.3 and 0.7, and in the last iteration $x_{t}$ was also numerical zero and $b_{t}$ was 0.7 . Note that the interesting "S-shaped" orbit occurs also if the individuals use this second learning system.

The analyses of one step move of difference equations (4.21) and (4.23) depending on the different input values of $b_{t}$ are identical. Thus the figure 18 holds also for this learning system. We remind, that this figure holds for positive $\gamma$ and for negative $\gamma$ there have to be interchanged $B_{1}$ with $\tilde{B}_{1}$ and $B_{2}$ with $\tilde{B}_{2}$.


Figure 24: An example of the dynamics with parameters: $\gamma=\frac{-5}{3}, \delta=\frac{91}{60}, \alpha=0.2, b_{0}=$ $1.28, x_{0}=2$.


Figure 25: An example of the dynamics with parameters: $\gamma=\frac{5}{3}, \delta=\frac{-91}{60}, \alpha=0.2, b_{0}=$ $-0.91, x_{0}=2$.

In the equation for $b_{t}(4.24)$, there is no more $x_{t-1}$ and the dynamics of $b_{t}$ can be studied separately. Using the transformation $b_{t}=y_{t+1}+\lambda_{2}$, (4.24) becomes:

$$
y_{t+1}+\lambda_{2}=y_{t}+\lambda_{2}-\alpha\left(\gamma y_{t}^{2}+2 \gamma \lambda_{2} y_{t}+\gamma \lambda_{2}^{2}+y_{t}+\lambda_{2}+\delta\right),
$$

and because $\lambda_{2}$ is the root of (4.2) we can simplify it to:

$$
y_{t+1}=y_{t}-\alpha\left(\gamma y_{t}^{2}+2 \gamma \lambda_{2} y_{t}+y_{t}\right) .
$$

We know that $\lambda_{2}=\frac{-1-\sqrt{1-4 \gamma \delta}}{2 \gamma}$, therefore it can be further rewritten as

$$
y_{t+1}=-\alpha \gamma y_{t}^{2}+y_{t}(1+\alpha \sqrt{1-4 \gamma \delta})
$$

To transform it into the logistic equation, we use the substitution $y_{t}=\frac{\mu}{\alpha \gamma} z_{t}$ where $\mu=$ $1+\alpha \sqrt{1-4 \gamma \delta}$ and obtain:

$$
\frac{\mu}{\alpha \gamma} z_{t+1}=-\alpha \gamma \frac{\mu^{2}}{(\alpha \gamma)^{2}} z_{t}^{2}+\mu \frac{\mu}{\alpha \gamma} z_{t}
$$

or after simplifying

$$
\begin{equation*}
z_{t+1}=\mu z_{t}\left(1-z_{t}\right) \tag{4.33}
\end{equation*}
$$

Our scalar parameter $\mu$ is always greater than 1. From Subsection 3.2 about the logistic equation we know that for such $\mu$ there exist two stationary equilibria $\bar{z}_{1}=0$ and $\bar{z}_{2}=$ $1-1 / \mu$. It can be verified by backward transformation that these points are $\lambda_{2}$ and $\lambda_{1}$ respectively. For $0 \leq \mu \leq 4$ one has $z_{t} \in\langle 0,1\rangle$. If $z_{t}=0$ then $b_{t}=\lambda_{2}$. If $z_{t}=1$, then $y_{t}=\frac{\mu}{\alpha \gamma}$ and $b_{t}=\lambda_{1}+\frac{1}{\alpha \gamma}$. Thus, for positive $\gamma, \lambda_{1}+\frac{1}{\alpha \gamma}>\lambda_{1}$ and orbits of initial points from the interval $\left\langle\lambda_{2}, \lambda_{1}+\frac{1}{\alpha \gamma}\right\rangle$ stay in this interval. Orbits of initial points which do not belong to this interval have a divergent orbit to $-\infty$ (lemma 1, figure 26). For $\gamma<0$ this interval changes to $\left\langle\lambda_{1}+\frac{1}{\alpha \gamma}, \lambda_{2}\right\rangle$. If we assume $\mu$ to be smaller than 4 , in fact we assume


Figure 26: Depicted (4.24) for $\gamma>0$ and parameters as in figure 25.
that the parameter $\alpha$, which weights the error, is bounded by

$$
\begin{equation*}
\alpha \leq \frac{3}{\sqrt{1-4 \gamma \delta}} \tag{4.34}
\end{equation*}
$$

for the parameters $\gamma$ and $\delta$ given by economy. Thus if want to determine the stability, from the part about logistic equation we see that $\bar{z}_{1}=0$ is unstable since $\mu>1$ and $\bar{z}_{2}=1-1 / \mu$ is asymptotically stable for $0<\alpha<\frac{2}{\sqrt{1-4 \gamma \delta}}$ (after transformation this corresponds to $1<\mu<3$ ) and unstable for $\frac{2}{\sqrt{1-4 \gamma \delta}}<\alpha \leq \frac{3}{\sqrt{1-4 \gamma \delta}}$ (this corresponds to $\mu>3$ ). If the individuals set the parameter $\alpha$ from the interval $\left(\frac{2}{\sqrt{1-4 \gamma \delta}}, \frac{3}{\sqrt{1-4 \gamma \delta}}\right\rangle \lambda_{1}$ becomes unstable as well as $\lambda_{2}$. For these values of $\alpha$ there exists only periodic orbits or chaos.

1. If $\alpha \in\left(\frac{2}{\sqrt{1-4 \gamma \delta}}, \frac{\sqrt{6}}{\sqrt{1-4 \gamma \delta}}\right)$ (i.e. $\left.\mu \in(3,1+\sqrt{6})\right)$, then for the parameters of our simulation in figure 25 it is $0.6<\alpha<\frac{3 \sqrt{6}}{10}$. Therefore if $\alpha<0.6$, then $\lambda_{1}=0.7$ is an asymptotically stable and if $\alpha=0.7$, then there is asymptotically stable periodic orbit $\{0.0420641,1.07222\}$.
2. If $\alpha \in\left(\frac{\sqrt{6}}{\sqrt{1-4 \gamma \delta}}, \frac{2.544}{\sqrt{1-4 \gamma \delta}}\right)$ (this corresponds to $\mu \in(1+\sqrt{6}, 3.544)$ ), then it is $\frac{3 \sqrt{6}}{10}<$ $\alpha<0.7632$ for values of our simulation. Simulations for $\alpha=0.75$ show a periodic orbit $\{1.4999,-0.228105,1.01543,0.102476\}$.
3. If $\alpha \in\left(\frac{2.544}{\sqrt{1-4 \gamma \delta}}, \frac{2.564}{\sqrt{1-4 \gamma \delta}}\right)$ (i.e. $\mu \in(3.544,3.564)$ ), then for our parameters it is $0.7632<$ $\alpha<0.7692$ and for example if $\alpha=0.765$ then the iterations converge to the periodic orbit $\{-0.268895,1.00487,0.108943,1.17072,-0.312125,0.962688,0.204853,1.15489\}$. 4.If $\alpha>\frac{2.570}{\sqrt{1-4 \gamma \delta}}($ i.e. $\mu>3.570$ ), then chaos appears (in our simulation for $\alpha>0.771$ ).

From this analysis we can conclude that the dynamics depends greatly on the correcting parameter $\alpha$. For $0<\alpha<\frac{2}{\sqrt{1-4 \gamma \delta}}$ there exist a unique asymptotically stable stationary equilibrium, which the individuals will find unless the initial guess $b_{0}$ was outside the inter-
$\operatorname{val}\left(\lambda_{2}, \lambda_{1}+\frac{1}{\alpha \gamma}\right)$ (for $\gamma$ positive and for $\gamma$ negative this interval is $\left.\left(\lambda_{1}+\frac{1}{\alpha \gamma}, \lambda_{2}\right)\right)$. Therefore, they will set the economy on the convergent orbit. For $\frac{2}{\sqrt{1-4 \gamma \delta}}<\alpha<\frac{3}{\sqrt{1-4 \gamma \delta}}$, this becomes more complicated because cycles occurs. If the periodic points are $\left\{p_{1}, p_{2}, \ldots, p_{k}\right\}$, from (4.23) we see that $x_{t+k-1}=(-1)^{k}\left(\gamma p_{1}+\delta\right)\left(\gamma p_{2}+\delta\right) \ldots\left(\gamma p_{k}+\delta\right) x_{t-1}$ and thus convergence depends on the moduli of the product $\left(\gamma p_{1}+\delta\right)\left(\gamma p_{2}+\delta\right) \ldots\left(\gamma p_{k}+\delta\right)$. We do not know whether the economy will converge to the stationary equilibrium $\bar{x}=0$. There is also the case that the dynamics of $b_{t}$ has the chaotic behavior and the dynamics of $x_{t}$ characterizing the economy is questionable.

The simulations show the possibility of both, convergence and divergence. In our example from figure 25 with the parameter $\alpha=0.7$ the asymtotically stable periodic orbit is given in point 1. above. For these values $\left|\left(\gamma p_{1}+\delta\right)\left(\gamma p_{2}+\delta\right)\right|<1$ holds and therefore the economy converges to its stationary equilibrium. On the contrary if we set $\gamma=2, \delta=-2, \alpha=0.55$, the asymtotically stable periodic orbit is $\{-1.14494,-0.173237\}$, and for these parameters $\left|\left(\gamma p_{1}+\delta\right)\left(\gamma p_{2}+\delta\right)\right|>1$ holds. Thus the economy do not converge to the stationary equilibrium $\bar{x}=0$ in this case.

Finally if $\alpha>\frac{3}{\sqrt{1-4 \gamma \gamma}}$, the orbit of $b_{0}$ does not stay in the interval $\left(\lambda_{2}, \lambda_{1}+\frac{1}{\alpha \gamma}\right)$, so there is a time $t^{\prime}$ for which $b_{t}^{\prime} \notin\left(\lambda_{2}, \lambda_{1}+\frac{1}{\alpha \gamma}\right)$ and from time $t^{\prime}$ the orbit diverges to $-\infty$ (lemma $1)$. So the economy diverges too.

## 5 Conclusion

A study of linear systems with rational expectations is rather known, because then Certainty Equivalence holds and the model is simplified into the difference system. The equilibrium orbit under Rational expectations then differs only in small random fluctuations from the solution of the difference system. In this master thesis we have tried to provide the analysis of the learning system with rational expectations, which is nonlinear.

We have supposed a linear model for the state of the economy $x$, which is unknown for the individuals. They form their rational expectations about the future by the formula $x_{t}=\beta_{t} x_{t-1}$. We have defined the values $\lambda_{1}$ and $\lambda_{2}$ and explained why we focus on the case when $\left|\lambda_{1}\right|<1<\left|\lambda_{2}\right|$. We have found the conditions on $\gamma$ and $\delta$ such that $\left|\lambda_{1}\right|<1<\left|\lambda_{2}\right|$ holds (lemma 2). We have shown that if the individuals set $\beta_{t}=\lambda_{1}$, they will form such rational expectations that the economy will move along the convergent orbit which is unique under perfect foresight.

However the nonlinear learning system, which individuals use for finding the correct $\beta_{t}$ and $x_{t}$, does not have to provide these desirable values and the convergent orbit is rather different from the perfect foresight orbit. Firstly, we have found that if the economy converges $\left(x_{t} \rightarrow 0\right)$, the individuals may not find $\beta_{t} \rightarrow \lambda_{1}$. Secondly if they even find out that $x_{t} \rightarrow 0$ and $\beta_{t} \rightarrow \lambda_{1}$, sometimes it is for the price that $\left|x_{t}\right|$ was decreasing, then $\left|x_{t}\right|$ had to increase for a certain time to finally converge to zero. In the real world this would cause uncertainty about $x_{t}$ and therefore is not desirable. Thirdly if the initial guess $b_{0}$ was in specified intervals ( $b_{0}<\tilde{B}_{2}$ or $b_{0}>\tilde{B}_{1}$ for $\gamma>0, b_{0}<B_{2}$ or $b_{0}>B_{1}$ for $\gamma<0$ ), then the economy will not converge to its stationary equilibrium $\bar{x}=0$.

Furthermore, the second learning system was sensitive to the revision parameter $\alpha$. If it was badly set, the periodic orbits of $b_{0}$ occurs or even the orbit of $b_{0}$ can have chaotic behavior. The simulations show that the economy may converge to its stationary equilibrium $\bar{x}=0$ as well as may diverge from this stationary equilibrium. The proof of this convergence or divergence for certain parameters would be interesting for further study of this model.

## References

[1] D. K. H. Begg: The Rational Expectations Revolution in Macroeconomics. Deddington, Oxford: Phillip Allan, 1982.
[2] O. J. Blanchard, C. M. Kahn: The Solution of Linear Difference Models under Rational Expectations.
Econometrica, 1980.
[3] F. Brauer, J. A. Nohel: The Qualitative Theory of Ordinary Differential Equations.
W.A.BENJAMIN, INC., New York, 1969.
[4] P. Brunovský: Difference and differential equations. http://pc2.iam.fmph.uniba.sk/skripta/brunovsky.
[5] S. Gauthier: Determinacy and Stability under Learning of Rational Expectations Equilibria.
Journal of Economic Theory, 2002.
[6] S. Gauthier: Learning and the saddle point property. Economic Letters, 2001.
[7] R. Guesnerie: Successes and failures in coordinating expectations. European Economic Review, 1993.
[8] J. Hale, H. Koçak: Dynamics and Bifurcations. Springer Verlag, 1991.
[9] C.H. Hommes: Chaotic Dynamics in Economic Models. Wolters-Noordhoff bv Groningen, The Netherlands, 1991.
[10] M. Woodford: Learning to believe in sunspots. Econometrica, 1990.


[^0]:    ${ }^{1}$ The possibility when it does not project surjectively is not treated both in [1] and [2]. Further explanation of this possibility is given in Subsection 4.1 and in figure 12.

