Mean-Variance Hedging for Exotic Options

MASTER THESIS

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9.1.9 Economic and Financial Mathematics

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I declare this thesis was written on my own, with the only help provided by my supervisor and the referred-to literature and sources.

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Abstract

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The task is to apply the standard mean-variance hedging theory to special case of lookback options, considering both fixed and floating strike contracts. The model for stock returns will be obtained by discretizing a Lévy process (that is we will consider independent and identically distributed stock returns). In this model we have to identify the relevant exogenous state variables, consider possible state space reduction by change of numeraire, and implement the model numerically for plausible parameter values.

Keywords: mean-variance hedging, lookback options, locally optimal strategy, levy processes

Abstrakt

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Práca si kladie za úlohu aplikovať štandardnú teóriu kvadratického zaistenia na špeciálny prípad lookback opcií s pevnou a plávajúcou expiračnou cenou. Ako model pre výnosy akcie volíme diskretizované Lévyho procesy (uvažujeme nezávisle a identicky rozdelené výnosy akcie). V modeli musíme identifikovať relevantné exogénne stavové premenné, zvážiť prípadnú redukciu priestoru stavových premenných zmenou numeraire a implementovať model pre hodnoverné hodnoty parametrov.

Kľúčové slová: kvadratické zaistenie, lookback opcia, lokálne optimálna stratégia, lévyho procesy

Contents

Introduction

Purpose of this thesis is to implement results of the classical mean-variance hedging theory for the special case of lookback options. This involves computation of mean-value process, locally optimal hedge and the unconditional hedging error. First chapter reviews general theory and presents further auxiliary theoretical results required for the implementation. Second chapter describes the implementation itself and demonstrates the necessity to increase efficiency by a change of numeraire, also presented in this chapter. The choice of appropriate model for stock returns is contained in the third chapter, where we introduce Itô and Lévy processes. Finally, the fourth chapter provides results and analysis of convergence. Program codes can be found on enclosed CD.

Chapter 1

Mean-Variance Hedging

The most commonly used tool for pricing option contracts is the Black Scholes model. One of its results is that under certain assumptions, it is possible to create a perfect self-financing hedging strategy to replicate an option. This, of course, does not guarantee elimination of risk for option issuer, since the model assumes stock price to be a pure diffusion process and the possibility of continuous trading. If we want to analyze hedging error, we need to step out of the Black Scholes model into an environment that permits trading only at certain times and allows for price jumps.

Our objective will be to minimize unconditional expected squared hedging error, which is equivalent to choice of quadratic utility function. In order to formulate this task mathematically, let us first introduce some notation.

1.1 Notation and Assumptions

We assume contingent claim H with time to maturity $T \in N$, set of trading dates $\tau := \{0, 1, ..., T\}$, a probability space $(\Omega, P, \mathcal{F})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \tau}$, where $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ is trivial. Assume that contingent claim H is $\mathcal{F}_T$ measurable and $H \in L^2(P)$.

S represents discounted stock price process. We assume that S is adapted to introduced filtration $\mathbb{F}$ and that it is locally square integrable:

$$E_t\left[(\Delta S_{t+1})^2\right] < \infty$$

for $t < T$, where $\Delta S_{t+1} := S_{t+1} - S_t$ and

$$E_t[X] = E[X|\mathcal{F}_t]. \tag{1.1}$$

As we have stated above, our objective will be to minimize unconditional expected squared hedging error

$$\min_{\vartheta} E\left[(v + \vartheta \bullet S_T - H)^2\right], \tag{1.2}$$

where v is an admissible initial endowment, ϑ is an admissible trading strategy and $\vartheta \bullet S_T$ represents gains from trading in the time interval $[0, T]$.

Definition 1. *We say that (v, ϑ) is an admissible endowment-strategy pair iff v is F_0 measurable, ϑ is predictable (meaning that ϑ_t is $\mathcal{F}_{t-1}$ measurable for $t \in \{1, ..., T\}$) and*

$$v + \vartheta \bullet S_T := v + \sum_{t=1}^{T} \vartheta_t \Delta S_t \in L^2(P).$$

Definition 2. *We say that process S is arbitrage-free iff for all $\mathcal{F}_{t-1}$ measurable portfolios ϑ_t*

$$P(\vartheta_t \Delta S_t \geq 0) = 1 \Rightarrow P(\vartheta_t \Delta S_t = 0) = 1. \tag{1.3}$$

Further on we assume S to be arbitrage-free. Denote

$$G_t^{v,\vartheta(v)} := v + \vartheta(v) \bullet S_t \tag{1.4}$$

a value of portfolio at time t with a self-financing property $G_{t+1}^{v,\xi} = G_t^{v,\xi} + \xi_{t+1} \Delta S_{t+1}$. Since now we have formulated the problem mathematically, we can analyze the solution.

1.2 Globally optimal strategy

According to [2]

Theorem 1.2.1. *Under the assumptions of section 1.1 the process L given by*

$$
\begin{aligned}
L_T &= 1, \\
L_{t-1} &= E_{t-1}\left[L_t\left(1 - E_{t-1}\left[L_t \Delta S_t\right]^\top E_{t-1}\left[\left(L_t \Delta S_t \Delta S_t^\top\right)^{-1}\right] \Delta S_t\right)\right],
\end{aligned}
$$

is $(0,1]$-valued and the opportunity-neutral measure P^*,

$$\frac{dP^*}{dP} := \prod_{t=1}^{T} \frac{L_t}{E_{t-1}\left[L_t\right]},$$

is well-defined. The processes $\tilde{\lambda}^*$, V^* and ξ^* given by

$$
\begin{aligned}
\tilde{\lambda}_t^* &= E_{t-1}\left[L_t \Delta S_t\right]^\top E_{t-1}\left(L_t \Delta S_t \Delta S_t^\top\right)^{-1} && (1.5)\\
&= E_{t-1}^{P^*}\left[\Delta S_t\right]^\top E_{t-1}^{P^*}\left(\Delta S_t \Delta S_t^\top\right)^{-1}, && (1.6)\\
V_{t-1}^* &= E_{t-1}\left[\frac{1 - \tilde{\lambda}_t^* \Delta S_t}{1 - \Delta \tilde{K}_t^*} V_t^*\right], \; V_T^* = H, && (1.7)\\
\Delta \tilde{K}_t^* &= E_{t-1}^{P^*}\left[\Delta S_t\right]^\top E_{t-1}^{P^*}\left(\Delta S_t \Delta S_t^\top\right)^{-1} E_{t-1}^{P^*}\left[\Delta S_t\right], && (1.8)\\
\xi_t^* &= E_{t-1}^{P^*}\left[\left(V_t^* - V_{t-1}^*\right)\Delta S_t\right]^\top E_{t-1}^{P^*}\left(\Delta S_t \Delta S_t^\top\right)^{-1} && (1.9)
\end{aligned}
$$

are well-defined. For a fixed admissible initial endowment $v \in \mathbf{R}$ the strategy $\varphi(v)$ given by

$$\varphi_t(v) = \xi_t^* + \tilde{\lambda}_t^*\left(V_{t-1}^* - G_{t-1}^{v,\varphi(v)}\right), \qquad (1.10)$$

is admissible and minimizes the expected squared hedging error among all admissible strategies with initial endowment v, while $\left(V_0^*, \varphi\left(V_0^*\right)\right)$ is the optimal endowment-strategy pair if the hedging error is minimized over the initial endowment as well.

Value of unconditional expected squared hedging error for $(v, \varphi(v))$ is

$$E\left[\left(G_T^{v,\varphi(v)} - V_T^*\right)^2\right] = L_0\left(v - V_0^*\right)^2 + \sum_{t=1}^{T} E\left[L_t \psi_t^*\right], \qquad (1.11)$$

where $\psi_t^* = E_{t-1}^{P^*}\left[\left(V_t^* - \xi_t^* \Delta S_t - V_{t-1}^*\right)^2\right]$.

For proof see [2, Theorem 8.7].

Remark 3. *Theorem presents globally optimal solution of a special case of 1.2 with a finite number of trading dates. General solution is presented in [1], where assumptions are that S is a locally square-integrable semimartingale and there exist some equivalent σ-martingale measure with square-integrable density, which in the model of section 1.1 is implied by the no-arbitrage assumption, see [10]. Moreover, [2] shows that the general notion of admissibility presented in [1] simplifies to that of Definition 1 in a model with finite number of trading dates.*

Further on, we consider S to have only one dimension. In this setting, V_{t-1}^* is the initial endowment and ξ_t^* a trading strategy of a locally optimal hedge

$$\begin{aligned} \left\{V_{t-1}^*, \xi_t^*\right\} &:= \arg \min_{v_{t-1}, \vartheta_t \in L^0(\Omega, \mathcal{F}_{t-1}, P)} E_{t-1}\left[L_t\left(v_{t-1} + \vartheta_t \Delta S_t - V_t^*\right)^2\right] && (1.12) \\ &:= \arg \min_{v_{t-1}, \vartheta_t} E_{t-1}^{P^*}\left[\left(v_{t-1} + \vartheta_t \Delta S_t - V_t^*\right)^2\right]. && (1.13) \end{aligned}$$

This problem is identical to finding coefficients of a linear regression with V_t^* being explained variable and v_{t-1} and ϑ_t being coefficients of explanatory variables 1 and ΔS_t respectively. Standard approach yields

$$\begin{aligned} \xi_t^* &= \frac{E_{t-1}^{P^*}\left[V_t^* \Delta S_t\right]}{E_{t-1}^{P^*}\left[(\Delta S_t)^2\right]}, \\ V_{t-1}^* &= E_{t-1}^{P^*}\left[V_t^*\right] - \xi_t^* E_{t-1}^{P^*}\left[\Delta S_t\right]. \end{aligned}$$

However, it is also possible to use Frisch-Waugh-Lovell theorem [5] to obtain the estimate ξ_1. This theorem suggests that if we rewrite a regression as

$$Y = X_1 \beta_1 + X_2 \beta_2 + \epsilon$$

and define $Y^* = M_1 Y$ and $X_2^* = M_1 X_2$ as OLS residuals of regressing Y and X_2 on X_1 where M_1 is a idempotent projection matrix into the space orthogonal to the space generated by columns of X_1, we can obtain OLS estimate of $\hat{\beta}_2$ by regressing Y^* on X_2^*. Furthermore, since we are interested only in $\hat{\beta}_2$, we can use idempotency of matrix M_1 and drop the transformation of Y. This implies that we will need only one auxiliary regression. In our case, $Y = V_t^*$, $X_2 = 1$ and $X_1 = \Delta S_t$.

Let us create a univariate auxiliary regression of 1 on explanatory variable ΔS_t. Thus we get an estimate

$$\tilde{\lambda}_t^* := \arg\min_{\vartheta_t} E_{t-1}^{P^*}\left[(\vartheta_t \Delta S_t - 1)^2\right] = \frac{E_{t-1}^{P^*}[\Delta S_t]}{E_{t-1}^{P^*}\left[(\Delta S_t)^2\right]}, \tag{1.14}$$

and a sum of squares of residuals

$$1 - \Delta\tilde{K}_t^* := \min_{\{\vartheta_1\}} E_{t-1}^{P^*}\left[(\vartheta_1 \Delta S_t - 1)^2\right] = 1 - \frac{\left(E_{t-1}^{P^*}[\Delta S_t]\right)^2}{E_{t-1}^{P^*}\left[(\Delta S_t)^2\right]}. \tag{1.15}$$

Residuals from this auxiliary regression are $1 - \tilde{\lambda}_t^* \Delta S_t$. Now we apply results of Frisch-Waugh-Lovell theorem and express V_{t-1}^* as a coefficient of regressing V_t^* on $1 - \tilde{\lambda}_t^* \Delta S_t$.

$$V_{t-1}^* = E_{t-1}^{P^*}\left[\frac{1 - \tilde{\lambda}_t^* \Delta S_t}{1 - \Delta\tilde{K}_t^*} V_t\right] \tag{1.16}$$

It is straightforward now to evaluate ξ_t^* with known V_{t-1}^*. Regression 1.13 reduces to regression of $V_t^* - V_{t-1}^*$ on ΔS_t and therefore

$$\xi_t^* = \frac{E_{t-1}^{P^*}\left[\left(V_t^* - V_{t-1}^*\right)\Delta S_t\right]}{E_{t-1}^{P^*}\left[(\Delta S_t)^2\right]} \tag{1.17}$$

Denote residuals of this equation

$$e_t^* = V_{t-1}^* + \xi_t^* \Delta S_t - V_t^*.$$

Remark 4. *Define probability measure Q^* as*

$$\frac{dQ^*}{dP^*} := \prod_{t=1}^{T} \frac{1 - \tilde{\lambda}_t^* \Delta S_t}{1 - \Delta\tilde{K}_t^*}. \tag{1.18}$$

Then Q^* is a risk neutral measure under P and we can write $V_{t-1}^* = E^{Q^*}[V_t^*]$

In order for Q^* to be well defined it must have a mass of 1

$$\begin{aligned} E_{t-1}^{P^*}\left[\frac{1 - \tilde{\lambda}_t^* \Delta S_t}{1 - \Delta\tilde{K}_t^*}\right] &= \frac{1 - \tilde{\lambda}_t^* E_{t-1}^{P^*}[\Delta S_t]}{1 - \Delta\tilde{K}_t^*} \\ &= 1 \end{aligned}$$

and it must price S correctly under P

$$
\begin{aligned}
E_{t-1}^{P^*}\left[\frac{1-\tilde{\lambda}_t^*\Delta S_t}{1-\Delta\tilde{K}_t^*}\Delta S_t\right] &= E_{t-1}\left[\frac{L_t}{E_{t-1}\left[L_t\right]}\frac{1-\tilde{\lambda}_t^*\Delta S_t}{1-\Delta\tilde{K}_t^*}\Delta S_t\right] \\
&= \frac{E_{t-1}\left[L_t\Delta S_t\right]-\frac{E_{t-1}\left[L_t\Delta S_t\right]}{E_{t-1}\left[L_t\left(\Delta S_t\right)^2\right]}E_{t-1}\left[L_t\left(\Delta S_t\right)^2\right]}{E_{t-1}\left[L_t\right]-\frac{\left(E_{t-1}\left[L_t\Delta S_t\right]\right)^2}{E_{t-1}\left[L_t\left(\Delta S_t\right)^2\right]}} \\
&= 0.
\end{aligned}
$$

Remark 5. *If stock returns $R_t = \frac{S_t}{S_{t-1}}$ are IID distributed, L_t is deterministic for $t \in \tau$ and P and P^* are equivalent.*

This follows from the property of process L_t. If stock returns are IID and L_t is deterministic, then L_{t-1} is also deterministic, since

$$
\frac{\left(E_{t-1}\left[\Delta S_t\right]\right)^2}{E_{t-1}\left[\left(\Delta S_t\right)^2\right]}
$$

is in the presence of IID stock returns deterministic. L_t is defined recursively from $L_T = 1$ and thus L_t is deterministic for all t. It follows that

$$
\frac{dP^*}{dP} = 1,
$$

so P and P^* are equivalent.

1.3 Locally optimal strategy

Locally optimal strategy aims to minimize local expected squared hedging error. It means we search for processes V and ξ that satisfy

$$
\{V_{t-1}, \xi_t\} := \arg \min_{v_{t-1}, \vartheta_t \in L^0(\Omega, \mathcal{F}_{t-1}, P)} E_{t-1}\left[\left(v_{t-1} + \vartheta_t \Delta S_t - V_t\right)^2\right] \tag{1.19}
$$

$$
V_T := H. \tag{1.20}
$$

where ϑ is admissible according to definition 1. Equation 1.19 is known from Follmer and Schweizer [6], but in our setting we consider ξ to be a self-financing strategy.

Assumptions introduced in section 1.1 are not sufficient to guarantee existence and admissibility of strategy ξ.

Theorem 1.3.1. *Assume*

$$S_t := \prod_{i=1}^{t} R_i \quad \text{for } t = 1, ..., T, \tag{1.21}$$

where $S_0 > 0$ *and* $\{R_t\}_{t\in\tau}$ *is a collection of IID random variables with finite second moment and* $P(R_t > 0) = 1$.

Locally optimal strategy ξ *is then well defined and admissible.*

Proof. First of all, it is useful to notice that optimal ednowment V and optimal trading strategy ξ are the same as V^* and ξ^* in theorem 1.2.1 when P and P^* are equivalent, which in this case holds true according to remark 5. Theorem 1.2.1 then claims, that both processes V and ξ are well defined.

As a consequence, we have that $V_0 < \infty$, so $E\left[\left(G_0^{V_0,\xi}\right)^2\right] < \infty$. We continue with mathematical induction. Assume that $E\left[\left(G_t^{V_0,\xi}\right)^2\right] < \infty$. Then we have

$$E\left[\left(G_{t+1}^{V_0,\xi}\right)^2\right] = E\left[\left(G_t^{V_0,\xi}\right)^2\right] + E\left[\xi_t^2\left(\Delta S_t\right)^2\right] + 2E\left[\xi_t \Delta S_t G_t^{V_0,\xi}\right]. \tag{1.22}$$

Observe that it is sufficient to prove $E\left[\xi_t^2\left(\Delta S_t\right)^2\right] < \infty$. According to the special case of Holder's inequality

$$E\left[\xi_t \Delta S_t G_t^{V_0,\xi}\right] \leq E\left[\left|\xi_t \Delta S_t G_t^{V_0,\xi}\right|\right] \leq \sqrt{E\left[\left(G_t^{V_0,\xi}\right)^2\right] E\left[\xi_t^2\left(\Delta S_t\right)^2\right]} < \infty.$$

According to Lemma 4.4 in [1], process LV^2 is a submartingale, i.e. $L_{t-1}V_{t-1}^2 \leq E_{t-1}\left[L_t V_t^2\right]$. According to remark 5, L is deterministic and we know from theorem 1.2.1 that it is $(0,1]$-valued. Thus $V_{t-1}^2 \leq cE_{t-1}\left[V_t^2\right]$ where $c = L_t/L_{t-1}$. We know that $V_T = H \in L^2(P)$. Assume $V_t = H \in L^2(P)$ and by mathematical induction we have

$$E\left[V_{t-1}^2\right] \leq cE\left[E_{t-1}\left[V_t^2\right]\right] = cE\left[V_t^2\right] \leq \infty,$$

so $V_t \in L^2(P)$ for all $t \in \tau$ and also $V_t \in L^2(P, \mathcal{F}_{t-1})$ for $t = 1, ..., T$. Then it follows from the properties of linear regression 1.19

$$E_{t-1}\left[(V_{t-1} + \xi_t \Delta S_t)^2\right] \leq E_{t-1}\left[V_t^2\right] < \infty$$

Observe that $E_{t-1}\left[(V_{t-1} + \xi_t \Delta S_t)^2\right]$ is not only finite, but since

$$E\left[E_{t-1}\left[V_t^2\right]\right] = E\left[V_t^2\right] < \infty$$

it also has an integrable majorant. Therefore

$$E\left[(V_{t-1} + \xi_t \Delta S_t)^2\right] = E\left[E_{t-1}\left[(V_{t-1} + \xi_t \Delta S_t)^2\right]\right] < \infty$$

i.e. $V_{t-1} + \xi_t \Delta S_t \in L^2(P)$. Since V_{t-1} is also square integrable, it follows that $\xi_t \Delta S_t$ is square integrable. This proves the induction 1.22. Hence, locally optimal strategy ξ_t is well defined and admissible.

□

This approach finds the perfectly replicable portfolio if it exists. However in general its existence is not guaranteed. It is also worth mentioning that when creating a self-financing hedging strategy, we are not allowed to determine initial endowment in each step. Initial endowment is determined by past trading strategy, so this strategy is not necessarily self-financing. Only the mean value of portfolio value at time t equals V_t, the true value of portfolio may be higher or lower than V_t. Therefore, trading strategy ξ is not globally optimal in general.

In order to investigate the unconditional squared hedging error, define a self-financing strategy with initial endowment v and trading strategy $\{\xi_t\}$ for $t = 1, ..., T$. Denote

$$e_t = V_{t-1} + \xi_t \Delta S_t - V_t$$

as residuals of local optimization.

Then we have for fixed initial endowment v

$$
\begin{aligned}
E\left[\left(G_T^{v,\xi}-V_T\right)^2\right] &= E\left[E_{T-1}\left[\left(G_T^{v,\xi}-V_T\right)^2\right]\right]\\
&= E\left[E_{T-1}\left[\left(G_{T-1}^{v,\xi}-V_{T-1}+V_{T-1}+\xi_T\Delta S_T-V_T\right)^2\right]\right]\\
&= E\left[E_{T-1}\left[\left(G_{T-1}^{v,\xi}-V_{T-1}+e_T\right)^2\right]\right]\\
&= E\left[E_{T-1}\left[\left(G_{T-1}^{v,\xi}-V_{T-1}\right)^2+e_T^2+2\left(G_{T-1}^{v,\xi}-V_{T-1}\right)(e_T)\right]\right]\\
&= E\left[\left(G_{T-1}^{v,\xi}-V_{T-1}\right)^2+E_{T-1}\left[e_T^2\right]\right].
\end{aligned}
$$

The last equality follows from the property of least squares residuals. Since e_T represents residuals of 1.13, $E_{T-1}\left[e_T\right]=0$.

Recursive application of this procedure yields

$$
E\left[\left(G_T^{v,\xi}-V_T\right)^2\right]=(v-V_0)^2+\sum_{t=1}^{T}E\left[\psi_t\right] \tag{1.23}
$$

where $\psi_t=E_{t-1}\left[e_t^2\right]$.

Since L is $(0,1]$-valued, it follows from theorem 1.2.1 that $\varphi(v)$ performs better than ξ. However, in practice it produces similar unconditional expected squared hedging errors as locally optimal hedging strategy ξ. Since the performance of $\varphi(v)$ does not compensate for the computational difficulty, we will use locally optimal strategy.

1.4 Change of Numeraire

In the following chapter a change of numeraire is necessary. We prove correctness of a special case of such a change.

Theorem 1.4.1. *We make the same assumptions as in Theorem 1.3.1.*

Minimization problem

$$
\min_{\vartheta} E^{\hat{P}}\left[\left(\frac{G_T^{v,\vartheta}}{S_T}-\frac{H_T}{S_T}\right)^2\right] \tag{1.24}
$$

where we define

$$\frac{d\hat{P}}{dP} := \prod_{t=1}^{T} \frac{S_t^2}{E_{t-1}\left[S_t^2\right]}.$$

has the same admissible, globally optimal endowment-strategy pair $V_0, \varphi\left(V_0\right)$ *as problem 1.2.*

Proof. Let us remark that assumptions imply $S_t > 0$ almost surely for all $t \in \tau$. Denote $\Theta\left(S, P\right)$ set of admissible trading strategies, meaning that $\vartheta \in \Theta\left(S, P\right)$ iff $\vartheta \bullet S_T \in L^2\left(P\right)$. Also, denote $\hat{S} := 1/S$ and $\hat{G}_t^{v,\vartheta} := v + \vartheta \bullet \hat{S}_t$.

We start by employing results of theorem 1.2.1 and showing that

$$\min_{\vartheta} E^{\hat{P}}\left[\left(\hat{G}_t^{v,\vartheta} - \frac{H_T}{S_T}\right)^2\right] \tag{1.25}$$

has optimal and admissible trading strategy.

First, we show $\hat{S}$ is $\hat{P}$-locally square integrable

$$E_{t-1}^{\hat{P}}\left[\left(\frac{1}{S_t} - \frac{1}{S_{t-1}}\right)^2\right] = \frac{E_{t-1}\left[\left(1-R_t\right)^2\right]}{E_{t-1}\left[S_t^2\right]} < \infty,$$

since R_t has finite second moment and denominator is non-zero, which follows from $S_t > 0$. Next, we show $\frac{H}{S_T} \in L^2\left(\hat{P}\right)$.

$$E^{\hat{P}}\left[\left(\frac{H}{S_T}\right)^2\right] = \frac{E\left[H^2\right]}{E\left[S_T^2\right]} < \infty$$

since we know that $H \in L^2\left(P\right)$ and denominator is again non-zero. It follows from theorem 1.2.1 that 1.25 has optimal and admissible trading strategy.

Assume $\vartheta \in \Theta\left(S, P\right)$. Set $\hat{\vartheta}_t := G_{t-1} - \vartheta_t S_{t-1}$. We prove for all $t \in \tau$

$$\hat{G}_t^{v/S_0,\hat{\vartheta}} = \frac{G_t^{v,\vartheta}}{S_t}.$$

It is obvious that

$$\hat{G}_0^{v/S_0,\hat{\vartheta}} = \frac{v}{S_0} = \frac{G_0^{v,\vartheta}}{S_0}.$$

We proceed with mathematical induction. Assume

$$\hat{G}_{t-1}^{v/S_0,\hat{\vartheta}} = \frac{G_{t-1}^{v,\vartheta}}{S_{t-1}}.$$

It follows that

$$\begin{aligned}
\hat{G}_t^{v/S_0,\hat{\vartheta}} &= \frac{G_{t-1}^{v,\vartheta}}{S_{t-1}} + \left(G_{t-1} - \vartheta_t S_{t-1}\right)\left(\frac{1}{S_t} - \frac{1}{S_{t-1}}\right) \\
&= \frac{G_{t-1}^{v,\vartheta}}{S_{t-1}} + \frac{G_{t-1}^{v,\vartheta}}{S_{t-1}}\left(R_t^{-1} - 1\right) + \vartheta_t \frac{S_t - S_{t-1}}{S_t} \\
&= \frac{G_{t-1}^{v,\vartheta} + \vartheta_t \Delta S_t}{S_t} \\
&= \frac{G_t^{v,\vartheta}}{S_t}
\end{aligned}$$

Employing this result we show that $\vartheta \in \Theta\left(S, P\right) \;\Rightarrow\; \hat{\vartheta} \in \Theta\left(\hat{S}, \hat{P}\right)$. We know that for arbitrary $v \in \mathbb{R}$, $\vartheta \in \Theta\left(S, P\right)$ implies $G_T^{v,\vartheta} \in L^2\left(P\right)$. Thus

$$E^{\hat{P}}\left[\left(\hat{G}_t^{v/S_0,\hat{\vartheta}}\right)^2\right] = \frac{E\left[\left(G_T^{v,\vartheta}\right)^2\right]}{E\left[S_T^2\right]} < \infty,$$

which implies $\hat{\vartheta} \in \Theta\left(\hat{S}, \hat{P}\right)$. Moreover, we can show that spaces of admissible strategies $\Theta\left(\hat{S}, \hat{P}\right)$ and $\Theta\left(S, P\right)$ are identical, meaning that

$$\vartheta \in \Theta\left(S, P\right) \;\Leftrightarrow\; \hat{\vartheta} \in \Theta\left(\hat{S}, \hat{P}\right). \tag{1.26}$$

We have already proved ($\Rightarrow$). Let us now assume $\hat{\vartheta} \in \Theta\left(\hat{S}, \hat{P}\right)$. Again, for arbitrary $v/S_0 \in \mathbb{R}$, $\hat{\vartheta} \in \Theta\left(\hat{S}, \hat{P}\right)$ implies $\hat{G}_t^{v/S_0,\hat{\vartheta}} \in L^2\left(\hat{P}\right)$. Thus

$$E\left[\left(G_T^{v,\vartheta}\right)^2\right] = E^{\hat{P}}\left[\left(\hat{G}_t^{v/S_0,\hat{\vartheta}}\right)^2\right] E\left[S_T^2\right] < \infty,$$

which implies $\vartheta \in \Theta\left(S, P\right)$. It is useful to notice

$$\begin{aligned}
E\left[\left(v + \vartheta \bullet S_T - H\right)^2\right] &= E\left[S_T^2\right] E^{\hat{P}}\left[\left(\frac{v + \vartheta \bullet S_T}{S_T} - \frac{H}{S_T}\right)^2\right] \\
&= E\left[S_T^2\right] E^{\hat{P}}\left[\left(v/S_0 + \hat{\vartheta} \bullet \hat{S}_T - \frac{H}{S_T}\right)^2\right].
\end{aligned}$$

Using this result and 1.26, we claim that when setting $\hat{v} = v/S_0$ and $\hat{\vartheta}_t := G_{t-1}^{v,\vartheta} - \vartheta_t S_{t-1}$ it follows

$$\begin{aligned}
\min_{\hat{\vartheta}\in\Theta\left(\hat{S},\hat{P}\right)} E^{\hat{P}}\left[\left(\hat{v} + \hat{\vartheta} \bullet \hat{S}_T - \frac{H}{S_T}\right)^2\right] &= \min_{\vartheta\in\Theta(S,P)} E^{\hat{P}}\left[\left(\frac{G_T^{v,\vartheta}}{S_T} - \frac{H}{S_T}\right)^2\right] \\
&= \min_{\vartheta\in\Theta(S,P)} E\left[\left(\frac{G_T^{v,\vartheta}}{S_T} - \frac{H}{S_T}\right)^2\right]\frac{1}{E\left[S_T^2\right]}
\end{aligned}$$

and since $1/E\left[S_T^2\right]$ is constant, finite and non-zero and 1.26 holds, it follows that optimal and admissible endowment-strategy pair of 1.25 is $\hat{V}_0 = V_0/S_0$ and $\hat{\varphi}_t = G_{t-1}^{v,\varphi} - \varphi_t S_{t-1}$ and consequently optimal endowment-strategy pair of 1.24 is V_0/S_0, φ.

$\square$

Chapter 2

Implementation of Mean Variance Hedging

Key task one has to solve when implementing procedures described above is the efficiency of computations. Therefore, we introduce the following model for stock price S.

Denote S_t a stock price at certain time $t \in \tau$

$$S_t = e^{(X_t)}, \tag{2.1}$$

where

$$\Delta X_t = X_t - X_{t-1}, t = 1, ..., T \tag{2.2}$$

are iid discrete random variables with n possible values that are assigned certain probabilities. Let us denote $x \in R^n$ a vector of these values and P a vector of probabilities assigned to these values. Furthermore we assume

- values of x are in descending order,
- $x_i - x_{i-1}$ equals a constant h for $i = 2, ..., n$,

While the first assumption is only for our convenience, the second one creates an efficient lattice where the number of possible states of S_t increases linearly with time. Figure 2.1 shows an example of such a lattice. Denote u number of elements in x greater than 0 and d number of elements in x smaller than 0.

t = 0	t = 1	t = 2	t = 3	t = 4
				117.3511
				115.0274
			112.7497	112.7497
			110.5171	110.5171
		108.3287	108.3287	108.3287
		106.1837	106.1837	106.1837
	104.0811	104.0811	104.0811	104.0811
	102.0201	102.0201	102.0201	102.0201
100	100	100	100	100
	98.0199	98.0199	98.0199	98.0199
	96.0789	96.0789	96.0789	96.0789
		94.1765	94.1765	94.1765
		92.3116	92.3116	92.3116
			90.4837	90.4837
			88.692	88.692
				86.9358
				85.2144

Figure 2.1: Efficient lattice

$$S_t \in \left\{e^{X_0+kh}; k \in Z, -td \leq k \leq tu\right\}.$$

Since we do not want the efficient lattice to be dependent on risk free rate, we do not use discounted values.

Denote R_f risk free rate over one hedging period. Self-financing property changes to

$$G_t^{v,\vartheta} = R_f G_{t-1}^{v,\vartheta} + \vartheta_t \left(S_t - R_f S_{t-1}\right). \tag{2.3}$$

We can now rewrite 1.13 as

$$\{V_{t-1}, \xi_t\} \equiv \arg \min_{v_{t-1},\vartheta_t} E_{t-1}\left(\left(R_f v_{t-1} + \vartheta_t \left(S_t - R_f S_{t-1}\right) - V_t\right)^2\right) \tag{2.4}$$

To make better use of the iid characteristic of stock returns, let us denote excess return $\tilde{R}_t$ as

$$S_t - R_f S_{t-1} = S_{t-1}\left(e^{\Delta X_t} - R_f\right) \equiv S_{t-1}\tilde{R}_t \tag{2.5}$$

Following the same steps as in 1.2, we obtain

$$\begin{aligned}
\tilde{\lambda}_t &= \frac{E\left[\tilde{R}_t\right]}{S_{t-1}E\left[\left(\tilde{R}_t\right)^2\right]}, &(2.6)\\
1-\Delta\tilde{K}_t &= 1-\frac{\left(E\left[\tilde{R}_t\right]\right)^2}{E\left[\tilde{R}_t^2\right]}, &(2.7)\\
\frac{dQ}{dP} &= \prod_{t=1}^{T}\frac{1-\tilde{\lambda}_t S_{t-1}\tilde{R}_t}{1-\Delta\tilde{K}_t} = \prod_{t=1}^{T}\frac{1-\frac{E\left[\tilde{R}_t\right]}{E\left[\left(\tilde{R}_t\right)^2\right]}\tilde{R}_t}{1-\Delta\tilde{K}_t}. &(2.8)
\end{aligned}$$

We reformulate solution and residuals as

$$\begin{aligned}
R_f V_{t-1} &= E^Q\left[V_t\right], &(2.9)\\
\xi_t &= \frac{E\left[\left(V_t - R_f V_{t-1}\right)\tilde{R}_t\right]}{S_{t-1}E\left[\left(\tilde{R}_t\right)^2\right]}, &(2.10)\\
e_t &= R_f V_{t-1} + \xi_t S_{t-1}\tilde{R}_t - V_t. &(2.11)
\end{aligned}$$

Last equality leads to expected squared hedging error

$$E\left[\left(G_T^{v,\xi} - V_T\right)^2\right] = R_f^{2T}\left(v - V_0\right)^2 + \sum_{t=1}^{T} R_f^{2(T-t)}E\left[\psi_t\right]. \tag{2.12}$$

An example of implementation for a european call option is shown in *eurocall.m.*

2.1 Lookback options

Lookback option is a path dependent option which allows the owner to review the path of the stock price during the life of the option and use the most suitable stock price. This definition implies that we are going to need the notions of maximum

$$\mathrm{MS}_t \equiv max\left\{S_u : u \in 0, 1, ...t\right\}$$

and minimum

$$\mathrm{mS}_t \equiv min\left\{S_u : u \in 0, 1, ...t\right\}$$

of the stock price S during a period of time $(0, t)$.

We distinquish between two main types of lookback options.

- Fixed strike price

  Strike price of these options is agreed at the time of issuance but the option is exercised in case of a put option at the minimum stock price and in case of a call option at the maximum stock price during the life of the option.

- Floating strike price

  Strike price of this option is determined at the time of maturity. In case of a put option it depends on the maximum price and in case of a call option on the minimum stock price during the life of the option.

Payoff of Fixed strike call option is $H_T = (\mathrm{MS}_T - K)^+$ and of Floating strike call option $H_T = (S_T - \lambda \mathrm{mS}_T)^+$.

The problem we encounter here is that maximum of stock price is not a Markov process, i.e. the distribution of MS_{t+1} does not depend only on MS_t, but also on the stock price S_t. For instance, if S_t is well below MS_t it might happen that the maximum stock price is not going to change in the next step no matter which path does the stock price take from S_t. On the other hand, MS_t is very likely to change if $S_t = \mathrm{MS}_t$.

Therefore, various combinations of stock price and its maximum must be taken into consideration. While in case of european call option we could imagine the lattice as a sequence of vectors, we now have a sequence of matrices. The amount of states is not going to increase linearly with time now, but quadratically.

This rises the need for an efficient lattice that would consider only plausible combinations of S_t and MS_t.

2.2 Efficient lattice for lookbacks

Let us assume for simplification $\exists j \in \{1, ..., n\}$ such that $x(j) = 0$. Even if this assumption is not crucial and all the calculations are possible without it, it is not too restrictive for small values of h and it facilitates computations.

It is more simple, in the process of determining the plausible combinations of S_t and MS_t, to work with logarithms of stock price X_t and maximum stock price MX_t. These are easier to work with when we know the initial stock price S_0, difference between price jumps h and time t.

The first obvious statement we can claim is that the maximum stock price will always be above the stock price. For instance

$$X_t = X_0 + sh \Rightarrow \mathrm{MX}_t \in \left\{X_0 + mh, m \in Z : s^+ \leq m \leq tu\right\}.$$

Value of the stock price may also impose a restriction on maximum stock price from above. For instance if $X_t = X_0 - td$, it is obvious that the maximum can not be higher than $\mathrm{MX}_t = X_0$.

In order to determine this upper bound of MX_t, we will find a path of the stock price beginning in X_0 and ending in $X_0 + sh$ with the highest possible maximum. We will let the stock price rise at rate of uh for $\tilde{t}$ periods, make a jump of $\tilde{n}h$ and then fall at rate $-dh$ for $t - \tilde{t} - 1$ periods. The upper bound for MX_t will be $\left(\tilde{t}u + \tilde{n}^+\right) h$.

We now have to solve equation

$$s = \tilde{t}u + \tilde{n} + \left(t - \tilde{t} - 1\right) d \tag{2.13}$$

for integers $\tilde{t} \in (0, t)$ and $-d < \tilde{n} \leq u$. After rearranging this equation we get

$$\frac{s + td - 1}{n - 1} = \tilde{t} + \frac{\tilde{n} + d - 1}{n - 1} \tag{2.14}$$

where $n = d + 1 + u$.

This form is very convenient. We can split the fraction on the left side into a natural number and a remainder after division by $n - 1$ which can take values $\left\{\frac{0}{n-1}, \frac{1}{n-1}, ..., \frac{n-2}{n-1}\right\}$. Since the range of $\tilde{n} + d - 1$ is $(0, n - 2)$ we can set the fraction on the right side to any of the values of the remainder on the left side. This allows

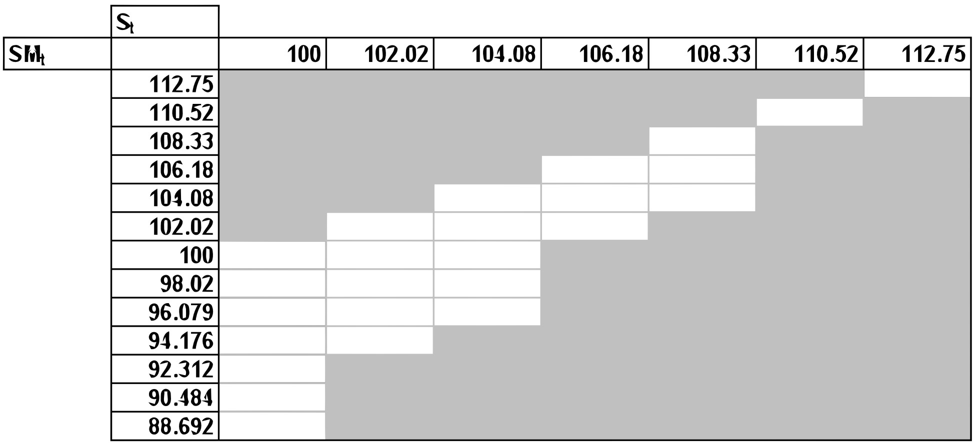


Figure 2.2: Plausible states of S_T and MS_T

us to set $\tilde{t} = \left\lfloor \frac{s+td-1}{n-1} \right\rfloor$ (floor of the fraction on the left side) and be sure that we can find $\tilde{n}$ that satisfies equation 2.14 and its own restrictions

$$\tilde{n} = s + d\,(t-1) - \tilde{t}\,(n-1)\,.$$

An example of an efficient lattice of possible states for

$$\begin{aligned} x &= (-0.04, -0.02, 0, 0.02, 0.04) \\ S_0 &= 100 \\ T &= 3 \end{aligned}$$

is depicted on Fig 2.2. Numbers of states in individual columns create a sequence of numbers starting with term $1 + td$. Difference between terms of this sequence is d and one term is repeating itself u times (with the exception of the first term).

This approach is implemented in *lookback_efficient_lattice.m*.

Now that we can find all the possible combinations of S_t and MS_t it is obvious that storing this information in a matrix would be very space inefficient. Since we can create the sequence of numbers of elements in individual columns, we can arrange all the necessary values in one vector and use this sequence to keep track of which value is related to which stock price and maximum stock price. This approach enables us to work with higher dimensions of x and more hedging intervals, however, it increases computational difficulty.

2.3 Dimension reduction

Another possibility we may consider in order to increase efficiency of computing is to decrease the number of necessary state variables. We demonstrate this reduction on a special case of floating strike put option. Let us remark that the same reduction is easily achieved also for floating strike call options if we replace maximum for minimum.

We have already stated that the problems we encountered when evaluating fixed strike lookback options were due to process MX_t not having the Markov property. Furthermore, the payoff of a floating strike lookback option depends not only on MX_t, but also on X_t. Therefore, we have to find a process with Markov property that would contain the necessary information to determine the option price at time of maturity.

Let us define process

$$\mathrm{DX} = \mathrm{MX} - X, \tag{2.15}$$

where DX_t represents the distance of the stock price from the maximum stock price at time t. It is obvious that DX_t has the Markov property and it behaves just as $-X_t$ except when it is close to zero, since this difference can not be negative. If, for instance, the distance from maximum DX_t equals 0.2 and X_t in the next period rises by 0.6, DX_{t+1} drops to zero and will remain zero until the stock price drops again, creating a gap between the stock price and the maximum.

After introducing DX_T into the payoff of floating price lookback option we get

$$\begin{aligned}(\lambda \mathrm{MS}_T - S_T)^+ &= \left(\lambda e^{\mathrm{MX}_T} - e^{X_T}\right)^+ &(2.16)\\ &= S_T\left(\lambda e^{\mathrm{DX}_T} - 1\right)^+. &(2.17)\end{aligned}$$

We have obtained a Markov process, but clearly it does not contain enough information to evaluate the option and we need S_T also. It turns out that in the process of evaluating the option, it is more efficient to work with $\frac{H_T}{S_T}$ rather than H_T. This leads us to the idea of changing numeraire to stock price S.

Let us remark that this reduction is also possible for fixed strike lookback options. In this case however, we need to define new process $\bar{X}$ as $\bar{X}_t = X_{T-t} - X_T$. We can then rewrite $\mathrm{MX}_T - X_0 = M\bar{X}_T - \bar{X}_T = D\bar{X}_T$. This leads to the form of payoff (for call option) $\left(e^{\mathrm{MX}_T} - K\right)^+ = S_0\left(e^{D\bar{X}_T} - \frac{K}{S_0}\right)^+$. Using process $D\bar{X}$ again reduces the dimension.

2.4 Change of Numeraire

Our goal is to transform 2.4 to a form where we can use $\frac{H_T}{S_T}$. We employ results of theorem 1.4.1 and we write

$$
\begin{aligned}
\{V_{t-1}, \xi_t\} &= \arg\min_{v_{t-1},\vartheta_t} E_{t-1}\left[\left(R_f v_{t-1} + \vartheta_t\left(S_t - R_f S_{t-1}\right) - V_t\right)^2\right] \\
&= \arg\min_{v_{t-1},\vartheta_t} E_{t-1}\left[S_t^2\left(\frac{R_f v_{t-1} + \vartheta_t S_{t-1}\tilde{R}_t}{S_t} - \frac{V_t}{S_t}\right)^2\right] \\
&= \arg\min_{v_{t-1},\vartheta_t} E_{t-1}\left[S_t^2\right] E_{t-1}^{\tilde{P}}\left[\left(\frac{R_f v_{t-1} + \vartheta_t S_{t-1}\tilde{R}_t}{S_t} - \frac{V_t}{S_t}\right)^2\right] \\
&= \arg\min_{v_{t-1},\vartheta_t} E_{t-1}^{\tilde{P}}\left[\left(\frac{R_f v_{t-1} + \vartheta_t S_{t-1}\tilde{R}_t}{S_t} - \frac{V_t}{S_t}\right)^2\right],
\end{aligned}
$$

where we define change of measure from P to $\hat{P}$ as

$$
\frac{d\hat{P}}{dP} := \prod_{t=1}^{T} \frac{S_t^2}{E_{t-1}\left[S_t^2\right]}.
$$

It follows from the IID property of stock returns that change of measure can be rewritten as

$$
\frac{d\hat{P}}{dP} := \prod_{t=1}^{T} \frac{R_t^2}{E_{t-1}\left[R_t^2\right]}.
$$

To simplify calculations we present auxiliary results

$$\frac{S_t}{S_{t-1}} = R_t,$$

$$\frac{R_f v_{t-1} + \vartheta_t S_{t-1}\tilde{R}_t}{S_t} = R_f \frac{v_{t-1}}{S_{t-1}} R_t^{-1} + \vartheta_t R_t^{-1}\tilde{R}_t.$$

This leads to

$$\left\{\frac{V_{t-1}}{S_{t-1}}, \xi_t\right\} = \arg\min_{v_{t-1},\vartheta_t} E_{t-1}^{\hat{P}}\left[\left(R_f\frac{v_{t-1}}{S_{t-1}}R_t^{-1} + \vartheta_t R_t^{-1}\tilde{R}_t - \frac{V_t}{S_t}\right)^2\right],$$

$$\frac{V_T}{S_T} := \frac{H_T}{S_T}.$$

Applying the same procedure with Frisch-Waugh-Lovell theorem as in section 1.2, auxiliary regression yields

$$\begin{aligned}\hat{\lambda}_t &= \arg\min_{\vartheta_t} E_{t-1}^{\hat{P}}\left[\left(R_t^{-1} - \vartheta_t\tilde{R}_t R_t^{-1}\right)^2\right]\\ &= \frac{E_{t-1}^{\hat{P}}\left[\tilde{R}_t R_t^{-2}\right]}{E_{t-1}^{\hat{P}}\left[\tilde{R}_t^2 R_t^{-2}\right]}\end{aligned}$$

and sum of squared residuals

$$1 - \Delta\hat{K}_t = E_{t-1}^{\hat{P}}\left[R_t^{-2}\right] - \hat{\lambda}_t E_{t-1}^{\hat{P}}\left[\tilde{R}_t R_t^{-2}\right].$$

Results of Frisch-Waugh-Lovell theorem imply

$$R_f\frac{V_{t-1}}{S_{t-1}} = E_{t-1}^{\hat{P}}\left[\frac{R_t^{-1} - \hat{\lambda}_t\tilde{R}_t R_t^{-1}}{1-\Delta\hat{K}_t}\frac{V_t}{S_t}\right], \tag{2.18}$$

$$\xi_t = \frac{E_{t-1}^{\hat{P}}\left[\left(\frac{V_t}{S_t} - R_f R_t^{-1}\frac{V_{t-1}}{S_{t-1}}\right)\tilde{R}_t R_t^{-1}\right]}{E\left[\tilde{R}_t^{\,2} R_t^{-2}\right]}, \tag{2.19}$$

$$\frac{e_t}{S_t} = R_f\frac{V_{t-1}}{S_{t-1}}R_t^{-1} + \xi_t\tilde{R}_t R_t^{-1} - \frac{V_t}{S_t}. \tag{2.20}$$

It is interesting to notice that by rewriting 2.18 as

$$R_f V_{t-1} = E_{t-1}^{\hat{P}} \left[\frac{R_t^{-2} - \hat{\lambda}_t \tilde{R}_t R_t^{-2}}{1 - \Delta \hat{K}_t} V_t \right],$$

would yield again a risk neutral measure.

The only value left to calculate is the unconditional expected squared hedging error. First,we observe how self financing condition changes.

$$\frac{G_t^{v,\vartheta}}{S_t} = R_f \frac{G_{t-1}^{v,\vartheta}}{S_t - 1} R_t^{-1} + \vartheta_t \tilde{R}_t R_t^{-1}. \tag{2.21}$$

Then we rewrite hedging error in our setup.

$$E\left[\left(G_T^{v,\xi} - V_T\right)^2\right] = E\left[E_{T-1}\left[S_T^2\left(\frac{G_T^{v,\xi}}{S_T} - \frac{V_T}{S_T}\right)^2\right]\right]$$

$$\begin{aligned}
&= E\left[E_{T-1}\left[S_T^2\left(\frac{G_T^{v,\xi}}{S_T} - \frac{V_T}{S_T}\right)^2\right]\right] \\
&= E\left[E_{T-1}\left[S_T^2\left(R_f \frac{G_{T-1}^{v,\xi}}{S_{T-1}} R_T^{-1} - R_f \frac{V_{T-1}}{S_{T-1}} R_T^{-1} + R_f \frac{V_{T-1}}{S_{T-1}} R_T^{-1} + R_f \xi_T \tilde{R}_T R_T^{-1} - \frac{V_T}{S_T}\right)^2\right]\right] \\
&= E\left[E_{T-1}\left[S_T^2\left(R_f^2 R_T^{-2}\left(\frac{G_{T-1}^{v,\xi}}{S_{T-1}} - \frac{V_{T-1}}{S_{T-1}}\right)^2 + \left(\frac{e_T}{S_T}\right)^2\right)\right]\right] \\
&= E\left[S_{T-1}^2 R_f^2\left(\frac{G_{T-1}^{v,\xi}}{S_{T-1}} - \frac{V_{T-1}}{S_{T-1}}\right)^2 + E_{T-1}\left[S_T^2\left(\frac{e_T}{S_T}\right)^2\right]\right] \\
&= E\left[S_{T-1}^2 R_f^2\left(\frac{G_{T-1}^{v,\xi}}{S_{T-1}} - \frac{V_{T-1}}{S_{T-1}}\right)^2 + S_{T-1}^2 E_{T-1}\left[R_T^2\right] E_{T-1}^{\hat{P}}\left[\left(\frac{e_T}{S_T}\right)^2\right]\right]
\end{aligned}$$

Bearing in mind IID property of stock returns we can denote

$$ER := E_{t-1}\left[R_t^2\right] \tag{2.22}$$

for all $t = 1, ..., T$. By recursive application we have

$$E\left[\left(G_T^{v,\xi} - V_T\right)^2\right] = (v - V_0)^2 + S_0^2 \sum_{t=1}^{T} R_f^{2(T-t)} E\left[\tilde{\psi}_t\right]. \tag{2.23}$$

$$\tilde{\psi}_t = ER\ E_0^{\hat{P}}\left[ER\ E_1^{\hat{P}}\left[...ER\ E_{t-1}^{\hat{P}}\left[\left(\frac{e_t}{S_t}\right)^2\right]...\right]\right] \tag{2.24}$$

Even if seemingly difficult, it proves easy enough to implement in an algorithm. Not only have we transformed the problem to a form where we can use $\frac{H_T}{S_T}$, but introducing processes $\frac{V}{S}$ and $\frac{e}{S}$ ensures that we do not need to keep track of S during calculations. Therefore, we only need DX as a state variable and the difficulty of this computation is the same as with european call options.

This approach is implemented in *lookback_dimension_reduction.m*.

Chapter 3

Model for Stock Returns

In order to apply the described framework, we need to find feasible values of ΔX and their respective probabilities. One of the possible solutions is to use empirical distribution and estimate these values directly from the data set using histogram. However, in case we want to change the hedging interval or h, this approach forces us to store all the empirical data so that we could recalculate weights of individual bins of the histogram.

If we make assumptions about the density of returns of stock price process S, we can determine respective probabilities of x using numerical integration. Changes in initial setting would then become only changes in parameters.

3.1 Brownian motion

First we will stay in the framework of Black and Scholes and we assume stock price S to be a geometric Brownian motion

$$\frac{dS}{S} = \mu dt + \sigma dW$$

This implies that logarithms of returns over period t^* are normally distributed

$$(X_{t+t^*} - X_t) \sim N\left(\mu t^* - \frac{\sigma^2 t^*}{2}, \sigma^2 t^*\right).$$

Values μ and σ^2 are easily estimated from a data set using estimates of first and second moments of log returns. On the other hand, this model does not allow us

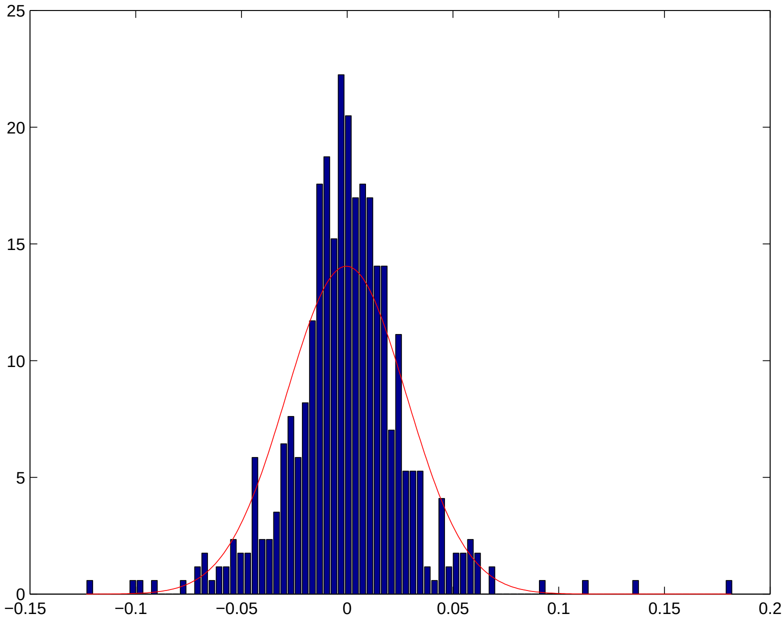


Figure 3.1: Data fitted with normal distribution

to model skewness or kurtosis of the data and it does not allow for price jumps. Empirical evidence show skewness and fat tails of the returns that deviate from normality. A more general framework is therefore desirable.

Figure 3.1 shows a histogram of Google daily returns over a period of two years fitted with normal distribution.

3.2 Levy Processes

Levy processes are becoming very popular in the fields of physics, engineering, economics and, of course, mathematical finance. They have the capacity to desribe observed reality more accuretly than Brownian motion based models.

According to [7] we define Levy process as follows

Definition 6. *A cadlag, adapted, real valued stochastic process $L = (L_t)_{0 \le t \le T}$ with $L_0 = 0$ almost surely is called a Levy process if the following conditions are satisfied:*

- *L has independent increments, i.e. $L_t - L_s$ is independent of F_s for any $0 \le s < t \le T$.*

- *L has stationary increments, i.e. for any $0 \leq s,\ t \leq T$ the distribution of $L_{t+s} - L_t$ does not depend on t.*
- *L is stochastically continuous, i.e. for every $0 \leq t \leq T$ and $\epsilon > 0$:*

$$\lim_{s \to t} P\left(|L_t - L_s| > \epsilon\right) = 0.$$

Processes are connected with their distributions by the famous Lévy-Khintchine formula ([7]).

3.2.1 Normal Inverse Gaussian

The normal inverse Gaussian distribution (NIG) is a special case of general hyperbolic distribution. It is a normal variance-mean mixture with mixing density being the inverse Gaussian distribution. It was introduced to finance by Barndorff-Nielsen in 1995 as a model for stock returns, later also used in modeling turbulence.

We choose this distribution because of its properties. First of all, density function of NIG is known explicitly, so we can employ numerical integration in order to determine values of x and their respective probabilities. NIG is also closed under convolution, which is important when rescaling the hedging period.

The density of NIG is

$$f_{NIG} = \frac{\alpha\delta}{\pi} \frac{\exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta\left(x - \mu\right)\right)}{\sqrt{\left(x-\mu\right)^2 + \delta^2}} K_1\left(\alpha\sqrt{\left(x-\mu\right)^2 + \delta^2}\right), \tag{3.1}$$

where K_1 is the modified Bessel function of the second kind.

Density function is dependent on four parameters $(\alpha, \beta, \mu, \delta)$, where $0 \leq |\beta| < \alpha$, $\delta > 0$ and $-\infty < \mu < \infty$. Parameter α controls for the steepness of the density and tail behaviour. Large values of α imply light tails. β is the skewness parameter, where $\beta < 0$ implies skewness to the left and $\beta > 0$ implies skewness to the left. For $\beta = 0$ means that density function is symmetric around μ, which is therefore interpreted as translation parameter. Finally, δ is interpreted as the scale parameter.

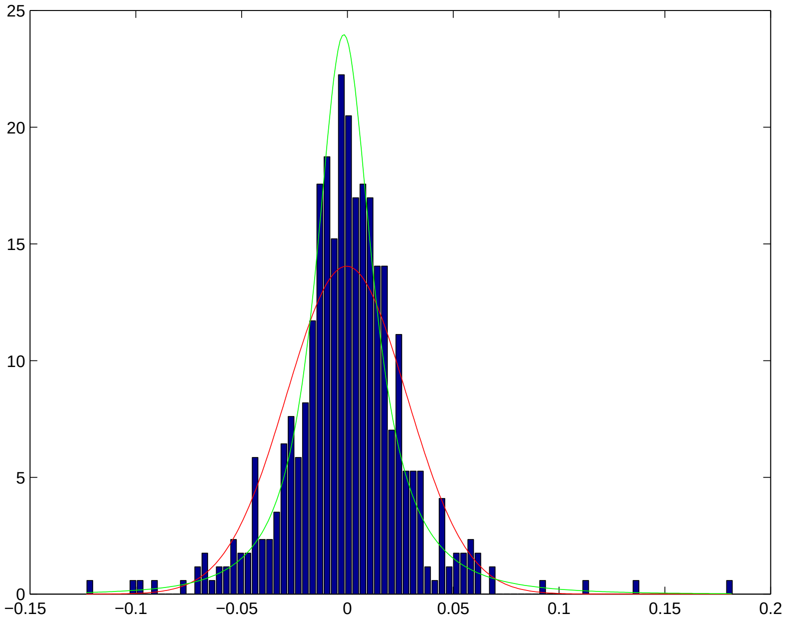


Figure 3.2: Data fitted with normal and NIG distribution

Characteristic function is defined as

$$\Phi(\omega) = e^{\delta\sqrt{\alpha^2-\beta^2}} e^{-\delta\sqrt{\alpha^2-(\beta+i\omega)^2}} e^{i\mu\omega}. \tag{3.2}$$

Using characteristic function, we can calculate first four moments of NIG distribution

$$\text{mean} = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \tag{3.3}$$

$$\text{variance} = \frac{\alpha^2\delta}{(\alpha^2-\beta^2)^{3/2}} \tag{3.4}$$

$$\text{skewness} = \frac{3\beta}{\alpha\sqrt{\delta\sqrt{\alpha^2-\beta^2}}} \tag{3.5}$$

$$\text{kurtosis} = \frac{3\left(1+4\frac{\beta^2}{\alpha^2}\right)}{\delta\sqrt{\alpha^2-\beta^2}} \tag{3.6}$$

It is possible to estimate parameters of NIG by estimating these four moments and solving 4 equations with 4 unknowns. Necessary m.files and data can be found in folder *Normal Inverse Gaussian*. Method of moments should be according to [8] replaced by ohter methods due to its poor statistical behaviour, however, it is sufficient for our purposes.

When rescaling the hedging period, we use the property of NIG being closed under convolution. If $X \sim NIG(\alpha, \beta, \delta_1, \mu_1)$ and $Y \sim NIG(\alpha, \beta, \delta_2, \mu_2)$, then

$$X + Y \sim NIG(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2). \tag{3.7}$$

It means that a change of hedging interval is accounted for by changing parameters δ and μ in proper proportion.

Figure 3.2 shows again a histogram of Google daily returns over a period of two years fitted with normal (red line) and normal inverse Gaussian (green line) distribution.

3.3 Numerical Integration

Assume we know the density function $f(x)$ of ΔX. First we determine the relevant interval (A, B) for values of x. In case of normal distribution we use $(\mu - 5\sigma, \mu + \sigma)$ and for normal inverse Gaussian we take quantiles of 0.1% and 99.9% Numerical integration allows us to approximate integral of density function as a sum. We try to model this sum so that we can proclaim its terms to be respective probabilities of x.

3.3.1 Midpoint Rule

Midpoint rule is an approximation of integral of function $f(x)$ on interval (a, b) as

$$\int_a^b f(x)dx \approx (b - a) f(a + b - a/2). \tag{3.8}$$

It follows that the composite midpoint rule has the form of

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} hf(a + ih + h/2) \tag{3.9}$$

where $h = (b-a)/n$. Assuming that function $f(x)$ is continuous, we can derive the approximation error. We defer this derivation to Appendix. Approximation errors of the simple and composite rule are $O(h^3)$ and $O(h^2)$ respectively.

We rewrite composite midpoint rule in our setting as

$$\sum_{i=1}^{n} hf(x_i) \tag{3.10}$$

where h is an equal distance between values of x. Then we assign value x_i a probability $hf(x_i)$.

3.3.2 Trapezoid Rule

Trapezoid rule approximates integral of function f on interval (a,b) as

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2} \tag{3.11}$$

It follows that composite trapezoid rule has the form of

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} h\frac{f(a+ih)+f(a+(i+1)h)}{2} \tag{3.12}$$

where $h = (b-a)/n$.

Approximation error of trapezoid and composite trapezoid rule is derived in a similar way as for midpoint rule and we reach the same results. Error of trapezoid rule is $O(h^3)$ and error of composite trapezoid rule is $O(h^2)$.

Again, we rewrite composite trapezoid rule in our setting as

$$\sum_{i=1}^{n} h\frac{f(x_i-h/2)+f(x_i+h/2)}{2} \tag{3.13}$$

where h is an equal distance between values of x. Then we assign value x_i a probability $h\frac{f(x_i-h/2)+f(x_i+h/2)}{2}$.

3.3.3 Cumulative Distribution Function

When we stay in the framework of Black and Scholes, we can employ cumulative distribution function of normal distribution. We assign x_i probability $\Phi\left(\frac{x_i+h/2-\mu}{\sigma}\right) - \Phi\left(\frac{x_i-h/2-\mu}{\sigma}\right)$ where Φ is a cumulative distribution function of $N(0,1)$ distributed random variable.

Program codes for individual methods can be found in folder *Model for stock returns*.

Chapter 4

Results

In this chapter we finally present numerical results of above described calculations and compare them with results of Black and Scholes. The calculations are applicable to both fixed and floating strike options and we choose to demonstrate our results on floating strike put option. According to [9], price of a floating strike lookback put option in Black-Scholes model is

$$P_{\text{floating}} = -S_0 N\left(-d'\right) + e^{-rT} M S_0 N\left(-d' + \sigma\sqrt{T}\right) +$$
$$e^{-rT}\frac{\sigma^2}{2r} S_0 \left[-\left(\frac{S_0}{MS_0}\right)^{-\frac{2r}{\sigma^2}} N\left(d' - \frac{2r}{\sigma}\sqrt{T}\right) + e^{rT} N\left(d'\right)\right],$$

where

$$d' = \left(ln\frac{S_0}{MS_0} + rT + \frac{1}{2}\sigma^2 T\right)/\sigma\sqrt{T}$$

and N is the cumulative distribution function of $N(0,1)$. In Black-Scholes model, there is of course no hedging error.

We used Google historical prices to model stock returns. Since we compare our results only with Black-Scholes model, we can choose other parameters arbitrarily (within sensible bounds). We apply our framework to compute price of floating strike put option with time to maturity one year. We assume $S_0 = \mathrm{MS}_0 = 100$ and risk free rate of return is 2%. Let us denote length of rehedging interval Δt. We will change Δt (which is equivalent to changing the number of rehedging periods) and difference h and observe convergence of mean value process and unconditional expected squared hedging error.

4.1 Brownian motion

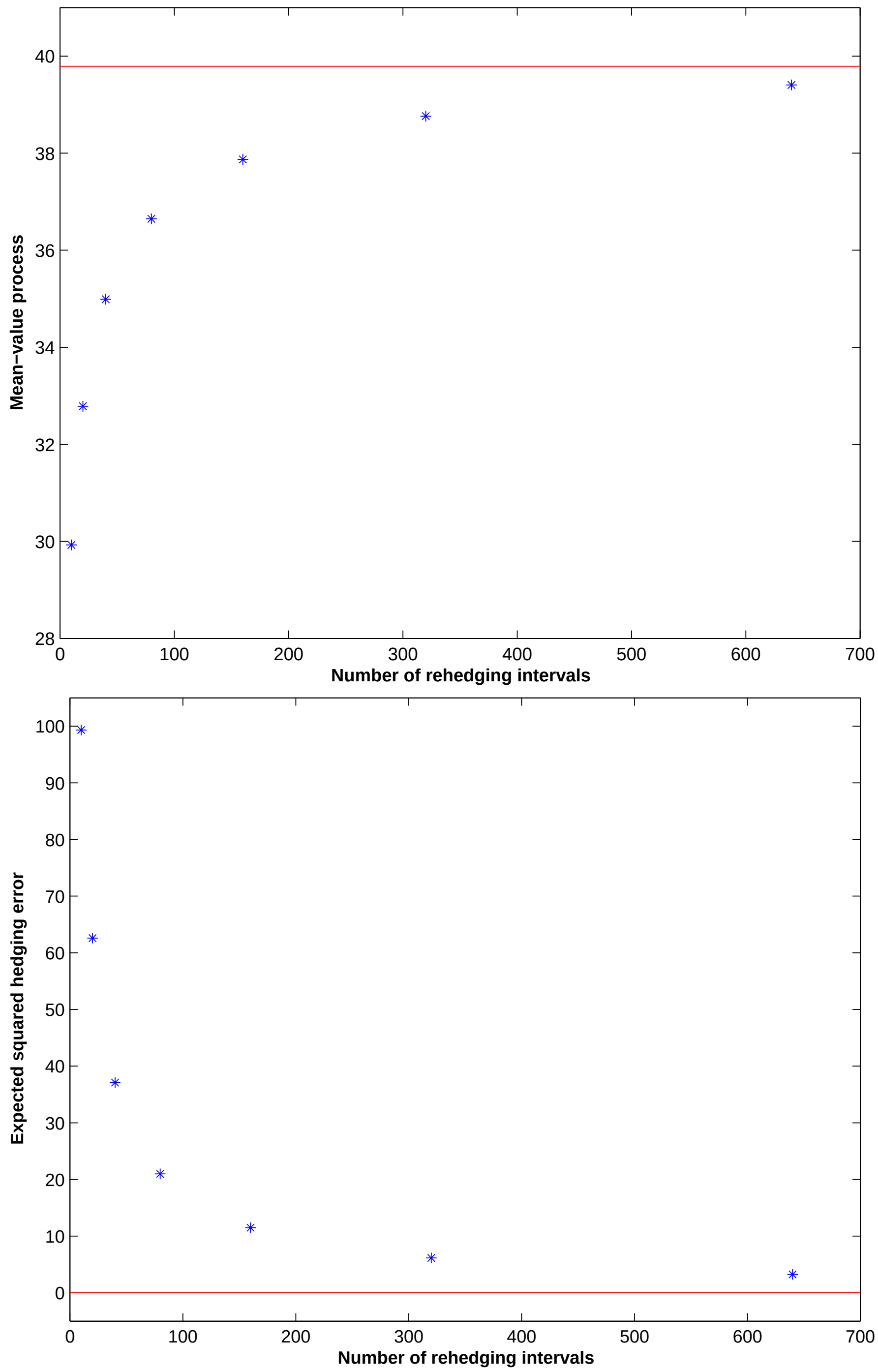
First we model data in accordance with Black-Scholes model (section 3.1). We therefore expect our model to behave in limit in accordance with Black-Scholes model. In other words, we expect V_0 to converge to price of the option as described above and expected squared hedging error to converge to 0.

4.1.1 Convergence in Number of Rehedging Intervals

In each iteration, we double the number of rehedging intervals starting with 10, while we keep the dimension of x constant. Let us remark that keeping h constant is not wise in this setting, since the relevant interval (A, B) diminishes with Δt. In other words, if we decrease length of rehedging interval ϵ times, we decrease h $\sqrt{\epsilon}$ times. Results are depicted in figures and table 4.1. Stars represent values of mean value process and expected squared hedging error. Red lines represent mean value process of Black-Scholes (price of the option) and its hedging error (0). Obviously, our model is in limit approaching values of standard Black-Scholes model. When dividing the length of rehedging interval by 2, differences between consecutive values of both, mean-value process and expected squared hedging error, decrease approximately by $1/\sqrt{2}$. We can say that the speed of convergence is directly proportional to square root of Δt.

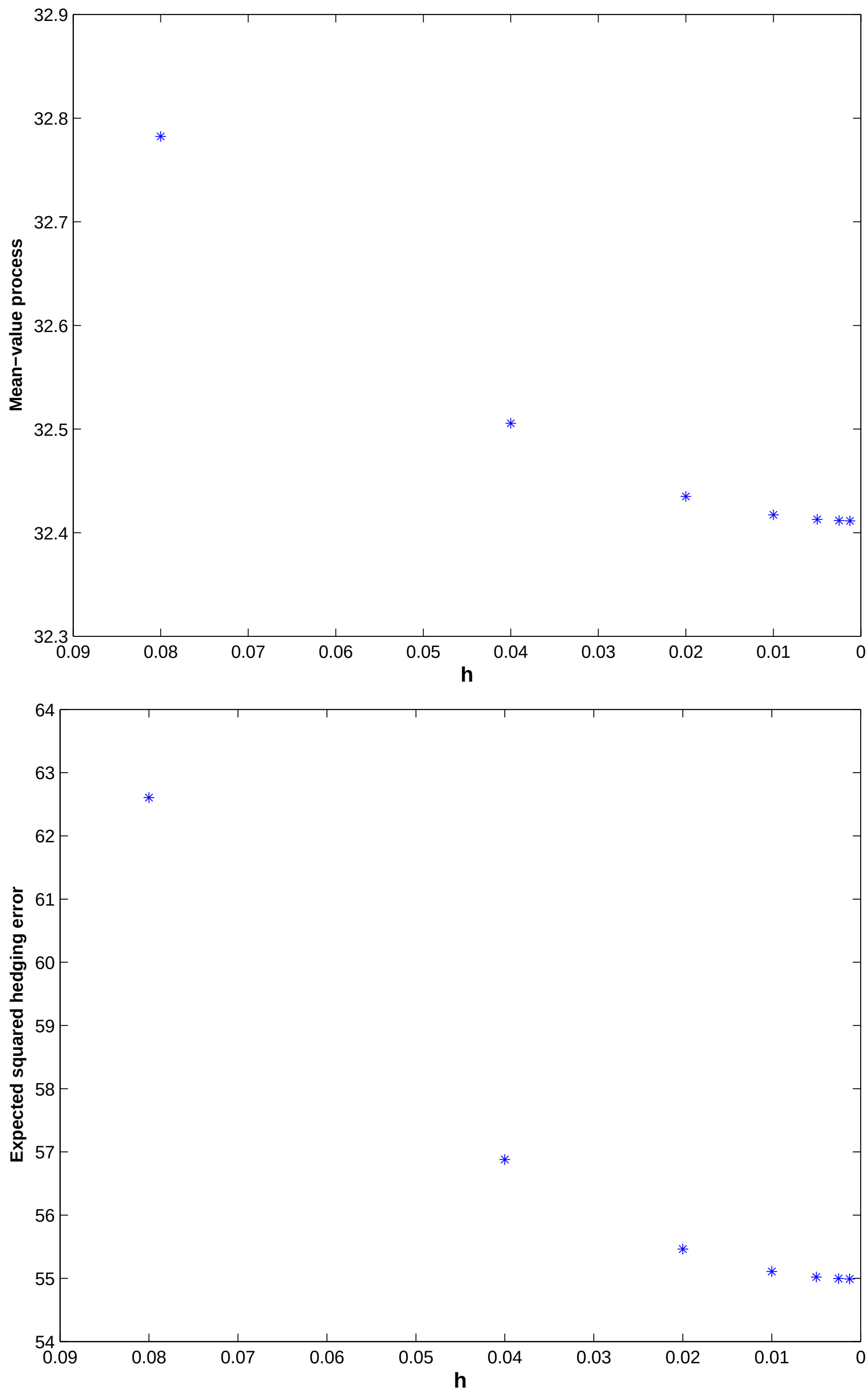
4.1.2 Convergence in Difference h

In this case we keep number of rehedging intervals constant, decrease length of difference h and observe how this richer stock lattice influences the solution. Mean-value process and expected squared hedging error really seem to have the same error as employed numerical integration methods, since differences between consecutive values are decreasing approximately four times with respect to the previous one, while we keep dividing the difference h by 2 (figure 4.2).



n =	10	20	40	80	160	320	640
Mean-value process	29.9265	32.7824	34.9882	36.6464	37.8699	38.7612	39.4047
Expected squared hedging error	99.3214	62.6062	37.0882	20.995	11.4976	6.146	3.2277

Figure 4.1: Convergence of mean-value process and expected squared hedging error



h =	**0.08**	**0.04**	**0.02**	**0.01**	**0.005**	**0.0025**	**0.00125**
Mean-value process	32.7824	32.5057	32.4351	32.4174	32.4129	32.4117	32.4115
Expected squared hedging error	62.6062	56.8807	55.4635	55.109	55.0197	54.9969	54.991

Figure 4.2: Convergence of mean-value process and expected squared hedging error

4.2 Normal Inverse Gaussian

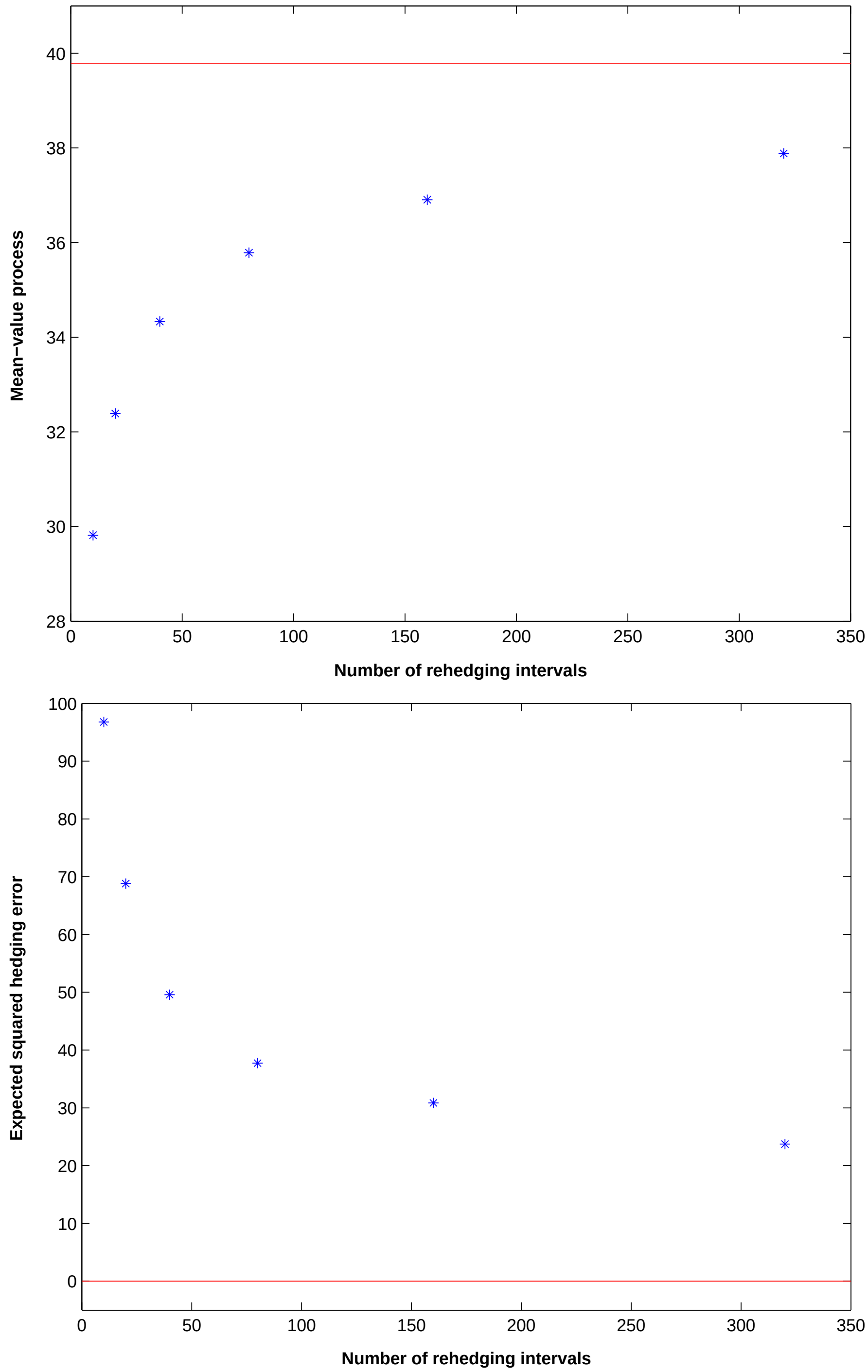
We perform the same calculations as in the previous section, but now we model stock returns as a NIG process (as described in section 3.2.1).

4.2.1 Convergence in Number of Rehedging Intervals

According to the results (figure4.3), values of this model are not converging to values of standard Black-Scholes model.

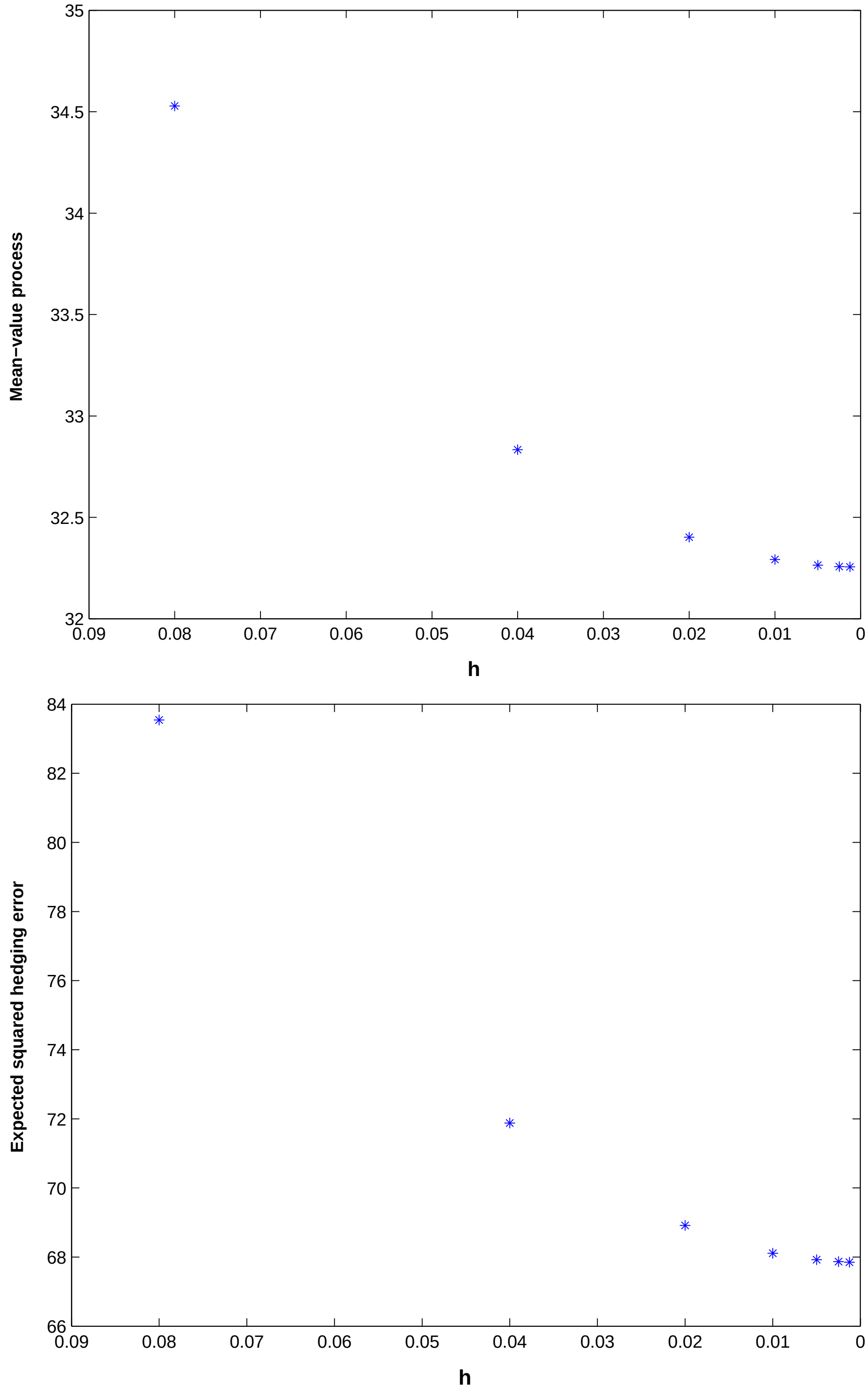
4.2.2 Convergence in Difference h

Since we use the same numerical integration methods as in section 4.1.2, we observe the same convergence (figure 4.4).



n =	10	20	40	80	160	320
Mean-value process	29.8166	32.3856	34.3304	35.7847	36.9045	37.8835
Expected squared hedging error	96.789	68.8016	49.5757	37.7401	30.8633	23.7295

Figure 4.3: Convergence of mean-value process and expected squared hedging error - NIG



h =	**0.08**	**0.04**	**0.02**	**0.01**	**0.005**	**0.0025**	**0.00125**
Mean-value process	34.5287	32.8336	32.4021	32.292	32.2645	32.2574	32.2556
Expected squared hedging error	83.543	71.8783	68.9148	68.1092	67.923	67.8665	67.8513

Figure 4.4: Convergence of mean-value process and expected squared hedging error - NIG

Chapter 5

Conclusions

We introduced general framework of unconditional expected squared hedging error minimization. We introduced simplifications and modifications (change of numeraire) required for efficient implementation for lookback options and we proved two theorems (1.3.1 and 1.4.1) to support existence of proposed solution. We suggested two models for stock returns (Brownian motion and normal inverse Gaussian) and estimated parameters of efficient lattice using numerical integration. We have then analysed convergence of mean-value process and expected squared hedging error in length of rehedging interval, while keeping the number of possible states of stock price over one rehedging period constant. When we stayed in the framework of Brownian motion for stock returns, calculated values of mean-value process and expected squared hedging error were limitely approaching values of standard Black-Scholes model and speed of this convergence was directly proportional to the square root of length of rehedging interval. This convergence was violated as a consequence of introducing normal inverse Gaussian for stock returns into the model.

Chapter 6

Appendix

6.1 Approximation Error of Midpoint Rule

It follows from continuity of f that there is a function $F(x)$ such that $F'(x) = f(x)$ and $\int_a^b f(x)dx = F(b) - F(a)$. Let us rewrite both terms as a Taylor expansion in the point $a + \frac{h}{2}$.

$$F(a) = F\left(a + \frac{h}{2}\right) - \frac{h}{2} f\left(a + \frac{h}{2}\right) + \frac{h^2}{8} f'\left(a + \frac{h}{2}\right) - \frac{h^3}{48} f''\left(a + \frac{h}{2}\right) + O\left(h^4\right)$$

$$F(b) = F\left(a + \frac{h}{2}\right) + \frac{h}{2} f\left(a + \frac{h}{2}\right) + \frac{h^2}{8} f'\left(a + \frac{h}{2}\right) + \frac{h^3}{48} f''\left(a + \frac{h}{2}\right) + O\left(h^4\right)$$

So we get

$$F(b) - F(a) - hf\left(a + \frac{h}{2}\right) = \frac{h^3}{24} f''\left(a + \frac{h}{2}\right) + O\left(h^4\right)$$

We have shown that the error of the midpoint rule is $O\left(h^3\right)$. It is very straightforward now to calculate the error of the composite midpoint rule. We will divide the interval (a, b) into n intervals with length $h = \frac{b-a}{n}$ and write the composite midpoint rule as

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} hf(a + ih + h/2) \tag{6.1}$$

Since this is a sum of simple rules, its error is n times the error of the simple rule.

$$\begin{aligned} n\left(\frac{h^3}{24}f''\left(a+\frac{h}{2}\right)+O\left(h^4\right)\right) &= \\ \frac{b-a}{h}\left(\frac{h^3}{24}f''\left(a+\frac{h}{2}\right)+O\left(h^4\right)\right) &= \\ h^2\frac{(b-a)}{24}f''\left(a+\frac{h}{2}\right)+O\left(h^3\right) & \end{aligned}$$

The approximation error of the composite midpoint rule is $O\left(h^2\right)$.

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